Note on the number of semistar operations, VIII

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We concern with the number of star operations or of semistar operations on an integral domain or on a semigroup. Since all the proofs in [M 4,7,8] and [MS] are not necessarily complete, it is necessary for us to make a self-contained, well got together and systematically written paper on the subject. Some results in this paper are new. We confer Badawi-Houston [BH], Hedstrom-Houston [HH], Heinzer [H], and many other references.

Throughout the paper, D denotes an integral domain, and S denotes a nonzero g -monoid. A g -monoid means a submonoid of a torsion-free abelian additive group.

The set of star operations on D is denoted by $\Sigma(D)$.

The set of star operations on S is denoted by $\Sigma(S)$.

The set of semistar operations on D is denoted by $\Sigma'(D)$.

The set of semistar operations on S is denoted by $\Sigma'(S)$.

Let T be an overring of D , and let d'_T be the d' -operation on T . Then the descent of d'_T to D is denoted by $*_T$ (or by $*(T)$).

Let T be an oversemigroup of S , and let d'_T be the d' -operation on T . Then the descent of d'_T to S is denoted by $*_T$ (or by $*(T)$).

Let Γ be a totally ordered set, and let X be a subset of Γ . Then ' $\inf$ ' of X in Γ is denoted by ' $\inf^\Gamma X$ ' (or simply by ' $\inf X$ '). ' $\sup$ ' of X in Γ is denoted by ' $\sup^\Gamma X$ ' (or simply by ' $\sup X$ '). Also $\inf^\Gamma X = -\infty$ means that X is not bounded below, and $\inf^\Gamma X > -\infty$ means that X is bounded below.

For any set X , the cardinality of X is denoted by $|X|$.

Lemma 1. *If $|\Sigma'(S)| < \infty$, then the integral closure $\bar{S}$ of S is a valuation semigroup.*

Proof. Suppose the contrary. Let $G = q(S)$. There exists an element $x \in G - \bar{S}$ such that $-x \notin \bar{S}$. Then we have $S[2^n x] \subsetneq S[2^{n+1} x]$ for each positive integer n . Let d'_n be the d' -operation on $S[2^n x]$, and let $*_n$ be the descent of d'_n to S . There arises an infinite number of semistar operations $*_1, *_2, \dots$ on S .

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Proposition 1. *Let V be a valuation semigroup with maximal ideal M .*

- (1) *If M is a principal ideal of V , then $|\Sigma(V)| = 1$.*
- (2) *If M is not a principal ideal of V , then $|\Sigma(V)| = 2$.*

Proof. Let v be a valuation on $G = q(V)$ belonging to V , and let Γ be the value group of v . Let I be an ideal of V .

The case where there exists $x \in I$ such that $v(x) = \min v(I)$: Then $I = (x)$.

The case where there exists $x \in V$ such that $v(x) = \inf_{\Gamma} v(I) \notin v(I)$: Then $I = x + M$.

The case where $\inf_{\Gamma} v(I)$ does not exist: Then $I = I^v$.

- (1) Each ideal of V is divisorial. It follows that $\Sigma(V) = \{d\}$.
- (2) Since $V = M^v$, we have $v \neq d$. Let $*$ be a star operation on V . Then, if $M^* = M$, then $* = d$, and if $M^* = V$, then $* = v$.

The following Proposition is a ring version of Proposition 1.

Proposition 2. *Let V be a valuation domain with maximal ideal M .*

- (1) *If M is a principal ideal, then $|\Sigma(V)| = 1$.*
- (2) *If M is not a principal ideal, then $|\Sigma(V)| = 2$.*

The set of nonzero fractional ideals of D is denoted by $F(D)$. The set of nonzero D -submodules of K is denoted by $F'(D)$, where K is the quotient field of D . The set of fractional ideals of S is denoted by $F(S)$. The set of S -submodules of G is denoted by $F'(S)$, where G is the quotient group of S .

Lemma 2. (1) *Let D be a valuation ring, or let D be a Prüfer ring with exactly two maximal ideals M and N such that there exists a prime ideal P of D with $M \cap N \not\supseteq P \not\supseteq (0)$. Then $F'(D) = F(D) \cup \{K\}$, where $K = q(D)$.*

(2) *Let S be a valuation semigroup. Then $F'(S) = F(S) \cup \{G\}$, where $G = q(S)$.*

Hence each star operation on D (resp. S) has a unique extension to a semistar operation on D (resp. S).

Proof. (1) Assume that D is not a valuation ring. Let v (resp. w) be a valuation belonging to $V = D_M$ (resp. $W = D_N$). Let $I \in F'(D)$.

The case where $v(I)$ and $w(I)$ are bounded below: There exists a nonzero element $a \in D$ such that $aI \subset V$ and $aI \subset W$. Then $I \in F(D)$.

The case where neither $v(I)$ nor $w(I)$ is bounded below: Let x be a nonzero element of K . There exist elements $a \in D$ and $\pi \in P$ such that $x = \frac{a}{\pi}$. There

exists an element $i \in I$ such that $v(i\pi^2) < 0$. We have $\frac{1}{i\pi^2} = \frac{b}{s}$ for elements $b \in D$ and $s \in D - M$. It follows that $\frac{x}{i} = \frac{ab\pi}{s} \in V \cap W$. Hence $x \in iD \subset I$, and hence $I = K$.

The case where one of $v(I)$ and $w(I)$, say $w(I)$, is bounded below and the other is not bounded below: There exists an element $a \in D - \{0\}$ such that $w(\frac{1}{a}) < w(i)$ for each $i \in I$. Let $\pi \in P - \{0\}$, and choose $i_0 \in I$ such that

$v(i_0) < v(\frac{1}{a\pi})$. Since $\frac{1}{a\pi i_0} \in V$, there exist $a \in D$ and $s \in D - M$ such that $\frac{1}{a\pi i_0} = \frac{d}{s}$. Then we have $w(i_0) < w(\frac{1}{a})$; a contradiction.

Let T be an overring of D , and let $*$ be a semistar operation on T . If we let $I^{\delta(*)} = (IT)^*$ for each $I \in F'(D)$, then $\delta(*)$ is a semistar operation on D , and is called the descent of $*$ to D .

Theorem 1. *Let D be an integrally closed local ring with dimension $n < \infty$. The following conditions are equivalent.*

- (1) D is a valuation ring.
- (2) $n + 1 \leq |\Sigma'(D)| \leq 2n + 1$.

Proof. (1) $\implies$ (2): We rely on the dimension of D , and suppose that the assertion holds for each integer $k < n$.

Let $M \supsetneq P_{n-1} \supsetneq \cdots \supsetneq P_1 \supsetneq \{0\}$ be the prime ideals of $V = D$. Let d'_i be the d' -operation on $U_i = V_{P_i}$, and let $*_i$ be the descent of d'_i to V . Then $\Sigma'(V)$ contains $\{d', *_1, \dots, *_{n-1}, e\}$.

Let $|\Sigma'(U_{n-1})| = k$ and let $\Sigma'(U_{n-1}) = \{*_1', \dots, *_k'\}$, where $k \leq 2(n-1) + 1$. Let $*$ be a semistar operation on V with $*$ $\notin \{\delta(*_1'), \dots, \delta(*_k')\}$, where $\delta(*'_i)$ is the descent of $*'_i$ to V . Then $V^* = V$. The restriction $*|_{F(V)}$ is a star operation on V . Proposition 2 implies that either $*$ = d' or $*$ = v' . Hence $|\Sigma'(V)| \leq k + 2 \leq 2n + 1$.

(2) $\implies$ (1): D is the intersection of a family of valuation overrings of D . Since $|\Sigma'(D)| < \infty$, D is the intersection of a finite family of valuation overrings of D . Since D is local, D is a valuation ring.

The following is a semigroup version of Theorem 1.

Theorem 2. *Let S be an integrally closed semigroup with dimension $n < \infty$. The following conditions are equivalent.*

- (1) S is a valuation semigroup.
- (2) $n + 1 \leq |\Sigma'(S)| \leq 2n + 1$.

We have the following,

Theorem 3. *Let D be an integrally closed local ring. The following conditions are equivalent.*

- (1) $|\Sigma'(D)| < \infty$.
- (2) D is a valuation ring with dimension $< \infty$.

The following is a semigroup version of Theorem 3.

Theorem 4. *Assume that S is integrally closed. The following conditions are equivalent.*

- (1) $|\Sigma'(S)| < \infty$.
- (2) S is a valuation semigroup with dimension $< \infty$.

Lemma 3. *Let V be a valuation semigroup, and let Γ be its value group. Let P be a prime ideal of V , and let H be the convex subgroup of Γ associated to P .*

- (1) *The ordered additive group Γ/H belongs to the valuation semigroup V_P .*

(2) The ordered additive group H belongs to the valuation semigroup $V - P$.

Proof. Let v be a valuation on $G = q(V)$ onto Γ belonging to V .

(1) Let w be the composition of the mappings v and the canonical mapping $\Gamma \rightarrow \Gamma/H$. Then w is a valuation on G with value group Γ/H . Let $w(\alpha) \geq 0$ for $\alpha \in G$. There exists an element $a \in V$ such that $v(\alpha) - v(a) \in H$. There exists an element $s \in V - P$ such that either $v(\alpha) - v(a) = v(s)$ or $v(\alpha) - v(a) = -v(s)$. Hence we have $\alpha \in V_P$. Therefore the valuation semigroup of w is V_P .

(2) The restriction $v|_{V-P}$ from $V - P$ to H determines a valuation w on $q(V - P)$. The valuation semigroup belonging to w is $V - P$.

Theorem 5. Assume that D has dimension $n < \infty$. The following conditions are equivalent.

- (1) D is a discrete valuation ring.
- (2) $|\Sigma'(D)| = n + 1$.

Proof. (1) $\implies$ (2): Let M be the maximal ideal of $V = D$, and let $M \supsetneq P_{n-1} \supsetneq \cdots \supsetneq P_1 \supsetneq (0)$ be the prime ideals of V . Then $U_i = V_{P_i}$ is a discrete valuation ring for each i . Let $*_i = *_{U_i}$.

We will show that $\Sigma'(V) = \{e, *_1, \dots, *_{n-1}, d'\}$ by the induction on $\dim(V)$. Suppose that the assertion holds for each $i < n$.

Let $*$ be a semistar operation on V .

The case where $V^* = K$, where $K = q(V)$: Then we have $* = e$.

The case where $V^* = U_i$ for some i : Let $\alpha(*)$ be the ascent of $*$ to U_i . Then we have $\alpha(*) = d'_{U_i}$ by the induction hypothesis. For each $I \in F'(V)$, we have

$$I^* = (IV)^* = (IU_i)^{\alpha(*)} = IU_i = I^{*_i}.$$

That is, $* = *_i$.

The case where $V^* = V$: Since V is discrete, M is principal. The restriction $*|_{F(V)}$ coincides with d by Proposition 2. Hence $* = d'$.

Therefore $|\Sigma'(V)| = n + 1$.

(2) $\implies$ (1): Let N be a maximal ideal of $\bar{D}$ with height n . We have $D = (\bar{D})_N$ by the assumption. That is, D is an integrally closed local ring. Then Theorem 3 implies that $V = D$ is a valuation ring. Let $M \supsetneq P_{n-1} \supsetneq \cdots \supsetneq P_1 \supsetneq (0)$ be the prime ideals of V , and let $U_i = V_{P_i}$ for each i . Let $*_i = *_{U_i}$ for each i .

We will show that V is a discrete valuation ring by the induction on $\dim(V)$. Suppose that the assertion holds for each $i < n$.

Let Γ be a value group belonging to V , and let H_i be the convex subgroup of Γ associated to P_i for each i .

The case where M is principal: Then we have $|\Sigma'(U_{n-1})| = n$, Γ has a minimal positive element, and H_{n-1} is order-isomorphic with $\mathbf{Z}$. Since U_{n-1} is discrete, Γ is discrete. That is, V is discrete.

The case where M is not principal: Then v' differs from each of $e, *_1, \dots, *_{n-1}, d'$. Hence we have $|\Sigma'(V)| > n + 1$; a contradiction.

The following is a semigroup version of Theorem 5.

Theorem 6. Assume that S has dimension $n < \infty$. The following conditions are equivalent.

- (1) S is a discrete valuation semigroup.
- (2) $|\Sigma'(S)| = n + 1$.

Theorem 7. Let V be a valuation semigroup with dimension $n < \infty$, let v be its valuation, and let Γ be the value group of v . Let $M = P_n \supsetneq P_{n-1} \supsetneq \cdots \supsetneq P_1$ be the prime ideals of V , let $H_n = \{0\} \subsetneq H_{n-1} \subsetneq \cdots \subsetneq H_1 \subsetneq \Gamma$ be the convex subgroups of Γ , and let $U_i = V_{P_i}$ for each i . Let m be a positive integer with $n + 1 \leq m \leq 2n + 1$. Then the following conditions are equivalent.

- (1) $|\Sigma'(V)| = m$.
- (2) The maximal ideal of U_i is principal for exactly $2n + 1 - m$ of i in $\{1, \dots, n\}$.
- (3) The ordered additive group Γ/H_i has a minimal positive element for exactly $2n + 1 - m$ of i in $\{1, \dots, n\}$.

Proof. (2) $\implies$ (1): We rely on the induction on $\dim(V)$. Suppose that the assertion holds for each positive integer $< n$.

(i) The case where M is not principal: Since the assertion holds for U_{n-1} , and since $2n + 1 - m = 2(n - 1) + 1 - (m - 2)$, we have $|\Sigma'(U_{n-1})| = m - 2$. Let $\Sigma'(U_{n-1}) = \{*_1, \dots, *_{m-2}\}$. Then $\delta(*_1), \dots, \delta(*_{m-2})$, d' , and v' are distinct with each other, where $\delta(*_i)$ is the descent of $*_i$ to V . Let $*$ be a semistar operation on V .

If $V^* = V$, then either $* = d'$ or $* = v'$.

If $q(V) \supsetneq V^* \supsetneq V$, we have $V^* = V_{P_i}$ for some $i < n$. Let $\alpha(*)$ be the ascent of $*$ to V_{P_i} . Then $*$ is the descent of $\alpha(*)$ to V .

It follows that $|\Sigma'(V)| = m$.

(ii) The case where M is principal: Since the assertion holds for U_{n-1} , and since $2n - m = 2(n - 1) + 1 - (m - 1)$, we have $|\Sigma'(U_{n-1})| = m - 1$. Let $\Sigma'(U_{n-1}) = \{*_1, \dots, *_{m-1}\}$. Then $\delta(*_1), \dots, \delta(*_{m-1})$ and d' are distinct with each other. Let $*$ be a semistar operation on V .

If $V^* = V$, then $* = d'$.

If $q(V) \supsetneq V^* \supsetneq V$, we have $V^* = V_{P_i}$ for some $i < n$. Let $\alpha(*)$ be the ascent of $*$ to V_{P_i} . Then $*$ is the descent of $\alpha(*)$ to V .

Therefore $|\Sigma'(V)| = m$.

(2) $\implies$ (3): Since the value group of V_{P_i} is Γ/H_i , Γ/H_i has a minimal positive element for exactly $2n + 1 - m$ of i .

The following is a ring version of Theorem 7.

Theorem 8. Let V be a valuation domain with dimension n , let v be its valuation, and let Γ be the value group of v . Let $M = P_n \supsetneq P_{n-1} \supsetneq \cdots \supsetneq P_1 \supsetneq (0)$ be the prime ideals of V , and let $H_n = \{0\} \subsetneq H_{n-1} \subsetneq \cdots \subsetneq H_1 \subsetneq \Gamma$ be the convex subgroups of Γ . Let m be a positive integer with $n + 1 \leq m \leq 2n + 1$. Then the following conditions are equivalent.

- (1) $|\Sigma'(V)| = m$.
- (2) The maximal ideal of V_{P_i} is principal for exactly $2n + 1 - m$ of i in $\{1, \dots, n\}$.
- (3) Γ/H_i has a minimal positive element for exactly $2n + 1 - m$ of i in $\{1, \dots, n\}$.

Example 1. Let n and m be positive integers with $n + 1 \leq m \leq 2n + 1$. Let $k_1, \dots, k_{2n+1-m} \in \{1, \dots, n\}$ be distinct with each other (The set $\{k_1, \dots, k_{2n+1-m}\}$ may be empty). Let $G_i = \mathbf{Z}$ with the standard order for each $i \in \{k_1, \dots, k_{2n+1-m}\}$, and let $G_j = \mathbf{R}$ with the standard order for each $j \in \{1, \dots, n\} - \{k_1, \dots, k_{2n+1-m}\}$. Let $\Gamma = G_1 \oplus \dots \oplus G_n$ be the direct sum of the G_i , and introduce the lexicographic order on Γ ; Let $x = (a_1, \dots, a_n), y = (b_1, \dots, b_n)$ be elements of $G_1 \oplus \dots \oplus G_n$ with $a_n < b_n$, then let $x < y$. There exists a valuation domain V with the value group Γ . Let $P_n \supsetneq P_{n-1} \supsetneq \dots \supsetneq P_1 \supsetneq \{0\}$ be the prime ideals of V . Then the maximal ideal of V_{P_i} is principal if and only if G_{n-i+1} is discrete. And G_{n-i+1} is discrete if and only if $i \in \{n + 1 - k_1, \dots, n + 1 - k_{2n+1-m}\}$.

Lemma 4. Let D be a Prüfer ring with exactly two maximal ideals M and N . Then D is a Bezout ring.

Proof. Let v (resp. w) be valuations belonging to $V = D_M$ (resp. $W = D_N$). Let a and b be nonzero elements of D .

The case (1) where $v(a) \leq v(b)$ and $w(a) \leq w(b)$: Then $d = \frac{b}{a} \in D$. Hence $(a, b) = (a)$.

The case (2) where $v(a) < v(b)$ and $w(a) > w(b)$: Then $v(a + b) = v(a), w(a + b) \leq w(a)$. The case (1) implies that $(a + b, a) = (a + b)$. Hence $(a, b) = (a + b)$.

Proposition 3. Let D be an n -dimensional Prüfer ring with $n > 2$ and with exactly two maximal ideals M and N . Assume that there exist prime ideals $P_1, P_2, \dots, P_{n-1}$ of D such that $M \cap N \supsetneq P_{n-1} \supsetneq \dots \supsetneq P_2 \supsetneq P_1 \supsetneq (0)$, and that there exist elements $\pi_1, \pi_2, \dots, \pi_{n-1}, p$ and q of D such that $P_i D_{P_i} = \pi_i D_{P_i}$ for each i , $M = (p)$ and $N = (q)$.

Define the fractional ideals $A_{n-1} = \cup_1^\infty (\frac{1}{\pi_{n-1}^m}), \dots, A_2 = \cup_1^\infty (\frac{1}{\pi_2^m}), A = \cup_1^\infty (\frac{1}{p^m}), B = \cup_1^\infty (\frac{1}{q^m})$ and $C = \cup_1^\infty (\frac{1}{p^m q^m})$ of D .

For each nonzero fractional ideal I of D , let $I^{*1} = I$ if I is of the form xA , and let $I^{*1} = I^v$ otherwise, where $x \in K = \mathbf{q}(D)$. Let $I^{*2} = I$ if I is of the form xB , and let $I^{*2} = I^v$ otherwise.

(1) Each element x of $K - \{0\}$ can be expressed as $\pi_1^{a_1} \pi_2^{a_2} \dots \pi_{n-1}^{a_{n-1}} p^b q^c u$ with integers a_i, b, c and a unit u of D . This expression is unique for the element x .

(2) Each non-principal fractional ideal I of D is of the form xA_2 or $\dots$ or xA_{n-1} or xA or xB or xC with $x \in K - \{0\}$.

(3) We have $P_1 = P_1^v = \pi_1 A_2, P_2 = P_2^v = \pi_2 A_3, \dots$, and $P_{n-2} = P_{n-2}^v = \pi_{n-2} A_{n-1}$.

(4) We have $P_{n-1} = P_{n-1}^v = \pi_{n-1} C = \pi_{n-1} A^v = \pi_{n-1} B^v, A \neq A^v, B \neq B^v$, and $A \neq xB$ for each $x \in K$.

(5) $*_1$ and $*_2$ are star operations on D .

(6) $*_1, *_2, d$ and v are distinct with each other.

Proof. Let $V = D_M, W = D_N, U_i = D_{P_i}$ for each i , let $M_i = P_i U_i$, let v (resp. w, u_i) be a valuation belonging to V (resp. W, U_i) with $v(p) = 1$ (resp.

$w(q) = 1, u_i(\pi_i) = 1$).

(1) Let $y \in D - \{0\}$. Since $u_1(M_1)$ has a minimal element $u_1(\pi_1)$, we have $y = \pi_1^k u$ with $k \geq 0$ and unit u of U_1 .

Let $y \in D - P_1$. Since $u_2(M_2)$ has a minimal element $u_2(\pi_2)$, we have $y = \pi_2^k u$ with $k \geq 0$ and unit u of U_2 .

...

Let $y \in D - P_{n-2}$. Since $u_{n-1}(M_{n-1})$ has a minimal element $u_{n-1}(\pi_{n-1})$, we have $y = \pi_{n-1}^k u$ with $k \geq 0$ and unit u of U_{n-1} .

Let $y \in D - P_{n-1}$. Since $v(MV)$ (resp. $v(NW)$) has a minimal element $v(p)$ (resp. $w(q)$), we have $y = p^{k_1} q^{k_2} u$ with $k_1 \geq 0, k_2 \geq 0$ and unit u of D .

(2) We may assume that there exist principal fractional ideals $I_n = (x_n)$ of D such that $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ and that $I = \bigcup_1^\infty I_n$. Each x_i is of the form $\pi_1^{a_{i,1}} \pi_2^{a_{i,2}} \cdots \pi_{n-1}^{a_{i,n-1}} p^{b_i} q^{c_i} u_i$ with integers $a_{i,j}, b_i, c_i$ and unit u_i of D .

If $\inf_i a_{i,1} = -\infty$, then $I = K$. Hence we may assume that $x_i = \pi_2^{a_{i,2}} \cdots \pi_{n-1}^{a_{i,n-1}} p^{b_i} q^{c_i}$. If $\inf_i a_{i,2} = -\infty$, then $I = A_2$. If $\inf_i a_{i,2} > -\infty$, then we may assume that $x_i = \pi_3^{a_{i,3}} \cdots \pi_{n-1}^{a_{i,n-1}} p^{b_i} q^{c_i}$. If $\inf_i a_{i,3} = -\infty$, then $I = A_3$. If $\inf_i a_{i,3} > -\infty$, we may assume that $x_i = \pi_4^{a_{i,4}} \cdots \pi_{n-1}^{a_{i,n-1}} p^{b_i} q^{c_i}$ We may assume that $x_i = \pi_{n-1}^{a_{i,n-1}} p^{b_i} q^{c_i}$. If $\inf_i a_{i,n-1} = -\infty$, then $I = A_{n-1}$. If $\inf_i a_{i,n-1} > -\infty$, then we may assume that $x_i = p^{b_i} q^{c_i}$. If $\inf_i b_i > -\infty$, we may assume that $x_i = q^{c_i}$. Then $I = B$. If $\inf_i c_i > -\infty$, we may assume that $x_i = p^{b_i}$. Then $I = A$. If $\inf_i b_i = \inf_i c_i = -\infty$, then $I = C$.

(3) We have $P_1 = \bigcap_1^\infty (\pi_2^m)$, and hence $P_1 = P_1^v$; $P_2 = \bigcap_1^\infty (\pi_3^m)$, and hence $P_2 = P_2^v$; ...; $P_{n-2} = \bigcap_1^\infty (\pi_{n-1}^m)$, and hence $P_{n-2} = P_{n-2}^v$.

Next, $\pi_{n-2} A_{n-1} = \bigcup_1^\infty (\frac{\pi_{n-2}}{\pi_{n-1}^m}) = P_{n-2}$; ...; $\pi_1 A_2 = \bigcup_1^\infty (\frac{\pi_1}{\pi_2^m}) = P_1$.

(4) $\pi_{n-1} C = \bigcup_1^\infty (\frac{\pi_{n-1}}{p^m q^m}) = P_{n-1}$. $P_{n-1} = \bigcap_1^\infty (p^m q^m)$, and hence $P_{n-1} = P_{n-1}^v$. Assume that $\pi_{n-1} A \subset (\alpha)$ for an element $\alpha \in K$. It follows that $P_{n-1} \subset (\alpha)$. Hence $\pi_{n-1} A^v = P_{n-1}$. Similarly, $\pi_{n-1} B^v = P_{n-1}$.

Suppose that $A = xB$ for some $x \in K$. Then we have $x = p^b q^c u$, where b, c are integers and u is a unit of D ; a contradiction.

(5) Let I and J be nonzero fractional ideals of D such that $I \subset J$. We will show that $I^{*1} \subset J^{*1}$. We may assume that $I \neq xA$ for each $x \in K$, and that $J = A$.

We must show that $I^v \subset A$. We may assume that $I \neq I^v$. Then (2) and (3) imply that $I = xB$ for some $x \in K - \{0\}$; $xB \subset A$. Since $\frac{x}{q^m} \in A$ for each $m > 0$, we have $w(x) \geq m$ for each $m > 0$. Hence we have $x \in P_{n-1}$. Therefore $I \subset (\frac{1}{p})$, and hence $I^v \subset (\frac{1}{p}) \subset A$.

Similarly, $*_2$ is a star operation on D .

(6) Consider A and B .

Remark 1. Let D be a 2-dimensional Prüfer ring with exactly two maximal ideals M and N . Assume that there exists a prime ideal P of D such that $M \cap N \supsetneq P \supsetneq (0)$, and that there exist elements π, p and q of D such that

$PD_P = \pi D_P$, $M = (p)$ and $N = (q)$. Define the fractional ideals $A = \cup_1^\infty (\frac{1}{p^m})$, $B = \cup_1^\infty (\frac{1}{q^m})$ and $C = \cup_1^\infty (\frac{1}{p^m q^m})$ of D .

For each nonzero fractionl ideal I of D , let $I^{*1} = I$ if I is of the form xA , and let $I^{*1} = I^v$ otherwise, where $x \in K = q(D)$. Let $I^{*2} = I$ if I is of the form xB , and let $I^{*2} = I^v$ otherwise.

- (1) Each element x of $K - \{0\}$ can be expressed as $\pi^a p^b q^c u$ with integers a, b, c and unit u of D . This expression is unique for the element x .
- (2) Each non-principal fractional ideal I of D is of the form xA or xB or xC with $x \in K - \{0\}$.
- (3) We have $P = P^v = \pi C = \pi A^v = \pi B^v$, $A \neq A^v$, and $B \neq B^v$.
- (4) $A \neq xB$ for each $x \in K$.
- (5) $*_1$ and $*_2$ are star operations on D .
- (6) $*_1, *_2, d$ and v are distinct with each other.

Theorem 9. *Let D be an n -dimensional Prüfer ring with $n > 1$ and with exactly two maximal ideals M and N . Assume that there exist prime ideals $P_1, P_2, \dots, P_{n-1}$ of D such that $M \cap N \supsetneq P_{n-1} \supsetneq \dots \supsetneq P_1 \supsetneq (0)$, and that $P_i D_{P_i}$ is a principal ideal of $U_i = D_{P_i}$ for each i , M and N are principal ideals of D .*

Then we have $|\Sigma'(D)| = n+6$, and $\Sigma'(D) = \{e, *_{U_1}, \dots, *_{U_{n-1}}, *_{V}, *_{W}, d', v', *'_1, *'_2\}$, where $*'_1$ (resp. $*'_2$) is the canonical extension of $*_1$ (resp. $*_2$) in Proposition 3 or in Remark 1 to a semistar operation on D , and $V = D_M, W = D_N$.

Proof. Suppose that there exists a semistar operation $*'$ on D such that $*' \notin \{e, *_{U_1}, \dots, *_{U_{n-1}}, *_{V}, *_{W}, d', v', *'_1, *'_2\}$.

If $D^{*'} = K = q(D)$, then $*' = e$.

If $D \subsetneq D^{*'} \subsetneq K$, then $R = D^{*'}$ is a member in $\{U_1, \dots, U_{n-1}, V, W\}$. We have $\alpha_{R/D}(*') = d'_R$, and hence $*' = *_{R}$.

Hence we may suppose that $D^{*' = D}$. Then $*'|_{F(D)} = *$ is a star operation on D . Then $* \notin \{d, v, *_1, *_2\}$ by the assumption.

The case where $A^* = A, B^* = B$ in Proposition 3 or Remark 1: Then $* = d$; a contradiction.

The case where $A^* = A, B^* = B^v$: Then $* = *_1$; a contradiction.

The case where $A^* = A^v, B^* = B$: Then $* = *_2$; a contradiction.

The case where $A^* = A^v, B^* = B^v$: Then $* = v$; a contradiction.

The case where $A \subsetneq A^* \subsetneq A^v$: Note that $A^v = \cup (\frac{1}{p^m q^m})$.

If $A^* \subset (\frac{1}{p^m q^m})$ for some m , then $A^v \subset (\frac{1}{p^m q^m})$; a contradiction. Therefore, for each m , there exists $n > m$ such that $(\frac{1}{p^n q^n})$ contains an element x_m of A^* with $x_m \notin (\frac{1}{p^{n-1} q^{n-1}})$. Then $x_m = \frac{u}{p^k q^{k'}}$, where $k, k' \leq n$ and u is a unit of D .

If $k, k' \leq n-1$, then $x_m \in (\frac{1}{p^{n-1}q^{n-1}})$; a contradiction. Hence either k or k' , say k' , equals to n ; $x_m = \frac{u}{p^k q^n}$ for some $k \leq n$. Then we have

$$\frac{1}{p^n q^n} \in \frac{1}{p^n q^n} D = \frac{1}{p^n q^n} (p^{n-k}, q^n) = (\frac{1}{p^k q^n}, \frac{1}{p^n}) \subset A^*,$$

and hence we have $y_m = \frac{1}{p^n q^n} \in A^*$. Then we have $\cup_m (\frac{1}{p^m q^m}) \subset \cup_m (y_m) \subset A^*$, and hence $A^v \subset A^*$; a contradiction.

Proposition 4. *Let D be an n -dimensional Prüfer ring with $n > 1$ and with exactly two maximal ideals M and N . Assume that there exist prime ideals $P_1, P_2, \dots, P_{n-1}$ of D such that $M \cap N \supsetneq P_{n-1} \supsetneq \dots \supsetneq P_2 \supsetneq P_1 \supsetneq (0)$, and that there exist elements p and q of D such that $M = (p)$ and $N = (q)$.*

Define the fractional ideals $A = \cup_1^\infty (\frac{1}{p^m})$, $B = \cup_1^\infty (\frac{1}{q^m})$, and $C = \cup_1^\infty (\frac{1}{p^m q^m})$.

For each nonzero fractional ideal I of D , let $I^{*1} = I$ if I is of the form xA , and let $I^{*1} = I^v$ otherwise, where $x \in K = q(D)$. Let $I^{*2} = I$ if I is of the form xB , and let $I^{*2} = I^v$ otherwise.

- (1) $P_1 = P_1^v, \dots, P_{n-1} = P_{n-1}^v$.
- (2) $C = A^v = B^v, A \neq A^v$, and $B \neq B^v$. $A \neq xB$ for each $x \in K$.
- (3) $*_1$ and $*_2$ are star operations on D .
- (4) $*_1, *_2, d$ and v are distinct with each other.

Proof. (1) We have $P_1 = \cap \{(\pi) | \pi \in P_2 - P_1\}$, hence $P_1 = P_1^v$; $\dots$; $P_{n-2} = \cap \{(\pi) | \pi \in P_{n-1} - P_{n-2}\}$, hence $P_{n-2} = P_{n-2}^v$.

We have $P_{n-1} = \cap (p^m q^m)$, hence $P_{n-1} = P_{n-1}^v$.

(2) Let $x \in (\frac{p^m q^m}{\pi})$ for $\pi \in P_{n-1} - P_{n-2}$ and $m \geq 0$. Then $v(x^{-1}) \leq v(\pi) - mv(p)$. It follows that $v(x^{-1}) \leq v(p^n)$ for some n . Similarly, $w(x^{-1}) \leq w(q^{n'})$ for some n' . Therefore $x \in C$. It follows that $C = \cap \{(\frac{p^m q^m}{\pi}) | \pi \in P_{n-1} - P_{n-2} \text{ and } m \geq 0\}$.

(3) Let I be a nonzero fractional ideal of D with $I \subset A$ such that $I \neq xA$ for each $x \in K$. We must show that $I^v \subset A$. We may assume that $I \not\subset (\frac{1}{p^m})$ for each $m > 0$. Choose $x_m \in I - (\frac{1}{p^m})$. Then $x_m \in (\frac{1}{p^n}) - (\frac{1}{p^{n-1}})$ for some $n > m$. Then $x_m = \frac{q^k u}{p^n}$ for $k \geq 0$ and unit u of D . Let $k_0 = \min \{k \geq 0 | q^k \in I\}$. It follows that $I = q^{k_0} A$; a contradiction.

Proposition 5. *Let D be an n -dimensional Prüfer ring with $n > 2$ and with exactly two maximal ideals M and N . Assume that there exist prime ideals $P_1, P_2, \dots, P_{n-1}$ of D such that $M \cap N \supsetneq P_{n-1} \supsetneq \dots \supsetneq P_2 \supsetneq P_1 \supsetneq (0)$, and*

that there exist elements $\pi_1, \pi_2, \dots, \pi_{n-1}$ and p of D such that $P_i D_{P_i} = \pi_i D_{P_i}$ for each i , $M = (p)$, and that N is not a principal ideal.

Let $A_{n-1} = \cup_1^\infty (\frac{1}{\pi_{n-1}^m})$, $\dots$, $A_2 = \cup_1^\infty (\frac{1}{\pi_2^m})$, $A = \cup_1^\infty (\frac{1}{p^m})$, $B = \cup \{(\frac{1}{q}) \mid q \in N - M\}$ and $C = \cup \{(\frac{1}{p^m q}) \mid q \in N - M \text{ and } m \geq 0\}$.

For each nonzero fractional ideal I of D , let $I^* = I$ if $I = xB$ for some $x \in K = q(D)$, and let $I^* = I^v$ otherwise.

(1) Each element x of $K - \{0\}$ can be expressed as $\pi_1^{a_1} \pi_2^{a_2} \dots \pi_{n-1}^{a_{n-1}} p^b u$ with integers a_i, b and with unit u of D_M . This expression is unique for the element x .

(2) We have $P_1 = P_1^v = \pi_1 A_2$, $P_2 = P_2^v = \pi_2 A_3, \dots$, and $P_{n-2} = P_{n-2}^v = \pi_{n-2} A_{n-1}$.

(3) $P_{n-1} = P_{n-1}^v = \pi_{n-1} C = \pi_{n-1} A^v = \pi_{n-1} B^v$, $A \neq A^v$, and $B \neq B^v$. $A \neq xB$ for each $x \in K$.

(4) $*$ is a star operation on D .

(5) $*$, d and v are distinct with each other.

Proof. Let $V = D_M$, $W = D_M$, and $U_i = D_{P_i}$ for each i , let $M_i = P_i U_i$, and let v (resp. w, u_i) be the valuation belonging to V (resp. W, U_i). Let $v(p) = 1$ and $u_i(\pi_i) = 1$ for each i , and let Γ' be the value group of w .

(2) We have $P_{n-1} = \cap \{(p^m q) \mid q \in N - M \text{ and } m \geq 0\}$.

(4) Assume that $I \subset J$ for $I, J \in F(D)$. It suffices to show that $I^* \subset J^*$.

We may assume that $J = B$ and $I \neq xB$ for each $x \in K$.

Let $S = \{x_\lambda \mid \lambda\}$ be a generating system for I . Let $x_\lambda = \pi_1^{a_{\lambda,1}} \dots \pi_{n-1}^{a_{\lambda,n-1}} p^{b_\lambda} u_\lambda$, where $a_{\lambda,i}, b_\lambda$ are integers and u_λ is a unit of V .

If $\inf_\lambda a_{\lambda,1} = -\infty$, then $I = K$; a contradiction. Therefore we may assume that $a_{\lambda,1} = a_1$ does not depend on λ .

If $\inf_\lambda a_{\lambda,2} = -\infty$, then $I \frac{1}{\pi_1^{a_1}} = A_2$. It follows that $I^v = I \subset B$. Therefore we may assume that $a_{\lambda,2} = a_2$ does not depend on λ .

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If $\inf_\lambda a_{\lambda,n-1} = -\infty$, then $I \frac{1}{\pi_1^{a_1} \dots \pi_{n-2}^{a_{n-2}}} = A_{n-1}$. It follows that $I^v = I \subset B$.

Therefore we may assume that $a_{\lambda,n-1} = a_{n-1}$ does not depend on λ .

We have $x_\lambda = \pi_1^{a_1} \dots \pi_{n-1}^{a_{n-1}} p^{b_\lambda} u_\lambda$.

Assume that $a_1 = \dots = a_{m-1} = 0 \neq a_m$ for some m . If $a_m < 0$, then $x_\lambda \notin B$; a contradiction. If $a_m > 0$, then $I \subset D$. Hence $I^v \subset B$.

Assume that $a_1 = \dots = a_{n-1} = 0$. We have $b_\lambda = k_\lambda \geq 0$ and $x_\lambda = p^{k_\lambda} u_\lambda$.

Assume that $\inf_\lambda \Gamma'_\lambda w(u_\lambda) > -\infty$. There exists $q_0 \in N - M$, such that $w(u_\lambda) > -w(q_0)$ for each λ . Then $x_\lambda \in (\frac{1}{q_0})$, and hence $I \subset (\frac{1}{q_0})$. It follows that $I^v \subset B$.

Assume that $\inf_\lambda \Gamma'_\lambda w(u_\lambda) = -\infty$. Since D is a Bezout ring, we may assume that $x_\lambda = \frac{p^{k_\lambda}}{q_\lambda}$ for each λ . We have $\sup_\lambda \Gamma'_\lambda w(q_\lambda) = \infty$. Let $k_0 = \min_\lambda \{k \geq$

$0|p^k \in I\}$. Since $\frac{p^{k_\lambda - k_0}}{q_\lambda} \in B$, we have $x_\lambda \in p^{k_0}B$. Hence $I \subset p^{k_0}B$.

Let $q_1 \in N - M$. Choose λ such that $w(q_\lambda) > w(q_1)$. We have $\frac{p^{k_0}}{q_1} \in (\frac{p^{k_0}}{q_\lambda}) = \frac{p^{k_0}}{q_\lambda}(q_\lambda, p^{k_\lambda - k_0}) = (p^{k_0}, \frac{p^{k_\lambda}}{q_\lambda}) \subset I$. Therefore $p^{k_0}B \subset I$. It follows that $I = p^{k_0}B$; a contradiction.

Remark 2. Let D be a 2-dimensional Prüfer ring with exactly two maximal ideals M and N . Assume that there exists a prime ideal P such that $M \cap N \not\supseteq P \supseteq (0)$, and that there exists elements π and p of D such that $PD_P = \pi D_P$ and $M = (p)$, and that N is not a principal ideal.

Let $A = \cup_1^\infty (\frac{1}{p^m})$, $B = \cup\{(\frac{1}{q}) \mid q \in N - M\}$ and $C = \cup\{(\frac{1}{p^m q}) \mid q \in N - M \text{ and } m \geq 0\}$.

For each nonzero fractional ideal I of D , let $I^* = I$ if $I = xB$ for some $x \in K = q(D)$, and let $I^* = I^v$ otherwise.

- (1) Each element x of $K - \{0\}$ can be expressed as $\pi^a p^b u$ with integers a, b and with unit u of D_M . This expression is unique for the element x .
- (2) We have $P = P^v = \pi C = \pi A^v = \pi B^v$, $A \neq A^v$, and $B \neq B^v$.
- (3) $A \neq xB$ for each $x \in K$.
- (4) $*$ is a star operation on D .
- (5) $*$, d and v are distinct with each other.

Proposition 6. Let D be a 1-dimensional Prüfer ring with exactly two maximal ideals M and N .

- (1) If either M or N , say N , is not principal, then $|\Sigma'(D)| \geq 6$.
- (2) Assume that $M = (p)$ and $N = (q)$ are principal ideals of D . Then $|\Sigma'(D)| = 7$.

Proof. Let v (resp. w) be a valuation on $K = q(D)$ belonging to $V = D_M$ (resp. $W = D_N$), and let $v(p) = 1$.

- (1) We have $\{e, *_V, *_W, \delta'(v'_W), d', v'\} \subset \Sigma'(D)$.
- (2) Let $A = \cup(\frac{1}{p^m})$ and $B = \cup(\frac{1}{q^m})$. Let $w(q) = 1$, and let $I \in F'(D)$.
 - (i) The case where $v(I)$ and $w(I)$ are bounded below: Let $a = \min v(I)$ and $b = \min w(I)$. Then $p^a q^b \in I$. It follows that $I = (p^a q^b)$.
 - (ii) The case where $v(I)$ is bounded below and $w(I)$ is not bounded below: Let $a = \min v(I)$. Then there exists a generating system $\{p^a q^{b_\lambda} \mid \lambda\}$ for I . Then $I = p^a (q^{b_\lambda} \mid \lambda) = p^a B$.
 - (iii) The case where $v(I)$ is not bounded below and $w(I)$ is bounded below: Let $b = \min w(I)$. Then we have $I = q^b A$.
 - (iv) The case where neither $v(I)$ nor $w(I)$ is bounded below: Then we have $I = K$.

It follows that each fractional ideal of D is principal, and hence $\Sigma(D) = \{d\}$. Furthermore, $F'(D) - F(D) = \{xA, xB, K \mid x \in K - \{0\}\}$.

Let $*$ be a semistar operation with $A^* \subsetneq K$. We have $A^* = xA$ for some $x \in K - \{0\}$. Since $(A^*)^* = A^*$, we have $A^* = A$. Hence A^* is either A or K . Similarly, B^* is either B or K .

Let $A^{*1} = B^{*1} = K$. Then there is determined a semistar operation $*_1$ on D extending d .

Let $A^{*2} = A$ and $B^{*2} = K$. Then there is determined a semistar operation $*_2$ on D extending d .

Let $A^{*3} = K$ and $B^{*3} = B$. Then there is determined a semistar $*_3$ on D extending d .

Let $A^{*4} = A$ and $B^{*4} = B$. Then there is determined a semistar $*_4 = d'$ on D extending d .

It follows that $|\Sigma'(D)| = 7$.

Proposition 7. (1) *Let D be a 2-dimensional Prüfer ring with exactly two maximal ideals M and N . Then we have $|\Sigma'(D)| \geq 7$.*

(2) *Let D be a 3-dimensional Prüfer ring with exactly two maximal ideals M and N . Then we have $|\Sigma'(D)| \geq 8$.*

(3) *Let D be a 4-dimensional Prüfer ring with exactly two maximal ideals M and N . Then we have $|\Sigma'(D)| \geq 10$.*

Proof. (1) Let $\text{height}(M) = 2$, and let P be a prime ideal of D such that $M \supsetneq P \supsetneq (0)$. Let $V = D_M, W = D_N$ and $U = D_P$.

(i) The case where U, V and W have principal maximal ideals and $P \subset N$: Then we have $|\Sigma'(D)| = 8$ by Theorem 9.

(ii) The case where either U or V or W , say T , has non-principal maximal ideal and $P \subset N$: There arises a semistar operation $\delta(v'_T)$ on D . It follows that $|\Sigma(D)| \geq 7$.

(iii) The case where either U or V or W , say T , has a non-principal maximal ideal and $\text{height}(N) = 1$: Let $R = U \cap W$. There arise semistar operations $*_R$ and $\delta(v'_T)$.

(iv) The case where U and W have the principal maximal ideals and $\text{height}(N) = 1$: Let $R = U \cap W$. We have $|\Sigma'(R)| = 7$ by Proposition 6.

(v) The case where $P \not\subset N$ and $\text{height}(N) = 2$: Let $R = U \cap W$. We have $|\Sigma'(R)| \geq 7$ by (iii) and (iv).

(2) Let $\text{height}(M) = 3$. There exist prime ideals P_1, P_2 such that $M \supsetneq P_2 \supsetneq P_1 \supsetneq (0)$. Let $V = D_M, U_2 = D_{P_2}, U_1 = D_{P_1}$, and $W = D_N$.

(i) The case where $\text{height}(N) \leq 2$: Let $R = U_2 \cap W$. We have $|\Sigma'(R)| \geq 7$ by (1).

(ii) The case where $\text{height}(N) = 3$ and $P_2 \not\subset N$: Let Q be a prime ideal of D with height 2 such that $Q \subset N$. Let $R = U_2 \cap D_Q$. Then $|\Sigma'(R)| \geq 7$ by (1).

(iii) The case where U_1, U_2, V , and W have principal maximal ideals and $P_2 \subset N$: Then $|\Sigma'(D)| \geq 9$ by Theorem 9.

(iv) The case where either U_1 or U_2 or V or W , say T , has a non-principal maximal ideal and $P_2 \subset N$: Then there arises the semistar operation $\delta(v'_T)$ on D .

(3) Let $\text{height}(M) = 4$, and let P_1, P_2 , and P_3 be prime ideals of D with $M \supsetneq P_3 \supsetneq P_2 \supsetneq P_1 \supsetneq (0)$, and let $V = D_M, U_3 = D_{P_3}, U_2 = D_{P_2}, U_1 = D_{P_1}$ and $W = D_N$.

(i) The case where $\text{height}(N) \leq 3$: Let $R = U_3 \cap W$. Then $|\Sigma'(R)| \geq 8$ by (2).

(ii) The case where $\text{height}(N) = 4$ and $P_3 \not\subset N$: Let Q be a prime ideal of D with height 3 such that $Q \subset N$. Let $R = U_3 \cap D_Q$. Then $|\Sigma'(R)| \geq 8$ by (2).

(iii) The case where neither MV nor NW is principal, and $P_3 \subset N$: Then $\Sigma'(D)$ contains $\{e, *_{U_1}, *_{U_2}, *_{U_3}, *_{V'}, \delta(v'_V), *_{W'}, \delta(v'_W), d', v'\}$, and hence $|\Sigma'(D)| \geq 10$.

(iv) The case where MV is principal, NW is not principal, some member T in $\{U_1, U_2, U_3\}$ has a non-principal maximal ideal, and $P_3 \subset N$: Then $\Sigma'(D)$ contains $\{e, *_{U_1}, *_{U_2}, *_{U_3}, \delta(v'_T), *_{V'}, *_{W'}, \delta(v'_W), d', v'\}$, and hence $|\Sigma'(D)| \geq 10$.

(v) The case where MV is principal, NW is not principal, each member in $\{U_1, U_2, U_3\}$ has a principal maximal ideal, and $P_3 \subset N$: Let $*$ be the star operation on D in Proposition 5, and let $*'$ be an extension of $*$ to a semistar operation on D . Then $\Sigma'(D)$ contains $\{e, *_{U_1}, *_{U_2}, *_{U_3}, *_{V'}, *_{W'}, \delta(v'_W), d', v', *'\}$, and hence $|\Sigma'(D)| \geq 10$.

(vi) The case where MV is principal, NW is principal, and $P_3 \subset N$: Let $*_1, *_2$ be the star operations on D in Proposition 4, and let $*'_1, *'_2$ be extensions of $*'_1, *'_2$ to semistar operations on D . Then $\Sigma'(D)$ contains $\{e, *_{U_1}, *_{U_2}, *_{U_3}, *_{V'}, *_{W'}, d', v', *'_1, *'_2\}$, and hence $|\Sigma'(D)| \geq 10$.

Theorem 10. *Let D be an integrally closed ring with $\dim n \leq 4$. If $n + 1 \leq |\Sigma'(D)| \leq 2n + 1$, then D is a valuation ring.*

Proof. Suppose the contrary. Then D is not a local ring by Theorem 3. We may assume that D is a Prüfer ring which has exactly two maximal ideals M and N with $\text{height}(M) = n$. Proposition 7 implies a contradiction.

Theorem 11. *Let D be n -dimensional with $n > 0$ which is not local. Then $|\Sigma'(D)| \geq n + 5$.*

Proof. We rely on the induction on $\dim(D)$. We may assume that the assertion holds for each $i < n$. We may assume that D is a Prüfer ring with exactly two maximal ideals M and N with $\text{height}(M) = n$. Let $M \supsetneq P_{n-1} \supsetneq \cdots \supsetneq P_1 \supsetneq (0)$ be prime ideals of D . Let $V = D_M, W = D_N, U_{n-1} = D_{P_{n-1}}, \dots$, and $U_1 = D_{P_1}$.

(1) The case where $P_{n-1} \not\subset N$ and $\text{height}(N) \leq n - 1$: Let $R = U_{n-1} \cap W$. Then $|\Sigma'(R)| \geq n + 4$.

(2) The case where $P_{n-1} \not\subset N$ and $\text{height}(N) = n$: Let Q be a prime ideal of D with height $n - 1$ such that $Q \subset N$. Let $R = U_{n-1} \cap D_Q$. We have $|\Sigma'(R)| \geq n + 4$.

(3) The case where $P_{n-1} \subset N$, and some member T in $\{U_1, \dots, U_{n-1}, V, W\}$ has a non-principal maximal ideal: Then $\Sigma'(D)$ contains $\{e, *_{U_1}, \dots, *_{U_{n-1}}, *_{V'}, *_{W'}, d', v', \delta(v'_T)\}$, and hence $|\Sigma'(D)| \geq n + 5$.

(4) The case where $P_{n-1} \subset N$, and each member in $\{U_1, \dots, U_{n-1}, V, W\}$ has a principal maximal ideal: Let $*_1, *_2$ be the star operations in Proposition 3. Then $\Sigma'(D)$ contains $\{e, *_{U_1}, \dots, *_{U_{n-1}}, *_{V'}, *_{W'}, d', v', *'_1, *'_2\}$, and hence $|\Sigma'(D)| \geq n + 6$.

Theorem 12. *For each positive integer n with $n \geq 5$, there exists an integrally closed domain D with dimension n such that $n + 1 \leq |\Sigma'(D)| \leq 2n + 1$ and which is not a valuation ring.*

Proof. Let D be an integral domain with dimension $n \geq 5$ in Theorem 9.

Example 2. Let n be a positive integer, let $A_1, A_2, B_1, \dots, B_{n-1}$ be nonzero subgroups of $\mathbf{R}$, and introduce on each of them the canonical order. Let $B = B_1 \oplus \dots \oplus B_{n-1}$ be the direct sum, and introduce on B the lexicographic order; Let $x = (b_1, \dots, b_{n-1}), x' = (b'_1, \dots, b'_{n-1})$ be elements of $B_1 \oplus \dots \oplus B_{n-1}$ with $b_{n-1} < b'_{n-1}$, then let $x < x'$. Let v_0 (resp. w_0) be the projection mapping of $A_1 \oplus A_2 \oplus B$ to the ordered group $A_1 \oplus B$ (resp. $A_2 \oplus B$). Let k be a field, and let $K = \mathbf{q}(k[X; A_1 \oplus A_2 \oplus B])$. Let v (resp. w) be the canonical extension of the valuation v_0 (resp. w_0) to the valuation on K , and let V (resp. W) be the n -dimensional valuation ring on K belonging to v (resp. w), and let M' (resp. N') be the maximal ideal of V (resp. W). For each i with $1 \leq i \leq n-1$, let u'_i be the projection mapping of $A_1 \oplus A_2 \oplus B_1 \oplus \dots \oplus B_{n-1}$ to the ordered group $B_{n-i} \oplus \dots \oplus B_{n-1}$, and let u_i be the canonical extension of the valuation u'_i to a valuation on K . Let U_i be the valuation ring on K belonging to u_i , and let M'_i be the maximal ideal of U_i . Let $D = V \cap W$, let $M = M' \cap D, N = N' \cap D$, and let $P_i = M'_i \cap D$ for each i . Then we have $V = D_M, M' = MV, W = D_N, N' = NW$, and $U_i = D_{P_i}, M'_i = P_i U_i$ for each i . It follows that M' (resp. N') is a principal ideal of V (resp. W) if and only if A_1 (resp. A_2) is a discrete subgroup of $\mathbf{R}$, and that M'_i is a principal ideal of U_i if and only if B_{n-i} is a discrete subgroup of $\mathbf{R}$.

Let P be a prime ideal of S . If $a + b \in P$ implies either $a \in P$ or $b \in P$ for elements $a, b \in \mathbf{q}(S)$, then P is called a strongly prime ideal of S . If each prime ideal of S is a strongly prime ideal, then S is called a pseudo-valuation semigroup (or a PVS).

Lemma 5. Let M be a maximal ideal of S , and let $G = \mathbf{q}(S)$.

- (1) If M is strongly prime, then S is a PVS.
- (2) If S is a PVS, then there exists a valuation oversemigroup V of S with maximal ideal M .
- (3) If S is a PVS, then $F'(S) = F(S) \cup \{G\}$.

Proof. (1) Let P be a prime ideal of S with $P \subsetneq M$. We will show that P is strongly prime. Let $a + b \in P$ for $a, b \in G$. We may assume that $a \notin S$. Since M is strongly prime, we have $b \in M$. Since $M \ni b = a + b - a$, we have $b - a \in M$. Then $2b = b - a + a + b \in P$, and hence $b \in P$.

(2) There exists a valuation oversemigroup V_1 with maximal ideal N such that $N \cap S = M$. Let $m \in M$, and let $w \in V_1$. Since $m = w + m - w$ and $-w \notin M$, we have $w + m \in M$. Hence M is a prime ideal of V_1 . Similarly, M is a prime ideal of the valuation semigroup $V = (V_1)_M$. Hence M is a maximal ideal of $(V_1)_M$.

(3) Let V be a valuation oversemigroup of S with maximal ideal M , and let v be a valuation on G belonging to V . Let $I \in F'(S)$. If $v(I)$ is bounded below, then $I \in F(S)$. If $v(I)$ is not bounded below, then $I = G$.

Lemma 6. Assume that S is a PVS with maximal ideal M which is not a valuation semigroup.

- (1) M is not a principal ideal of S .
- (2) $V = M^{-1}$ is a valuation oversemigroup with maximal ideal M .

- (3) M is a divisorial ideal of S .
- (4) Each prime ideal of S is a prime ideal of V , and $\dim(V) = \dim(S)$.
- (5) $M^{-1} = (M : M)$.

Proof. (1) Assume the contrary. We have $M = x + S$ for some $x \in M$. Let W be a valuation oversemigroup of S with maximal ideal M . Since $W + M = M$, we have $W = S$. Hence S is a valuation semigroup; a contradiction.

(2) Let $a \in V$. Since M is not a principal ideal of S , we have $a + M \subset M$. Hence M is an ideal of V .

Let $x, y \in V$. Then $x + y + M \subset x + M \subset M$. Hence $x + y \in V$, and hence V is an oversemigroup of S .

Let $z \in q(S) - V$. We have $z + m \notin M$ for some $m \in M$. Since $m = z + m - z$, we have $-z \in M$. Hence V is a valuation semigroup.

Let $\alpha \in V - M$. Since $m = -\alpha + m + \alpha$, we have $-\alpha + m \in M$. Hence $-\alpha \in V$. That is, α is a unit of V . Hence M is a maximal ideal of V .

(3) If $M^v \supsetneq M$, then $M^v = S$. It follows that $M^{-1} = S$, and hence S is a valuation semigroup; a contradiction.

(4) Let P be a prime ideal of S . It suffices to show that $V + P \subset P$. Let $x \in V$ and $p \in P$. If $x \in M$, then $x + p \in P$. If $x \notin M$, then $-x \in V - M$. Since $p = x + p - x$, we have $x + p \in P$.

(5) Suppose the contrary. There exists an element $x \in M^{-1}$ such that $x \notin (M : M)$. Then $x + M = S$, and hence M is a principal ideal of S ; a contradiction.

Lemma 7. *Let S be a PVS with maximal ideal M which is not a valuation semigroup, let $V = M^{-1}$, let H be the unit group of S , and let L be the unit group of V .*

- (1) *If $|L/H| = \infty$, or if $\dim(S) = \infty$, then $|\Sigma'(S)| = \infty$.*
- (2) *Assume that $|L/H| < \infty$. Let T be an oversemigroup of S . Then, either $T \supset V$, or there exists a subgroup K of L with $K \supset H$ such that $T = K \cup M$.*

Proof. (1) If $|L/H| = \infty$, there exists an infinite number of subgroups K of L containing H . Let $T = K \cup M$. Then T is an oversemigroup of S .

(2) Assume that $T \not\supset V$. Choose $t \in T - V$, and let $\alpha \in L$. Then $-t \in M$ and $\alpha - t \in M$. It follows that $\alpha = t + (\alpha - t) \in T$, and that $T \supset V$.

Let T be an oversemigroup of S with $T \subset V$. Let $K = T - M$. Since $|L/H| < \infty$, each element $\alpha \in K$ is integral over H . Therefore K is a subgroup of L with $K \supset H$, and $T = K \cup M$.

Theorem 13. *Let S be a PVS which is not a valuation semigroup, let M be the maximal ideal of S , and let $V = M^{-1}$. Let Σ'_1 be the set of semistar operations $*$ on S such that $S^* \supset V$, and let Σ'_2 be the set of semistar operations $*$ on S such that $S^* \subsetneq V$. Let H be the unit group of S , and let L be the unit group of V . Assume that $|L/H| < \infty$ and that $\dim(S) < \infty$. Let $H_1, \dots, H_l$ be the subgroups H' of L such that $L \supsetneq H' \supset H$, and let $S_i = H_i \cup M$ for each i .*

- (1) $\Sigma'(S) = \Sigma'_1 \cup \Sigma'_2$.
- (2) $|\Sigma'_1| = |\Sigma'(V)|$.
- (3) $|\Sigma'_2| = \sum_{i=1}^l |\Sigma(S_i)|$.

(4) $|\Sigma'(V)| < \infty$, and $|\Sigma(S_i)| < \infty$ for each i .

Proof. (1) follows from Lemma 7 (2).

(3) Let $*$ $\in \Sigma(S_i)$ for some i , let $*'$ be an extension of $*$ to a semistar operation on S_i , and let $\mu(*)$ be the descent of $*'$ to S . Then we have $S^{\mu(*)} = S_i$. Therefore $\Sigma(S_i) \cap \Sigma(S_j) = \emptyset$ for $i \neq j$, and $\mu(*) \in \Sigma'_2$. Then μ is a bijection from $\cup \Sigma(S_i)$ onto Σ'_2 .

(4) Let v be a valuation on $G = q(S)$ belonging to V , and let Γ be the value group of v .

It suffices to show that $|\Sigma(S)| < \infty$. Thus let $C = L/H$, and let Π be the set of nonempty subsets of C . Let Π^Π be the set of mappings from Π to Π . Then Π^Π is a finite set. An element c of C canonically generates a fractional ideal $c + S$ of S . And an element σ of Π canonically generates a fractional ideal $\sigma + S$ of S .

Let $*$ be a star operation on S . Since $V = M^{-1}$, V is a divisorial fractional ideal of S . Hence, for each $\sigma \in \Pi$, we have $(\sigma + S)^* \subset V^* \subset V^v = V$. Therefore $(\sigma + S)^* = \sigma_1 + S$ for some $\sigma_1 \in \Pi$. Then σ_1 is uniquely determined by σ . It follows that $*$ gives a mapping $\sigma \mapsto \sigma_1$ from Π to Π , that is, $*$ gives an element $F(*)$ of Π^Π .

Then F is a mapping from $\Sigma(S)$ to Π^Π . We will show that F is an injective mapping.

Assume that star operations $*_1$ and $*_2$ of S give the same element of Π^Π . We will show that $*_1 = *_2$. Let I be a fractional ideal of S . We must show that $I^{*1} = I^{*2}$.

There arise the following three cases: (i) $v(I)$ has a minimal element $v(x)$ with $x \in I$. (ii) $v(I)$ does not have a minimal element, and there exists $v(x) = \inf \Gamma v(I)$ with $x \in G$. (iii) There does not exist $\inf \Gamma v(I)$.

The case (i): We may assume that I is a fractional ideal of S with $I \ni 0$ such that $v(I)$ has the minimal element 0. Then $I = \sigma + S$ for some $\sigma \in \Pi$. It follows that $I^{*1} = I^{*2}$.

The case (ii): We may assume that I is a fractional ideal of S such that $\inf \Gamma v(I) = 0$. Then we have $I = M$. Hence $I = I^v$, and hence $I^{*1} = I^{*2}$.

The case (iii): Let $X = \{x \in G \mid v(x) \leq v(i) \text{ for each } i \in I\}$. Let $y \notin I$. Then $v(y) < v(i)$ for each $i \in I$. By the assumption, there exists $z \in X$ such that $v(y) < v(z)$. Therefore $y \notin \cap \{x + S \mid x \in X\}$. It follows that $I = \cap \{x + S \mid x \in X\}$, and hence $I = I^v$. Therefore $I^{*1} = I^{*2}$.

Proposition 8. *Let S be a PVS with maximal ideal M . The following conditions are equivalent.*

(1) $|\Sigma'(S)| < \infty$.

(2) $\dim(S) < \infty$, and $(M : M)$ is a finitely generated S -module.

Proof. $V = (M : M)$ is a valuation oversemigroup of S , and, if S is a valuation semigroup, the assertion holds.

Thus assume that S is not a valuation semigroup. Then $V = M^{-1}$ by Lemma 6.

(1) $\implies$ (2): Then V is a finitely generated semigroup over S , and $\bar{S}$ is a valuation semigroup (Lemma 1). Since V and $\bar{S}$ have the common maximal ideal M , we have $V = \bar{S}$. It follows that V is a finitely generated S -module.

(2) $\implies$ (1): Let H (resp. L) be the unit group of S (resp. V). Then we have $|L/H| < \infty$. Theorem 13 implies that $|\Sigma'(S)| < \infty$.

Theorem 14. *Assume that S and $\bar{S}$ have a common maximal ideal M . Let H (resp. $\bar{H}$) be the group of units of S (resp. $\bar{S}$). Then the following conditions are equivalent.*

- (1) $|\Sigma'(S)| < \infty$.
- (2) $\dim(S) < \infty$, $\bar{S}$ is a valuation semigroup, and $\bar{H}/H$ is a finite group.

Proof. (1) $\implies$ (2): If $|\bar{H}/H| = \infty$, there exists an infinite number of subgroups K with $K \supset H$ of $\bar{H}$. Then $T = K \cup M$ is an oversemigroup of S . Then there arises an infinite number of semistar operations on S .

(2) $\implies$ (1): Then S is a PVS, and $\bar{S} = (M : M)$. Proposition 8 implies $|\Sigma'(S)| < \infty$.

Let D be a domain, and let P be a prime ideal of D . If $ab \in P$ implies either $a \in P$ or $b \in P$ for elements $a, b \in q(D)$, then P is called a strongly prime ideal of D . If each prime ideal of D is strongly prime, then D is called a pseudo-valuation domain (or a PVD).

Lemma 8. *Let M be a maximal ideal of D , and let $K = q(D)$.*

- (1) *If D is a PVD, then D is a local ring.*
- (2) *If D is a local ring with M strongly prime. Then D is a PVD.*
- (3) *If D is a PVD, then there exists a valuation overring W of D with maximal ideal M .*
- (4) *If D is a PVD, then $F'(D) = F(D) \cup \{K\}$.*

Proof. (1) Suppose the contrary. There exists a prime ideal P of D with $P \not\subset M$. Choose $a \in P - M$ and $b \in M - P$. Since $M \ni b = a \cdot \frac{b}{a}$, we have $\frac{b}{a} \in M$. Hence $b \in P$; a contradiction.

The following is a ring version of Lemma 6.

Lemma 9. *Let D be a PVD with maximal ideal M which is not a valuation ring.*

- (1) *M is not a principal ideal of D .*
- (2) *$V = M^{-1}$ is a valuation ring with maximal ideal M .*
- (3) *M is a divisorial ideal of D .*
- (4) *Each prime ideal of D is a prime ideal of V , and $\dim(V) = \dim(D)$.*
- (5) *$M^{-1} = (M : M)$.*

Lemma 10. *Let D be a PVD with maximal ideal M which is not a valuation ring. Let $V = M^{-1}$, let $K = V/M$, and let $k = D/M$.*

- (1) *Let T be an overring of D . Then either $T \supset V$ or $T \subset V$.*
- (2) *Assume that $T \subset V$ be an overring of D , and that T is integral over D . Then T is a PVD with maximal ideal M .*
- (3) *If $|\Sigma'(D)| < \infty$, then V is the integral closure of D , and V is a finitely generated D -module.*
- (4) *If $[K : k] = \infty$, or if $\dim(D) = \infty$, then $|\Sigma'(D)| = \infty$.*

(5) *There exists only a finite number of intermediate rings between D and V if and only if K is a simple extension field of k with $[K : k] < \infty$.*

Proof. (1) Assume that $T \not\subseteq V$. Choose $t \in T - V$, and let $\alpha \in V$. Then $t^{-1} \in M$ and $\alpha t^{-1} \in M$. It follows that $\alpha = t(\alpha t^{-1}) \in T$, and hence $T \supset V$.

(2) Let $\alpha \in T - M$. Let $\alpha^n + a_1\alpha^{n-1} + \cdots + a_0 = 0$ be an integral equation of α over D with minimal degree n . If $a_0 \in M$, then $\alpha^{n-1} + a_1\alpha^{n-2} + \cdots + a_1 = m$ belongs to M ; a contradiction to the choice of n . Hence $a_0 \notin M$. It follows that α is a unit of T , and that M is the unique maximal ideal of T .

(3) $\bar{D}$ is a PVD with maximal ideal M by (2). Then $\bar{D}$ is a valuation ring by Theorem 3. It follows that $\bar{D} = V$. Since V is a finitely generated overring of D , V is a finitely generated D -module.

(4) If $[K : k] = \infty$, there exists an infinite number of intermediate rings T between D and V .

(5) If $[K : k] < \infty$, then V is integral over D .

Let k_1 be a field, and let k_2 be an extension field of k_1 . Then there exists only a finite number of intermediate fields between k_1 and k_2 if and only if k_2 is a simple extension of k_1 with finite degree.

Let T be an overring of D , and let $*$ be a semistar operation on D . If we let $J^{\alpha(*)} = J^*$ for each $J \in F'(T)$, then $\alpha(*)$ is a semistar operation on T , and is called the ascent of $*$ to T .

Theorem 15. *Let D be a PVD which is not a valuation ring, let M be a maximal ideal of D , and let $V = M^{-1}$. Assume that $\dim(D) < \infty$, and that V/M is a simple extension field of D/M with $[V/M : D/M] < \infty$. Let $\{D_1, \dots, D_l\}$ be the set of overrings T of D such that $T \subsetneq V$. Let Σ'_1 be the set of semistar operations $*$ on D such that $D^* \supset V$, and let Σ'_2 be the set of semistar operations $*$ on D such that $D^* \subsetneq V$.*

(1) $\Sigma'(D) = \Sigma'_1 \cup \Sigma'_2$.

(2) $|\Sigma'(V)| < \infty$.

(3) $|\Sigma'_1| = |\Sigma'(V)|$.

(4) *There is a canonical bijection from the disjoint union $\cup_1^l \Sigma(D_i)$ onto Σ'_2 .*

Proof. (1) follows from Lemma 10 (1).

(3) Let $*$ $\in \Sigma'(V)$, and let $\delta(*)$ be the descent of $*$ to D . Then $\delta(*) \in \Sigma'_1$. The mapping δ from $\Sigma'(V)$ to Σ'_1 is injective.

Let $*$ $\in \Sigma'_1$, and let $\alpha(*)$ be the ascent of $*$ to V . Then we have $*$ $= \delta(\alpha(*)$). Therefore $|\Sigma'_1| = |\Sigma'(V)|$.

(4) The proof is similar to that of Theorem 13 (3).

Remark 3. In Theorem 15, if $[V/M : D/M]$ is either 2 or 3, then $|\Sigma(D)| < \infty$.

Proof. Let v be a valuation belonging to V , Γ be the value group of v , and $K = V/M$, $k = D/M$, and assume that $[K : k] = 3$ (The case where $[K : k] = 2$ is similar).

Let U, W be distinct 2-dimensional k -subspaces of K containing k . Let $1, s$ (resp. $1, t$) be a basis of U (resp. W). Then $K = k + ks + kt$. Hence there exist

elements a_1, a_2, a_3 of k such that $st = a_1 + a_2s + a_3t$. Let $\alpha = t - a_2$. Then $\alpha s = a_1 + a_3t \in W$. It follows that $W = \alpha U$.

Let $D \subsetneq I_0 \subsetneq V$ be a fixed element of $F(D)$. Assume that $D \subsetneq I \subsetneq V$ for $I \in F(D)$. Then we have $I/M = \bar{x}(I_0/M)$ and $I_0/M = \bar{y}(I/M)$ for some $x, y \in V$ with $\bar{x}\bar{y} = \bar{1}$. Then xy is a unit of D . Since $I + M = I$, we have $xI_0 \subset I$. Similarly, $yI \subset I_0$. Therefore $xI_0 = I$ and $yI = I_0$.

Note that M and V are divisorial fractional ideals of D .

Let $*$ be a star operation on D . Then $I_0^* = I_0$ or $I_0^* = V$.

Let I be a nonzero fractional ideal of D .

If there does not exist $\inf^\Gamma(v(I))$, then $I = I^v$.

Assume that there exists $\inf^\Gamma(v(I)) = v(x)$ with $v(x) \notin v(I)$. Then we have $x^{-1}I = M$, and hence $I = I^v$.

Assume that there exists a minimal element $v(i_0)$ of $v(I)$ with $i_0 \in I$. Since $D \subset i_0^{-1}I \subset V$, we have either $I = i_0D$ or $I = i_0V$ or $I = iI_0$ with $i \in I$.

The case where $I_0^* = I_0$: Then $*$ = d.

The case where $I_0^* = V$: We have $V = I_0^* \subset I_0^v \subset V$. Hence $*$ = v.

Proposition 9. *Let V be a valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V . Let k be a subfield of K , and let $D = k + M$. Assume that $\dim(D) < \infty$, and K is a simple extension of k with finite degree. Then $|\Sigma(D)|$ need not be finite.*

Proof. Let k_0 be a field with characteristic 0, let $k = k_0(\pi^4)$, and let $K = k(\pi)$, where π is an indeterminate over k_0 . Let $V = K[[X]]$ be the ring of formal power series of an indeterminate X over K , let M be the maximal ideal of V , let $D = k + M$, let v be the canonical $\mathbf{Z}$ -valued valuation on $q(D)$ belonging to V .

Fact 1. There exists an infinite number of 2-dimensional k -subspaces $U \supset k$ of K .

For, let a, b be distinct elements of k , let $u_1 = a\pi + \pi^2$, $u_2 = b\pi + \pi^2$, and let $U_1 = k + ku_1$, $U_2 = k + ku_2$. Then $U_1 \neq U_2$.

Fact 2. Let a, b be elements of k with $b \neq \pm a$, let $u_1 = a\pi + \pi^2$, $u_2 = b\pi + \pi^2$, and let $U_1 = k + ku_1$, let $U_2 = k + ku_2$. Then there exist no elements $\alpha \in K$ such that $\alpha U_1 = U_2$.

For, the proof of Fact 1 shows that $U_1 \neq U_2$. Suppose that there exists $\alpha \in K$ such that $\alpha U_1 = U_2$. Then we have $\alpha = c_0 + c_1u_2$ and $\alpha^{-1} = d_0 + d_1u_1$ for some $c_0, c_1, d_0, d_1 \in k$. Since $U_1 \neq U_2$, we have $c_1 \neq 0$ and $d_1 \neq 0$. Seeing the coefficient of π^3 in the equation $\alpha\alpha^{-1} = (c_0 + c_1b\pi + c_1\pi^2)(d_0 + d_1a\pi + d_1\pi^2)$, we have $c_1d_1(a + b) = 0$; a contradiction.

Fact 3. Let I be a nonzero fractional ideal of D . Then there exist $x \in q(D)$ and a k -subspace $U \supset k$ of K such that $I = xUD$.

Moreover, let also $xUD = yWD$, where $W \supset k$ is a k -subspace of K and $y \in q(D)$. Then we have $\dim(W) = \dim(U)$, $W = \alpha U$ for some $\alpha \in K$, and $y = xu$ for some unit u of V .

For, let x be an element of I such that $v(x)$ is the minimal element of $v(I)$. Let $U = x^{-1}I \cap K$. Then we have $x^{-1}I = UD$, and hence $I = xUD$.

Fact 4. Let $U_0 \supset k$ be a fixed 2-dimensional k -subspace of K . Let $U \supset k$ be any k -subspace of K .

If $U = k$, let $(UD)^* = D$.

If $U = \alpha U_0$ for some $\alpha \in K$, let $(UD)^* = V$.

If $\dim(U) = 2$, and if $U \neq \alpha U_0$ for each $\alpha \in K$, let $(UD)^* = UD$.

If $\dim(U) \geq 3$, let $(UD)^* = V$.

Let $I = xUD$ be a nonzero fractional ideal of D , where $x \in q(D)$ and $U \supset k$ is a k -subspace of K . Let $I^* = x(UD)^*$.

Then there is defined a unique mapping $*$ from $F(D)$ to $F(D)$.

For, let also $I = yWD$ for $y \in q(D)$ and a k -subspace $W \supset k$ of K . We must show that $x(UD)^* = y(WD)^*$. We have $y = xu_0$ and $W = \alpha_0 U$, where u_0 is a unit of V and $\alpha_0 \in K$.

The case where $U = k$: Then $W = k$, and hence $xD = yD$. We have $x(UD)^* = xD$, and $y(WD)^* = yD$.

The case where $U = \alpha U_0$ for some $\alpha \in K$: Then $W = \alpha_0 \alpha U_0$. We have $x(UD)^* = xV$, and $y(WD)^* = yV = xu_0 V = xV$.

The case where $\dim(U) = 2$ and $U \neq \alpha U_0$ for each $\alpha \in K$: Then $W \neq \alpha_0 U_0$ for each $\alpha \in K$. We have $x(UD)^* = xUD$, and $y(WD)^* = yWD$.

The case where $\dim(U) \geq 3$: Then $\dim(W) \geq 3$. We have $x(UD)^* = xV$, and $y(WD)^* = yV = xu_0 V = xV$.

Let $*$ be the mapping from $F(D)$ to $F(D)$ defined in Fact 4.

Fact 5. Let $k \subset U \subset W$ be k -subspaces of K . Then $(UD)^* \subset (WD)^*$.

Fact 6. Let $x \in q(D) - \{0\}$, and let U, W be k -subspaces of K containing k . If $UD \subset xWD$, then $(UD)^* \subset (xWD)^*$.

For, we may assume that $x = X^{-k}\alpha$, where k is a nonnegative integer and $\alpha \in K$.

The case where $k > 0$: We have

$$X^k \alpha^{-1} (UD)^* \subset X^k \alpha^{-1} V \subset M \subset D \subset (WD)^*.$$

Hence $(UD)^* \subset (xWD)^*$.

The case where $k = 0$: Then $x = \alpha \in K$, and $U \subset \alpha W$. Fact 5 implies that $(UD)^* \subset x(WD)^*$.

Fact 7. The mapping $*$ from $F(D)$ to $F(D)$ is a star operation on D .

For, let $I \in F(D)$. There exist $x \in q(D)$ and a k -subspace $U \supset k$ of K such that $I = xUD$. Then $I^* = x(UD)^*$. It follows that $I \subset I^*$.

The case where $I = xD$ with $x \in q(D)$. We have $I^* = xD = I$.

Let $y \in q(D) - \{0\}$. Since $yI = yxUD$, we have $(yI)^* = yx(UD)^* = yI^*$.

If $U = \alpha U_0$ for some $\alpha \in K$, then $I^* = xV$. We have $(I^*)^* = (xKD)^* = xV = I^*$.

If U is a 2-dimensional k -subspace of K with $\alpha U_0 \neq U$ for each $\alpha \in K$, then $I^* = xUD$. Hence $(I^*)^* = xUD = I^*$.

If $\dim(U) \geq 3$, then $I^* = xV$. Since $I^* = xKD$, we have $(I^*)^* = xV$, and hence $(I^*)^* = I^*$.

Assume that $I \subset J$ for $J \in F(D)$. We have $J = yWD$, where $y \in q(D)$ and $W \supset k$ is a k -subspace of K . Fact 6 implies that $I^* \subset J^*$.

Let $U_0 \supset k$ be a 2-dimensional k -subspace of K , and let $*(U_0)$ be the star operation on D defined in Fact 7. Thus, for each 2-dimensional k -subspace $U \supset k$ of K , there is defined a star operation $*(U)$ on D . Let U_1 and U_2 be 2-dimensional k -subspaces of K containing k such that $U_2 \neq \alpha U_1$ for each $\alpha \in K$. Then we have $(U_1 D)^{*(U_1)} = V$ and $(U_1 D)^{*(U_2)} = U_1 D$. Hence $*(U_1) \neq *(U_2)$. Fact 2 implies that $|\Sigma(D)| = \infty$.

Remark 4. Let D be a PVD with maximal ideal M such that $\dim(D) < \infty$. If $V = (M : M)$ is a finitely generated D -module, $|\Sigma'(D)|$ need not be finite.

Let I be an ideal of S . If $a + b \in I$ and $b \notin I$ imply $na \in I$ for some $n > 0$ for elements $a, b \in q(S)$, then I is called a strongly primary ideal of S . If each prime ideal of S is strongly primary, then S is called an almost pseudo-valuation semigroup (or an APVS).

Let I be an ideal of S . If $a + b \in I$ implies either $a \in S$ or $b \in S$ for elements $a, b \in q(S)$, then I is called a powerful ideal in S .

Lemma 11. Assume that S has a strongly primary ideal I with $I \subsetneq S$.

- (1) I is an ideal of $\bar{S}$.
 - (2) I is a powerful ideal in $\bar{S}$.
 - (3) $3I$ is a powerful ideal in S .
 - (4) $V = (I : I)$ is a valuation oversemigroup of S .
 - (5) I is a primary ideal in V , and the radical M of I in V is the maximal ideal of V .
 - (6) $F'(S) = F(S) \cup \{G\}$, where $G = q(S)$.
- Let Q be a prime ideal of S with $Q \subsetneq I$. Then
- (7) $Q \subset 3I$, and Q is a powerful ideal in S .
 - (8) Q is strongly prime.
 - (9) Q is a prime ideal of V .

Proof. (1) Suppose the contrary. There exists $x \in \bar{S}$ with $x + I \not\subset I$. We have $mx \in S$ for some $m > 0$. Since $-x + x + I \subset I$, we have $-nx \in I$ for some $n > 0$. Then we have

$$0 = n(mx) + m(-nx) \in S + I = I,$$

and hence $I = S$; a contradiction.

(2) Let $x + y \in I$, and let $x \notin \bar{S}$. We have $ny \in I$ for some $n > 0$, and hence $y \in \bar{S}$.

(3) Let $x + y \in 3I$, and let $x \notin S$. Let $a, b \in I$. Since $(a + b - x) + (x - b) \in I$ and $x - b \notin S$, we have $n(a + b - x) \in I$ for some $n > 0$. Then $a + b - x \in \bar{S}$. Hence $2I \subset x + \bar{S}$. Then

$$y = (-x) + (x + y) \in -x + 3I = (-x + I) + 2I \subset (-x + I) + x + \bar{S} = I + \bar{S}.$$

Hence $y \in I$ by (1).

(4) Let $x \in G - V$. Then $\bar{S} \subset V$ by (1). Since $x - x + I = I$ and $x \notin \bar{S}$, we have $-x + I \subset I$. Therefore $-x \in V$, and hence V is a valuation semigroup.

(5) I is a primary ideal of V by the definition of V .

Suppose that $V_M \supsetneq V$. Choose $x \in V_M$ with $x \notin V$. Since $-x + x + I \subset I$, we have $-nx \in I$ for some $n > 0$. Then $0 = -nx + nx \in I + V_M$, and hence

$I + V_M = V_M$; a contradiction. Therefore $V = V_M$, and M is the maximal ideal of V .

(6) Let v be a valuation on G belonging to V . Let $I' \in F'(S)$.

The case where $v(I')$ is bounded below: There exists $x \in G$ such that $v(x) < v(j)$ for each $j \in I'$. Then $-x + I' \subset V$. Hence I' is a fractional ideal of S .

The case where $v(I')$ is not bounded below: Let $x \in G$. There exists $y \in I'$ such that $v(y) < v(x)$. Then $x = (x - y) + y \in V + y \subset I'$. Hence $I' = G$.

(7) Suppose that $Q \not\subset 3I$. Choose $b \in Q - 3I$ and $a \in 3I$. Since $a = (2a - b) + (b - a)$ and $b - a \notin S$, we have $2a - b \in S$ by (3). Then we have $2a \in b + S$, $2a \in Q$, $a \in Q$, $3I \subset Q$, and $I \subset Q$; a contradiction. Hence $Q \subset 3I$.

Since $3I$ is powerful in S , Q is powerful in S .

(8) Let $x + y \in Q$. We may assume that either $x \notin S$ or $y \notin S$, say $x \notin S$. Since Q is powerful in S , we have $y \in S$ and $2x \notin Q$.

If $2x \in S$; Since $2x + 2y \in Q$, we have $2y \in Q$, and hence $y \in Q$.

If $2x \notin S$; Since $(2y - x - y) + 2x \in Q$, we have $2y - x - y \in S$. Then $2y \in x + y + S \subset Q$, and hence $y \in Q$.

(9) Suppose the contrary. There exists an element $i \in I - Q$ such that $V + Q \ni i$. We have $i = x + q$ for some $x \in V$ and $q \in Q$. Since $i - x \in Q$, we have $-x \in Q$. It follows that $0 = x - x \in V + I = I$; a contradiction.

Lemma 12. *Let S be an APVS, let P be the maximal ideal of S , and let $V = (P : P)$.*

(1) *If S is not a valuation semigroup, then $V = P^{-1}$.*

(2) *The set of non-maximal prime ideals of S coincides with the set of non-maximal prime ideals of V , and $\dim(V) = \dim(S)$.*

(3) *Let T be an oversemigroup of S . Then either $T \supset V$ or $T \not\subseteq V$.*

(4) *Let $\Sigma'_1 = \{ * \in \Sigma'(S) \mid S^* \supset V \}$. Then there is a canonical bijection from $\Sigma'(V)$ onto Σ'_1 .*

(5) *Let $\Sigma'_2 = \{ * \in \Sigma'(S) \mid S^* \subsetneq V \}$. Then $\Sigma'(S) = \Sigma'_1 \cup \Sigma'_2$.*

(6) *If $|\Sigma'(S)| < \infty$, then $\dim(S) < \infty$ and V is a finitely generated S -module.*

Proof. Lemma 11 implies that V is a valuation oversemigroup of S , P is a primary ideal of V , and the radical M of P in V is the maximal ideal of V .

(1) Suppose the contrary. Choose an element $x \in P^{-1} - V$. Since $x + P \not\subset P$, we have $S = x + P$. It follows that $P = -x + S$, and $V = S$; a contradiction.

(2) If P is not a maximal ideal of V , then P is not a prime ideal of V . The assertion follows from Lemma 11 (9).

(3) Assume that $T \not\subset V$. There exists an element $t \in T - V$. Then $-t \in M$, and hence $-nt \in P$ for some $n > 0$. Let $x \in V$. Then $x = nt + (x - nt) \in T + P \subset T$, and hence $V \subset T$.

(6) Then V is a finitely generated oversemigroup of S . By Lemma 1, $\bar{S}$ is a valuation semigroup. Since $\bar{S} \subset V$, and M is a maximal ideal of $\bar{S}$, we have $V = \bar{S}$. Therefore V is a finitely generated S -module.

Lemma 13. *Let S be an APVS with dimension $< \infty$ which is not a PVS, P be the maximal ideal of S , $V = (P : P)$ which is a finitely generated S -module,*

M be the maximal ideal of V , L be the unit group of V , H be the unit group of S , v be the valuation belonging to V , and Γ be the value group of v .

- (1) $V = \bar{S}$.
- (2) $|L/H| < \infty$.
- (3) There exist no $g_i \in L$ such that $V = S[g_1, \dots, g_l]$.
- (4) There exist $g_i \in L$ and $x_0 \in M$ such that $V = S[g_1, \dots, g_l, x_0]$.
- (5) In (4), $v(x_0)$ is a minimal positive element of Γ .
- (6) $\mathbf{Z}v(x_0)$ is a rank 1 convex subgroup of Γ .
- (7) Let m be a minimal positive integer k such that $kx_0 \in S$. Then $P = \{x \in \mathbf{q}(S) \mid v(x) \geq mv(x_0)\}$.

Proof. (1) Since V is a valuation oversemigroup of S which is a finitely generated S -module, we have $V = \bar{S}$.

(2) Let $V = (S + x_1) \cup \dots \cup (S + x_n)$. Let $\alpha \in L$. Then $\alpha = s + x_i$ for some i with $s \in H$. Hence $|L/H| \leq n$.

(3) Suppose the contrary. For any $x \in M$, we have $x = s + \sum k_i g_i$ for $s \in S$ and $k_i \geq 0$. Then $s \in P$, hence $x \in P$. Therefore $P = M$, and hence P is strongly prime; a contradiction.

(4) There exist $g_1, \dots, g_l \in L$ and $x_1, \dots, x_m \in M$ such that $V = S[g_1, \dots, g_l, x_1, \dots, x_m]$ with $m > 0$. Assume that, for instance, $v(x_2) > v(x_1)$. We have $x_2 - x_1 = s + \sum k_i g_i + \sum k'_i x_i$ for $s \in S$ and $k_i \geq 0, k'_i \geq 0$. Hence $x_2 = s + \sum k_i g_i + (1 + k'_1)x_1 + \sum_{i=2}^m k'_i x_i$. It follows that $k'_2 = 0$, and $V = S[g_1, \dots, g_l, x_1, x_3, x_4, \dots, x_m]$. Therefore we may assume that $v(x_1) = \dots = v(x_m)$. It follows that $V = S[g_1, \dots, g_l, x_1]$.

(5) Suppose the contrary. There exists $x \in M$ such that $v(x) < v(x_0)$. Then $x_0 - x = s + \sum k_i g_i + kx_0$. If $k > 0$, then $x \in L$; a contradiction. If $k = 0$, then $s \in P$. Then $x_0 = x + s + \sum k_i g_i \in P$, and hence $V = S[g_1, \dots, g_l]$; a contradiction.

(7) Assume that $\gamma \geq mv(x_0)$. Then $\gamma - mv(x_0) = v(x)$ for some $x \in V$, and $\gamma = v(mx_0 + x) \in v(P)$.

A fractional ideal of S is also called an S -fractional ideal, and a divisorial fractional ideal of S is also called an S -divisorial fractional ideal.

Lemma 14. In Lemma 13, let T be a subsemigroup of V with $S \subset T \subsetneq V$. Then $|\Sigma(T)| < \infty$.

Proof. Let $g_1, \dots, g_l$ be a complete representative system of L modulo H with $g_1 = 0$. We have $V = S[g_1, \dots, g_l, x_0]$ with $x_0 \in M$. We may assume that $v(x_0) = 1$, and that $\mathbf{Z}$ is a rank 1 convex subgroup of Γ .

(1) The set $\{I \mid I \text{ is a } T\text{-fractional ideal with } T \subset I \subset V\}$ is of finite.

For, $I \supset P$, and $I - P \subset \{g_i + h + kx_0 \mid 1 \leq i \leq l, 0 \leq k < m, \text{ and } h \in H\}$.

Let $\{I_1, \dots, I_a\}$ be the set in (1).

(2) The set $\{I \mid I \text{ is a } T\text{-fractional ideal with } T \subset I \subset V - mx_0\}$ is of finite.

For, $I \supset P$, and $I - P \subset \{g_i + h + kx_0 \mid 1 \leq i \leq l, -m \leq k < m, \text{ and } h \in H\}$.

Let $\{J_1, \dots, J_b\}$ be the set in (2).

(3) Let I be a T -fractional ideal such that $\inf \Gamma v(I)$ does not exist. Then I is a T -divisorial fractional ideal, that is, $I = I^{v_T}$.

For, let $\{x \in q(S) \mid x \text{ is a lower bound of } v(I)\} = \{x_\lambda \mid \lambda \in \Lambda\}$. Let $i \in I$, and let $x \in \{x_\lambda \mid \lambda \in \Lambda\}$. There exists $i_1, \dots, i_m \in I$ such that $v(i_m) < v(i_{m-1}) < \dots < v(i_1) < v(i)$. Then we have $v(i_m) + m \leq v(i)$, and hence $v(i) - v(x) > m$. Therefore $i - x \in P$ and $i \in x + T$. It follows that $I \subset \cap_\lambda (x_\lambda + T)$. Conversely, let $y \in \cap_\lambda (x_\lambda + T)$. Then the similar argument as above shows that $y \in I$. Therefore $I = \cap_\lambda (x_\lambda + T)$.

(4) Let I be a T -fractional ideal. Then either $v(I)$ has a minimal element or $\inf^\Gamma v(I)$ does not exist.

For, suppose that $v(I)$ does not have a minimal element, and that there exists $x \in q(S)$ with $v(x) = \inf^\Gamma v(I)$. Then there exists $i \in I$ such that $v(x) < v(i) < v(x) + 1$; a contradiction.

(5) The set $\{* \in \Sigma(T) \mid I_i^* \in \{J_1, \dots, J_b\} \text{ for each } i\}$ is of finite.

For, let $*_1, *_2$ be elements of the set in (5), and assume that $I_i^{*1} = I_i^{*2}$ for each i . It suffices to show that $*_1 = *_2$. Let I be a T -fractional ideal.

The case where $v(I)$ has a minimal element $v(x)$ with $x \in I$: Since $T \subset I - x \subset V$, we have $I - x = I_i$ for some i . Since $I_i^{*1} = I_i^{*2}$, we have $I^{*1} = I^{*2}$.

The case where $\inf^\Gamma v(I)$ does not exist: Then $I = I^{vT}$. Hence $I^{*1} = I^{*2}$.

Let $\{*_1, \dots, *_c\}$ be the set in (5).

(6) $\Sigma(T) = \{*_1, \dots, *_c\}$.

For, let $* \in \Sigma(T)$. Since $mx_0 + V \subset P$, we have $V^* \subset T - mx_0$. Since $T \subset I_i^* \subset V^* \subset V - mx_0$, we have $* \in \{*_1, \dots, *_c\}$.

Theorem 16. *Let S be an APVS, let P be the maximal ideal of S , and let $V = (P : P)$. Then $|\Sigma'(S)| < \infty$ if and only if $\dim(S) < \infty$ and V is a finitely generated S -module.*

Proof. The assertion holds for each PVS S (Proposition 8). And the necessity of the assertion holds (Lemma 12 (6)).

Thus let S be an APVS with dimension $< \infty$ which is not a PVS, P be the maximal ideal of S , V be a finitely generated S -module, M be the maximal ideal of V , L be the unit group of V , H be the unit group of S , v be the valuation belonging to V , and Γ be the value group of v .

In Lemma 14, we have $V - S \subset \{g_i + h + kx_0 \mid 1 \leq i \leq l, 0 \leq k < m, \text{ and } h \in H\}$. Therefore the set $\{T \mid T \text{ is a subsemigroup of } V \text{ with } S \subset T \subsetneq V\} = \{T_1, \dots, T_\alpha\}$ is a finite set. Let $\Sigma'_2 = \{* \in \Sigma'(S) \mid S^* \subsetneq V\}$. It suffices to show that $|\Sigma'_2| < \infty$. By Lemma 14, we have $|\Sigma(T_i)| < \infty$, and $|\Sigma'_2| = \sum |\Sigma(T_i)| < \infty$.

Let I be an ideal of D . If $ab \in I$ and $b \notin I$ imply $a^n \in I$ for some $n > 0$ for elements $a, b \in q(D)$, then I is called a strongly primary ideal of D . If $ab \in I$ implies either $a \in D$ or $b \in D$ for $a, b \in q(D)$, then I is called a powerful ideal in D .

If each prime ideal of D is strongly primary, then D is called an almost pseudo-valuation domain (or an APVD).

Lemma 15. *Assume that D has a strongly primary ideal I with $(0) \subsetneq I \subsetneq D$.*

- (1) I is an ideal of $\bar{D}$.
- (2) I is a powerful ideal in $\bar{D}$.
- (3) I^3 is a powerful ideal in D .

- (4) $V = (I : I)$ is a valuation overring of D .
- (5) I is a primary ideal in V , and the radical of I in V is the maximal ideal of V .
- (6) $F'(D) = F(D) \cup \{q(D)\}$.
- (7) If Q is a prime ideal of D with $Q \subsetneq I$. We have $Q \subset I^3$, and Q is a powerful ideal in D .
- (8) Q is strongly prime.
- (9) Q is a prime ideal of V .

Lemma 16. Let D be an APVD, let P be a maximal ideal of D , let $V = (P : P)$, and let M be the maximal ideal of V .

- (1) D is a local ring.
- (2) If D is not a valuation ring, then $V = P^{-1}$.
- (3) The set of non-maximal prime ideals of D coincides with the set of non-maximal prime ideals of V , and $\dim(V) = \dim(D)$.
- (4) $\bar{D}$ is a PVD with maximal ideal M .
- (5) Let T be an overring of D . Then either $T \supset V$ or $T \subsetneq V$.
- (6) Let $\Sigma'_1 = \{ * \in \Sigma'(D) \mid D^* \supset V \}$. Then there exists a canonical bijection from $\Sigma'(V)$ onto Σ'_1 .
- (7) Let $\Sigma'_2 = \{ * \in \Sigma'(D) \mid D^* \subsetneq V \}$. Then $\Sigma'(D) = \Sigma'_1 \cup \Sigma'_2$.
- (8) If $|\Sigma'(D)| < \infty$, then $\dim(D) < \infty$, $V = \bar{D}$, V is a finitely generated D -module, and V/M is a simple extension field of D/P with $[V/M : D/P] < \infty$.

Proof. (1) Suppose the contrary. There exists a maximal ideal Q of D with $Q \not\subset P$. Choose $a \in Q - P$. Then $a \notin M$, that is, a is a unit of V . Therefore $a^{-1} \in (P : P)$, and hence $P \subset aP \subset Q$; a contradiction.

(4) Suppose that $\bar{D}$ has a maximal ideal N distinct from M . Choose $a \in M - N$ and $b \in N - M$. We have $a^n \in P$ for some $n > 0$. Since b is a unit of V , we have $b^{-1} \in (P : P)$, and hence $P \subset bP \subset N$. It follows that $a \in N$; a contradiction. Then $\bar{D}$ is a PVD by Lemma 8 (2).

(8) V is a finitely generated overring of D , and $\bar{D} \subset V$. Since $|\Sigma'(\bar{D})| < \infty$, $\bar{D}$ is a valuation ring by Theorem 3. Hence $V = \bar{D}$. Therefore V is a finitely generated D -module.

Theorem 17. Let D be an APVD which is not a PVD, let P be a maximal ideal of D , and let $V = (P : P)$. Assume that $\dim(D) < \infty$, and let $\{T_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings T of D with $T \subsetneq V$. Let Σ'_1 be the set of semistar operations $*$ on D such that $D^* \supset V$, and let Σ'_2 be the set of semistar operations $*$ on D such that $D^* \subsetneq V$.

- (1) $\Sigma'(D) = \Sigma'_1 \cup \Sigma'_2$.
- (2) $|\Sigma'(V)| < \infty$.
- (3) $|\Sigma'_1| = |\Sigma'(V)|$.
- (4) There is a canonical bijection from the disjoint union $\cup \Sigma(T_\lambda)$ onto Σ'_2 .

Proposition 10. Let S be 1-dimensional such that the set H of units of S coincides with the set of units of $\bar{S}$. The following conditions are equivalent.

- (1) $|\Sigma'(S)| < \infty$.

(2) $\bar{S}$ is a valuation semigroup on the quotient group G of S , and $\bar{S}$ is a finitely generated S -module.

Proof. (2) $\implies$ (1): We may assume that $S \subsetneq \bar{S}$. Let M (resp. N) be the maximal ideal of S (resp. $V = \bar{S}$), let v be the valuation on G belonging to V , and let Γ be the value group of v .

(i) Let $V = S[x_1, \dots, x_n]$ with $x_i \notin S$ for each i . Let $v(x_1) = \min \{v(x_1), \dots, v(x_n)\}$. Then we have $v(x_1) = \min \{v(N)\}$.

For, suppose that there exists an element $y \in N$ such that $v(y) < v(x_1)$. We have $y = s + \sum k_i x_i$ with $s \in S$ and $k_i \geq 0$. By the choice of y , it follows that $k_1 = \dots = k_n = 0$, and that $y = s \in M$. Put $y_1 = x_1 - y$. Since $y_1 \in N$ and $v(y_1) < v(x_1)$, also it follows that $y_1 \in M$. Then $x_1 = y + y_1 \in M$; a contradiction.

Thus V is a discrete valuation semigroup, and we may assume that $\Gamma = \mathbf{Z}$ with $v(x_1) = 1$.

(ii) There exists a positive integer h such that $h + \mathbf{Z}_0 \subset v(S)$.

(iii) The set $\{I \mid I \text{ is a fractional ideal of } S \text{ such that } S \subset I \subset V\}$ is of finite.

For, $V - S \subset \{kx_1 + h \mid 0 \leq k < h \text{ and } h \in H\}$.

Let $\{J_1, \dots, J_a\}$ be the set in (iii)

(iv) $F'(S) = F(S) \cup \{G\}$.

(v) The set $\{* \in \Sigma'(S) \mid J_i^* \in \{J_1, \dots, J_a\} \text{ for each } i\}$ is of finite.

For, let $*_1, *_2$ be elements of the set in (v), and assume that $J_i^{*1} = J_i^{*2}$ for each i . It suffices to show that $*_1 = *_2$. Let I be a fractional ideal of S . There exists an element $g \in G$ such that $S \subset g + I \subset V$. Hence $I = J_i - g$ for some i . Then we have $I^{*1} = J_i^{*1} - g = J_i^{*2} - g = I^{*2}$. It follows that $*_1 = *_2$.

Let $\{*_1, \dots, *_b\}$ be the set in (v). Let $*$ be a semistar operation on S which differs from the e -operation on S . Let $\alpha(*)$ be the ascent of $*$ to V . V has exactly two semistar operations d'_V and e_V . Therefore $\alpha(*)$ is the d' -operation on V . Hence $S \subset J_i^* \subset V^* = V^{\alpha(*)} = V$ for each i , and hence $J_i^* \in \{J_1, \dots, J_a\}$ for each i . It follows that $* \in \{*_1, \dots, *_b\}$, and hence $|\Sigma'(S)| < \infty$.

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