Caloric morphisms with respect to radial metrics on semi-euclidean spaces

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1. Introduction.

In [5], we defined the notion of caloric morphism, the transformation which preserves the solutions of the heat equation, between semi-riemannian manifolds, and obtained a general characterization theorem. The Appell transformation is the typical example in Euclidean spaces. We also gave some examples of caloric morphism on Euclidean space with respect to a radial metric of form

$$g_{ij} = \rho(|x|)\delta_{ij}, \quad i, j = 1, 2, \dots, n,$$

where ρ is a positive C^∞ -function defined on an interval $I \subset (0, \infty)$ and δ_{ij} is Kronecker's delta. In our previous paper [7], we determined the caloric morphism of form

$$f(t, x) = (f_0(t), \nu(t)R(t)x)$$

or

$$f(t, x) = (f_0(t), \nu(t)|x|^{-2}R(t)x),$$

where $\nu(t)$ is a positive C^∞ -function and $R(t)$ is a C^∞ -function of orthogonal matrix value.

In [5], we treated the case of semi-riemannian manifold to illustrate which property of caloric morphism depends on the ellipticity of the Laplacian. On the other hand, semi-riemannian metric arises from natural constructions of manifolds such as direct product (see Example 2 below). This fact leads us to extend our results in [7] to semi-riemannian case.

In this paper, we shall extend the results in [7] to the semi-riemannian case. Let $n \geq 2$ and $\gamma = (\gamma_{ij})$ be a non-degenerate (n, n) -symmetric matrix. We consider the semi-riemannian manifold $M = \{x \in \mathbb{R}^n; x \cdot \gamma x > 0\}$ with a metric of form

$$(*) \quad g_{ij} = \rho(\langle x \rangle)\gamma_{ij}, \quad i, j = 1, 2, \dots, n,$$

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where $\langle x \rangle = \sqrt{x \cdot \gamma x}$ and ρ is a positive C^∞ -function defined on an interval $I \subset (0, \infty)$. In this paper, we call this type of metric as *radial metric*. We shall determine the caloric morphism defined on a domain in $\mathbb{R} \times M$ which has one of the following forms (a) or (b):

- (a) $f(t, x) = (f_0(t), \nu(t)R(t)x),$
 (b) $f(t, x) = (f_0(t), \nu(t)\langle x \rangle^{-2}R(t)x).$

Here $\nu(t)$ is a positive C^∞ -function and $R(t)$ is a C^∞ -function of

$$O_\gamma(n) = \{R; R\gamma^{-1}R = \gamma^{-1}\}$$

value. We note that the existence of the caloric morphism of above forms puts strong restrictions on the metric in the case that the function ν is not constant. In fact, It is shown that ρ has the form $\rho(r) = pr^q$ in such cases. Our main result is formulated in the following theorem.

Theorem 1. *Let $M = \{x \in \mathbb{R}^n; x \cdot \gamma x > 0\}$ be the semi-riemannian manifold with metric of form (*) and (f, φ) be a caloric morphism of form (a) or (b) defined on a domain $D \subset \mathbb{R} \times M$ (see Definition 1 below). Assume that $f(D) \cap D \neq \emptyset$. Then (f, φ) and ρ have one of the following forms.*

Case 1-a. $n = 2$ and ρ is of form $\rho(r) = pr^{-2}$ ($p > 0$) and f is of form (a),

$$\begin{aligned} f(t, x) &= (t + d, ce^{at}R_0 \exp bt\gamma^{-1}Jx), \\ \varphi(t, r, \theta) &= Cr^{\frac{1}{2}ap} \exp \frac{p}{2} \left(\frac{b}{\sqrt{|\det \gamma|}} \theta + \frac{1}{2} \left(a^2 + \frac{b^2}{\det \gamma} \right) t \right), \end{aligned}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Case 1-b. $n = 2$ and ρ is of form $\rho(r) = pr^{-2}$ ($p > 0$) and f is of form (b),

$$\begin{aligned} f(t, x) &= (t + d, ce^{at}\langle x \rangle^{-2}R_0 \exp bt\gamma^{-1}Jx), \\ \varphi(t, r, \theta) &= Cr^{-\frac{1}{2}ap} \exp \frac{p}{2} \left(\frac{b}{\sqrt{|\det \gamma|}} \theta + \frac{1}{2} \left(a^2 + \frac{b^2}{\det \gamma} \right) t \right). \end{aligned}$$

In the cases 1-a and 1-b, $a, b, c, d, C \in \mathbb{R}$ ($c, C > 0$), $R_0 \in O_\gamma(2)$ and (r, θ) is the polar coordinate of $\mathbb{R}^2$ with respect to γ (see section 4).

Case 2-a. $n \geq 2$ and ρ is of form $\rho(r) = pr^{-2}$ ($p > 0$) and f is of form (a),

$$f(t, x) = (t + d, ce^{at}R_0x), \quad \varphi(t, x) = C\langle x \rangle^{\frac{1}{2}ap} \exp \left(\frac{1}{4}pa^2t \right).$$

Case 2-b. $n \geq 2$ and ρ is of form $\rho(r) = pr^{-2}$ ($p > 0$) and f is of form (b),

$$f(t, x) = (t + d, ce^{at}\langle x \rangle^{-2}R_0x), \quad \varphi(t, x) = C\langle x \rangle^{-\frac{1}{2}ap} \exp \left(\frac{1}{4}pa^2t \right).$$

In the cases 2-a and 2-b, $a, c, d, C \in \mathbb{R}$ ($c, C > 0$) and $R_0 \in O_\gamma(n)$.

Case 3. ρ is of form $\rho(r) = pr^q$ ($q \in \mathbb{R}$, $q \neq -2$), and

$$f(t, x) = \left(\frac{ct + d}{at + b}, |at + b|^{-2/(q+2)} R_0 x \right),$$

$$\varphi(t, x) = \frac{C}{|at + b|^{n/2}} \exp \left[- \frac{pa \langle x \rangle^{q+2}}{(q+2)^2 (at + b)} \right],$$

where $a, b, c, d, C \in \mathbb{R}$ ($bc - ad = 1$, $C > 0$) and $R_0 \in O_\gamma(n)$.

Case 4-a. ρ is a positive C^∞ -function satisfying $\rho(\nu r) = \lambda \rho(r)$ for all r with some positive constants ν and λ and f is of form (a), and

$$f(t, x) = (\lambda t + d, \nu R_0 x), \quad \varphi(t, x) = C,$$

where $C > 0$, $d \in \mathbb{R}$ and $R_0 \in O_\gamma(n)$.

Case 4-b. ρ is a positive C^∞ -function satisfying $\rho\left(\frac{\nu}{r}\right) = \frac{\lambda r^4}{\nu^2} \rho(r)$ for all r with some positive constants ν , λ and f is of form (b), and

$$f(t, x) = (\lambda t + d, \nu \langle x \rangle^{-2} R_0 x), \quad \varphi(t, x) = C,$$

where $C > 0$, $d \in \mathbb{R}$ and $R_0 \in O_\gamma(n)$.

Case 5. ρ is a positive C^∞ -function, and

$$f(t, x) = (t + d, R_0 x), \quad \varphi(t, x) = C,$$

where $C > 0$, $d \in \mathbb{R}$ and $R_0 \in O_\gamma(n)$.

Remark. We remark that Case 2-a can be obtained as the limit of Case 3. In fact, for arbitrary set of parameters $a, c, d, C \in \mathbb{R}$ satisfying $c, C > 0$, if we put

$$a_q = -\frac{q+2}{2} ac^{-\frac{q+2}{2}}, \quad b_q = c^{-\frac{q+2}{2}}, \quad c_q = c^{\frac{q+2}{2}} \left(1 + \frac{q+2}{2} ad\right),$$

$$d_q = c^{\frac{q+2}{2}} d, \quad C_q = C e^{-\frac{pa}{2(q+2)}},$$

then the set of parameters a_q, b_q, c_q, d_q, C_q satisfies $b_q c_q - a_q d_q = 1$. Put

$$f_{(q)}(t, x) = \left(\frac{c_q t + d_q}{a_q t + b_q}, |a_q t + b_q|^{-\frac{2}{q+2}} R x \right)$$

$$\varphi_{(q)}(t, x) = \frac{C_q}{|a_q t + b_q|^{n/2}} \exp \left[- \frac{pa_q \gamma(x, x)^{\frac{q+2}{2}}}{(q+2)^2 (a_q t + b_q)} \right].$$

Then the pair $(f_{(q)}, \varphi_{(q)})$ is a caloric morphism with respect to the metric $p \langle x \rangle^q \gamma$ and satisfies

$$\lim_{q \rightarrow -2} f_{(q)}(t, x) = (t + d, ce^{at} R x), \quad \lim_{q \rightarrow -2} \varphi_{(q)}(t, x) = C \langle x \rangle^{\frac{pa}{2}} \exp \left(\frac{pa^2 t}{4} \right).$$

2. Notations and Preliminaries.

Let (M, g) be an n -dimensional semi-riemannian manifold. In this paper, we always consider manifolds to be connected. We denote by Δ_g the Laplace-Beltrami operator on (M, g) which is given in local coordinates $(x_i)_{i=1}^n$ by

$$\Delta_g u = \sum_{i,j=1}^n \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x_i} \left(\sqrt{|\det g|} g^{ij} \frac{\partial u}{\partial x_j} \right),$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}) as usual. We define the gradient vector field $\nabla_g u$ of a C^∞ -function u by

$$\nabla_g u = \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Then the gradient of the product uv of two smooth functions u, v is given by

$$\nabla_g(uv) = u \nabla_g v + v \nabla_g u,$$

and the inner product of two gradients $\nabla_g u$ and $\nabla_g v$ is given by

$$g(\nabla_g u, \nabla_g v) = \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}.$$

We denote the heat operator $\frac{\partial}{\partial t} - \Delta_g$ on $\mathbb{R} \times M$ by H_g . A C^∞ -function $u(t, x)$ defined on an open set $D \subset \mathbb{R} \times M$ is said to be caloric, if u satisfies the heat equation

$$H_g u = 0$$

on D .

Definition 1. Let (M, g) be an n -dimensional semi-riemannian manifold. A pair (f, φ) of a C^∞ -mapping f from a domain $D \subset \mathbb{R} \times M$ to $\mathbb{R} \times M$ and a strictly positive C^∞ -function φ defined on D is said to be a caloric morphism, if f and φ satisfy the following conditions;

- (1) $f(D)$ is a domain in $\mathbb{R} \times M$.
- (2) For any subdomain $E \subset f(D)$ and any caloric function u defined on E , the function $\varphi(t, x)(u \circ f)(t, x)$ is caloric on $f^{-1}(E)$.

Let (f, φ) and (F, Φ) be caloric morphisms defined on D and E , respectively. If $f(D) \subset E$, then it is easily seen that the composition $(F \circ f, \varphi \cdot (\Phi \circ f))$ is a caloric morphism on D .

In our previous paper [5], we proved the following characterization theorem for caloric morphism (Theorem 2.2, Remark 1).

Theorem A. *Let (f, φ) be the pair of a smooth mapping f from $D \subset \mathbb{R} \times M$ to a domain $f(D)$ of $\mathbb{R} \times M$ and a positive smooth function φ defined on D .*

Then the following three conditions are equivalent.

(1) *(f, φ) is a caloric morphism.*

(2) *Let $(x_\alpha)_{\alpha=1}^n$ be a local coordinate system of (M, g) , and let*

$$f = (f_0, f_1, \dots, f_n)$$

be the expression of the mapping f in the coordinate system $(t, x_1, \dots, x_n)$ of the product manifold $\mathbb{R} \times M$. Then (f, φ) satisfies the following equations:

(E-1)

$$H_g \varphi = 0,$$

(E-2)

$$H_g f_\alpha = 2g(\nabla_g \log \varphi, \nabla_g f_\alpha) + \sum_{\beta, \gamma=1}^n g(\nabla_g f_\beta, \nabla_g f_\gamma) (\Gamma_{\beta\gamma}^\alpha \circ f),$$

$$\alpha = 1, \dots, n,$$

(E-3)

$$\nabla_g f_0 = 0,$$

(E-4)

$$g(\nabla_g f_\alpha, \nabla_g f_\beta) = f'_0(t)(g^{\alpha\beta} \circ f), \quad \alpha, \beta = 1, \dots, n,$$

where $\Gamma_{\beta\gamma}^\alpha = \sum_{l=1}^n \frac{1}{2} g^{\alpha l} \left(\frac{\partial g_{\gamma l}}{\partial x_\beta} + \frac{\partial g_{\beta l}}{\partial x_\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x_l} \right)$ ($\alpha, \beta, \gamma = 1, \dots, n$) is the Christoffel symbol of (M, g) .

(3) *There exists a smooth function λ on D such that*

$$H_g(\varphi \cdot u \circ f)(t, x) = \lambda(t, x) \varphi(t, x) ((H_g u) \circ f)(t, x)$$

holds for any C^2 -function u defined on a subdomain of $f(D)$.

Remark. In the above theorem, we can replace (E-2) by

$$(E-2') \quad H_g f_\alpha = 2g(\nabla_g \log \varphi, \nabla_g f_\alpha) - f'_0(t)[(\Delta_g x_\alpha) \circ f], \quad \alpha = 1, \dots, n.$$

3. Radial metric on $\mathbb{R}^n$.

In the following, we consider the caloric morphism on the semi-euclidean space with a radial metric. Let

$$\gamma(x, y) = \sum_{i,j=1}^n \gamma_{ij} x_i y_j$$

be a non-degenerate symmetric bilinear form. We define the domain $M \subset \mathbb{R}^n$ as $M = \{x \in \mathbb{R}^n; \gamma(x, x) > 0\}$, and define a positive function $\langle x \rangle$ as $\langle x \rangle = \sqrt{\gamma(x, x)}$ on M . We consider the metric g on M of form

$$g_{ij}(x) = \rho(\langle x \rangle) \gamma_{ij},$$

where ρ is a positive C^∞ -function defined on an interval $I \subset (0, \infty)$. We call this type of metric as *radial metric* here. Recall that $O_\gamma(n) = \{R; R\gamma^{-1}R = \gamma^{-1}\}$ and that for any $R \in O_\gamma(n)$ the equality $\langle Rx \rangle = \langle x \rangle$ holds.

Example 1. (hyperbolic metric) Let $\rho(r) = (1 - r^2)^{-2}$ ($0 \leq r < 1$) and $\gamma_{ij} = \delta_{ij}$ ($i, j = 1, 2$). Then we have the well known real hyperbolic space.

Example 2. The above hyperbolic space is isometric to the hyperbola $\mathbb{H} = \{(x_1, x_2, x_3); -x_1^2 - x_2^2 + x_3^2 = 1\}$ in the semi-euclidean space $(\mathbb{R}^3, \gamma)$ where $\gamma(x, x) = -x_1^2 - x_2^2 + x_3^2$. The metric of the direct product space $\mathbb{R} \times \mathbb{H}$ is isometric to the radial metric

$$g_{ij} = \frac{1}{r^2} (-(dx_1)^2 - (dx_2)^2 + (dx_3)^2)$$

on a semi-euclidean space $(\mathbb{R}^3, \gamma)$.

Hereafter, we use the following notations: for a symmetric matrix $A = (a_{ij})$,

$$(Ax)_i = \sum_j A_{ij}x_j, \quad A(x, y) = \sum_{i,j} A_{ij}x_iy_j.$$

Then we have

$$\begin{aligned} \frac{\partial \langle x \rangle}{\partial x_j} &= \frac{1}{2\sqrt{\gamma(x, x)}} \frac{\partial \gamma(x, x)}{\partial x_j} = \frac{(\gamma x)_j}{\langle x \rangle}, \\ \frac{\partial \rho(\langle x \rangle)}{\partial x_j} &= \rho'(\langle x \rangle) \frac{\partial \langle x \rangle}{\partial x_j} = \frac{\rho'(\langle x \rangle)}{\langle x \rangle} \sum_{i=1}^n (\gamma x)_i, \end{aligned}$$

$$\det g = \rho(\langle x \rangle)^n \det \gamma, \quad \sqrt{|\det g|} = \rho(\langle x \rangle)^{n/2} \sqrt{|\det \gamma|} \quad \text{and} \quad g^{ij} = \frac{1}{\rho(\langle x \rangle)} \gamma^{ij},$$

where (γ^{ij}) denotes the inverse matrix of (γ_{ij}) . We can choose the usual cartesian coordinate system as a local coordinate of M . Then the Laplacian of a function u is given by

$$(1) \quad \Delta_g u = \frac{1}{\rho(\langle x \rangle)} \sum_{i,j=1}^n \gamma^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{n-2}{2} \frac{\rho'(\langle x \rangle)}{\rho(\langle x \rangle)^2} \sum_{j=1}^n \frac{x_j}{\langle x \rangle} \frac{\partial u}{\partial x_j}.$$

The gradient of a function u is given by

$$\nabla_g u = \frac{1}{\rho(\langle x \rangle)} \sum_{i,j=1}^n \gamma^{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j},$$

and hence the inner product of the gradients of two functions u and v is given by

$$(2) \quad g(\nabla_g u, \nabla_g v) = \frac{1}{\rho(\langle x \rangle)} \sum_{i,j=1}^n \gamma^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}.$$

The Laplacian of the coordinate function x_α is given by

$$(3) \quad \Delta_g x_\alpha = \frac{n-2}{2} \frac{\rho'(\langle x \rangle)}{\rho(\langle x \rangle)^2} \frac{x_\alpha}{\langle x \rangle}, \quad \alpha = 1, 2, \dots, n.$$

Now we proceed to consider caloric morphisms with respect to the radial metric g . Let (f, φ) be a caloric morphism such that f is of form (a):

$$f(t, x) = (f_0(t), \nu(t)R(t)x).$$

Then f is expressed as

$$f(t, x) = (f_0(t), f_1(t, x), \dots, f_n(t, x)),$$

$$f_\alpha(t, x) = \sum_{j=1}^n \nu(t) r_{\alpha j}(t) x_j, \quad \alpha = 1, 2, \dots, n,$$

where $r_{\alpha j}(t)$ ($\alpha, j = 1, 2, \dots, n$) is the (α, j) -element of the $O_\gamma(n)$ -matrix $R(t)$. Their derivatives are given by

$$(4) \quad \begin{aligned} \frac{\partial f_\alpha}{\partial t} &= \sum_{j=1}^n (\nu'(t) r_{\alpha j}(t) + \nu(t) r'_{\alpha j}(t)) x_j, \\ \frac{\partial f_\alpha}{\partial x_j} &= \nu(t) r_{\alpha j}(t) \end{aligned}$$

for $\alpha, j = 1, 2, \dots, n$.

Next, if (f, φ) be a caloric morphism such that f is of form (b):

$$f(t, x) = (f_0(t), \frac{\nu(t)R(t)x}{\langle x \rangle^2}),$$

then f_α is given by

$$f_\alpha(t, x) = \sum_{j=1}^n \nu(t) r_{\alpha j}(t) \frac{x_j}{\langle x \rangle^2}, \quad \alpha = 1, 2, \dots, n.$$

Their derivatives are given by

$$(5) \quad \begin{aligned} \frac{\partial f_\alpha}{\partial t} &= \sum_{j=1}^n (\nu'(t) r_{\alpha j}(t) + \nu(t) r'_{\alpha j}(t)) \frac{x_j}{\langle x \rangle^2}, \\ \frac{\partial f_\alpha}{\partial x_i} &= \nu(t) \frac{r_{\alpha i}}{\langle x \rangle^2} - 2\nu(t) \frac{y_\alpha(\gamma x)_i}{\langle x \rangle^4}, \end{aligned}$$

where $y_\alpha = \sum_{j=1}^n r_{\alpha j} x_j$, for $\alpha, i = 1, 2, \dots, n$.

In the rest of this section, we prepare some lemmas. The first lemma gives the condition for the metric to have a caloric morphism of (a) or (b).

Lemma 1. (1) Let (f, φ) be a caloric morphism such that f is of form (a). Then the equation

$$(6) \quad \rho(\nu(t)\langle x \rangle) = \frac{f'_0(t)}{\nu(t)^2} \rho(\langle x \rangle)$$

holds for all $(t, x) \in D$.

(2) Let (f, φ) be a caloric morphism such that f is of form (b). Then the equation

$$(7) \quad \rho\left(\frac{\nu(t)}{\langle x \rangle}\right) = \frac{f'_0(t)\langle x \rangle^4}{\nu(t)^2} \rho(\langle x \rangle)$$

holds for all $(t, x) \in D$.

Proof. Let (f, φ) be a caloric morphism such that f is of form (a). Then by (2) and (4), the equalities

$$\begin{aligned} g(\nabla_g f_\alpha, \nabla_g f_\beta) &= \frac{1}{\rho(\langle x \rangle)} \sum_{i,j=1}^n \gamma^{ij} \frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_j} \\ &= \frac{1}{\rho(\langle x \rangle)} \sum_{i,j=1}^n \gamma^{ij} \nu(t) r_{\alpha i}(t) \nu(t) r_{\beta j}(t) \\ &= \frac{\nu(t)^2}{\rho(\langle x \rangle)} \gamma^{\alpha\beta} \end{aligned}$$

and

$$f'_0(t)(g^{\alpha\beta} \circ f)(t, x) = f'_0(t) \gamma^{\alpha\beta} \frac{1}{\rho(\nu(t)\langle x \rangle)}$$

hold for $\alpha, \beta = 1, \dots, n$, because $R(t)\gamma^{-1}R(t) = \gamma^{-1}$ holds for each t . The desired equation

$$\rho(\nu(t)\langle x \rangle) = \frac{f'_0(t)}{\nu(t)^2} \rho(\langle x \rangle), \quad (t, x) \in D$$

follows from (E-4).

Next let (f, φ) be a caloric morphism such that f is of form (b). Then by (2) and (5), we have

$$\begin{aligned} g(\nabla_g f_\alpha, \nabla_g f_\beta) &= \frac{\nu(t)^2}{\rho(\langle x \rangle)} \sum_{i,j=1}^n \frac{\gamma^{ij}}{\langle x \rangle^4} \left(r_{\alpha i} - 2 \frac{y_\alpha(\gamma x)_i}{\langle x \rangle^2} \right) \left(r_{\beta j} - 2 \frac{y_\beta(\gamma x)_j}{\langle x \rangle^2} \right) \\ &= \frac{1}{\rho(\langle x \rangle)} \frac{\nu^2}{\langle x \rangle^4} \sum_{j=1}^n \left(\sum_{i=1}^n r_{\alpha i} \gamma^{ij} - 2 \frac{y_\alpha(\gamma^{-1} \gamma x)_j}{\langle x \rangle^2} \right) \left(r_{\beta j} - 2 \frac{y_\beta(\gamma x)_j}{\langle x \rangle^2} \right) \\ &= \frac{1}{\rho(\langle x \rangle)} \frac{\nu^2}{\langle x \rangle^4} \left(\sum_{i,j=1}^n r_{\alpha i} \gamma^{ij} r_{\beta j} - 2 \frac{y_\beta y_\alpha}{\langle x \rangle^2} - 2 \frac{y_\alpha y_\beta}{\langle x \rangle^2} + 4 \frac{y_\alpha y_\beta}{\langle x \rangle^2} \right) \end{aligned}$$

$$= \frac{1}{\rho(\langle x \rangle)} \frac{\nu^2}{\langle x \rangle^4} \sum_{i,j=1}^n r_{\alpha i} \gamma^{ij} r_{\beta j},$$

where $y_\alpha = \sum_{k=1}^n r_{\alpha k} x_k$, $y_\beta = \sum_{l=1}^n r_{\beta l} x_l$. Since $R \in O_\gamma(n)$, $R\gamma^{-1}R = \gamma^{-1}$ holds. Therefore

$$g(\nabla_g f_\alpha, \nabla_g f_\beta) = \frac{\nu^2}{\langle x \rangle^4 \rho(\langle x \rangle)} \gamma^{\alpha\beta}$$

and

$$f'_0 \cdot g^{\alpha\beta} \circ f = f'_0(t) \gamma^{\alpha\beta} \frac{1}{\rho(\nu(t)/\langle x \rangle)}$$

hold for $i, j = 1, \dots, n$, because $R(t) \in O_\gamma(n)$ for each t . The desired equation

$$\rho\left(\frac{\nu(t)}{\langle x \rangle}\right) = \frac{f'_0(t) \langle x \rangle^4}{\nu(t)^2} \rho(\langle x \rangle), \quad (t, x) \in D$$

follows from (E-4). $\square$

The equations (6) and (7) put strong restrictions on ρ in the case that the function ν is not constant. In fact, the second lemma shows that ρ has the form $\rho(r) = pr^q$ in that case.

Lemma 2. *Let ρ be a positive smooth function defined on an open interval $I \subset (0, \infty)$, λ be a positive smooth function and ν be a non-constant positive smooth function, and let E be a connected open subset of $\mathbb{R} \times I$.*

(1) *The equation $\rho(\nu(t)r) = \frac{\lambda(t)}{\nu(t)^2} \rho(r)$ holds for all $(t, r) \in E$ if and only if there exists a constant q and positive constant c, p satisfying $\rho(r) = pr^q$ and $\lambda(t) = c\nu(t)^{q+2}$ for all $(t, r) \in E$. Moreover, if $(t_0, \nu(t_0)r_0) \in E$ holds for some $(t_0, r_0) \in E$, then $c = 1$.*

(2) *The equation $\rho\left(\frac{\nu(t)}{r}\right) = \frac{\lambda(t)r^4}{\nu(t)^2} \rho(r)$ holds for all $(t, r) \in E$ if and only if there exists a constant q and positive constant c, p satisfying $\rho(r) = pr^q$ and $\lambda(t) = c\nu(t)^{-(q+2)}$ for all $(t, r) \in E$. Moreover, if $(t_0, \nu(t_0)r_0) \in E$ holds for some $(t_0, r_0) \in E$, then $\lambda(t) = 1$ and $\rho(r) = pr^{-2}$ for all $(t, r) \in E$.*

Proof. First, we prove (1). Differentiate the equation by r and by t . We have the equations

$$\rho'(\nu(t)r)\nu(t) = \frac{\lambda(t)}{\nu(t)^2} \rho'(r), \quad \rho'(\nu(t)r)\nu'(t)r = \left(\frac{\lambda'(t)}{\nu(t)^2} - 2 \frac{\lambda(t)\nu'(t)}{\nu(t)^3} \right) \rho(r),$$

and hence

$$\nu'(t) \frac{r\rho'(r)}{\rho(r)} = \frac{\lambda'(t)\nu(t)}{\lambda(t)} - 2\nu'(t).$$

Therefore, both $\frac{r\rho'(r)}{\rho(r)}$ and $\frac{\lambda'(t)\nu(t)}{\lambda(t)\nu'(t)} - 2$ are equal to a constant q because $\nu'(t) \neq 0$ for some t . Then we have

$$\frac{\rho'(r)}{\rho(r)} = \frac{q}{r}, \quad \frac{\lambda'(t)}{\lambda(t)} = (q+2) \frac{\nu'(t)}{\nu(t)}.$$

These two equations imply

$$\rho(r) = pr^q \quad (r \in I), \quad \lambda(t) = c\nu(t)^{q+2}$$

with some positive constants p and c . Substituting the both into the given equation, we obtain $c = 1$, if $(t_0, \nu(t_0)r_0) \in E$.

To prove (2), differentiating the equation by r and by t , we have

$$\rho'(\frac{\nu(t)}{r}) = -\frac{\lambda(t)r^5}{\nu(t)^3}(4\rho(r) + r\rho'(r)), \quad \rho'(\frac{\nu(t)}{r}) = \frac{\lambda(t)r^5}{\nu(t)^3}\left(\frac{\lambda'(t)\nu(t)}{\lambda(t)\nu'(t)} - 2\right)\rho(r).$$

By the same argument as above, we have

$$\frac{r\rho'(r)}{\rho(r)} = -\frac{\lambda'(t)\nu(t)}{\lambda(t)\nu'(t)} - 2 = q$$

with some constant q . Then

$$\rho(r) = pr^q, \quad \lambda(t) = c\nu(t)^{-(q+2)}$$

with some positive constants p and c . Substituting the both into the given equation, in the case $(t_0, \nu(t_0)r_0) \in E$, we obtain

$$\rho(r) = pr^{-2}, \quad \lambda(t) = 1.$$

□

The next lemma enables us to reduce the case (b) to the case (a).

Lemma 3. *Assume that the metric ρ satisfies $\rho(\frac{\nu}{r}) = \frac{\lambda r^4}{\nu^2}\rho(r)$ with some positive constants ν and λ . Then the inversion*

$$f(t, x) = (\lambda t, \frac{\nu x}{\langle x \rangle^2}), \quad \varphi(t, x) = 1$$

is a caloric morphism from $\mathbb{R} \times M$ onto itself. Especially, if $\rho(r) = pr^{-2}$, then the simple inversion $(j, 1)$, where

$$j(t, x) = (t, \frac{x}{\langle x \rangle^2}),$$

is a caloric morphism from $\mathbb{R} \times M$ onto itself.

Proof. Clearly, (f, φ) satisfies the equations (E-1) and (E-3). We shall show the equation (E-2). Since $f_\alpha(t, x) = \frac{\nu x_\alpha}{\langle x \rangle^2}$, we have

$$\Delta_g f_\alpha = \frac{1}{\rho(\langle x \rangle)} \sum_{i,j=1}^n \gamma^{ij} \frac{\partial^2 f_\alpha}{\partial x_i \partial x_j} + \frac{n-2}{2} \frac{\rho'(\langle x \rangle)}{\rho(\langle x \rangle)^2} \sum_{i=1}^n \frac{x_i}{\langle x \rangle} \frac{\partial f_\alpha}{\partial x_i}$$

$$= \frac{2(2-n)\nu x_\alpha}{\langle x \rangle^4 \rho(\langle x \rangle)} - \frac{n-2}{2} \frac{\nu \rho'(\langle x \rangle) x_\alpha}{\langle x \rangle^3 \rho(\langle x \rangle)^2}.$$

By the assumption $\rho(\nu/r)^{-1} = \frac{\nu^2}{\lambda r^4} \rho(r)^{-1}$,

$$-\frac{\rho'(\nu/r)}{\rho(\nu/r)^2} \left(-\frac{\nu}{r^2}\right) = \left(\frac{-4\nu^2}{\lambda r^5 \rho(r)} - \frac{\nu^2 \rho'(r)}{\lambda r^4 \rho(r)^2}\right).$$

By (3),

$$\begin{aligned} -f'_0(t)[(\Delta_g x_\alpha) \circ f] &= -\lambda \frac{n-2}{2} \frac{\rho'(\nu/\langle x \rangle)}{\rho(\nu/\langle x \rangle)^2} \frac{(\nu x_\alpha)/\langle x \rangle^2}{\nu/\langle x \rangle} \\ &= \left(\frac{2(n-2)\nu x_\alpha}{\langle x \rangle^4 \rho(\langle x \rangle)} + \frac{n-2}{2} \frac{\nu \rho'(\langle x \rangle) x_\alpha}{\langle x \rangle^3 \rho(\langle x \rangle)^2}\right). \end{aligned}$$

Thus we have

$$\begin{aligned} &\Delta_g f_\alpha + 2g(\nabla_g \log \varphi, \nabla_g f_\alpha) - f'_0(t)((\Delta_g x_\alpha) \circ f) \\ &= \frac{2(2-n)\nu x_i}{\langle x \rangle^4 \rho(\langle x \rangle)} - \frac{n-2}{2} \frac{\nu \rho'(\langle x \rangle) x_i}{\langle x \rangle^3 \rho(\langle x \rangle)^2} + 0 + \left(\frac{2(n-2)\nu x_\alpha}{\langle x \rangle^4 \rho(\langle x \rangle)} + \frac{n-2}{2} \frac{\nu \rho'(\langle x \rangle) x_\alpha}{\langle x \rangle^3 \rho(\langle x \rangle)^2}\right) \\ &= 0 = \frac{\partial f_i}{\partial t}. \end{aligned}$$

To show (E-4), we have by (2) and (5),

$$\begin{aligned} g(\nabla_g f_\alpha, \nabla_g f_\beta) &= \frac{1}{\rho(\langle x \rangle)} \sum_{i,j=1}^n \gamma^{ij} \frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\beta}{\partial x_j} \\ &= \frac{1}{\rho(\langle x \rangle)} \sum_{i,j=1}^n \gamma^{ij} \nu \left(\frac{\delta_{\alpha i}}{\langle x \rangle^2} - 2 \frac{x_\alpha (\gamma x)_i}{\langle x \rangle^4} \right) \nu \left(\frac{\delta_{\beta j}}{\langle x \rangle^2} - 2 \frac{x_\beta (\gamma x)_j}{\langle x \rangle^4} \right) \\ &= \frac{\nu^2}{\langle x \rangle^4 \rho(\langle x \rangle)} \gamma^{\alpha\beta}. \end{aligned}$$

By the assumption on ρ , we have

$$g(\nabla_g f_\alpha, \nabla_g f_\beta) = \frac{\nu^2}{\langle x \rangle^4 \rho(\langle x \rangle)} \gamma^{\alpha\beta} = \lambda \frac{\gamma^{\alpha\beta}}{\rho(\nu/\langle x \rangle)} f'_0 \cdot g^{\alpha\beta} \circ f.$$

Therefore (f, φ) is a caloric morphism. $\square$

We close this section with the following lemma concerning the equation for the function $\log \varphi$.

Lemma 4. *Let (f, φ) be a caloric morphism such that f is of form (a).*

(1) *φ satisfies the following equations:*

$$(8) \quad \nabla_g \log \varphi = \frac{1}{2} \left[\frac{\nu'(t)}{\nu(t)} I + \gamma^{-1} S(t) \right] x,$$

$$(9) \quad \Delta_g \log \varphi = \frac{n}{4} \frac{\nu'(t)}{\nu(t)} \left(\frac{\langle x \rangle \rho'(\langle x \rangle)}{\rho(\langle x \rangle)} + 2 \right),$$

$$(10) \quad g(\nabla_g \log \varphi, \nabla_g \log \varphi) = \frac{\rho(\langle x \rangle)}{4} \left\{ \left(\frac{\nu'(t)}{\nu(t)} \right)^2 \langle x \rangle^2 + (x, {}^t S(t) \gamma^{-1} S(t) x) \right\},$$

where I is the identity matrix and $S(t)$ ($j, k = 1, \dots, n$) is the skew symmetric matrix

$$S(t) = \gamma R(t)^{-1} R'(t).$$

(2) *Unless both conditions $n = 2$ and $\rho(r) = pr^{-2}$ hold, $R'(t) = 0$ holds for all t . In this case, the equation (8) becomes*

$$(11) \quad \nabla_g \log \varphi = \frac{1}{2} \frac{\nu'(t)}{\nu(t)} x$$

and (10) becomes

$$(12) \quad g(\nabla_g \log \varphi, \nabla_g \log \varphi) = \rho(\langle x \rangle) \left(\frac{\nu'(t)}{\nu(t)} \right)^2 \frac{\langle x \rangle^2}{4}.$$

Proof. Since $f(t, x) = (f_0(t), \nu(t)R(t)x)$, where $\nu(t) > 0$ and $R(t) \in O_\gamma(n)$, we have

$$\begin{aligned} \Delta_g f_\alpha &= \frac{1}{\rho(\langle x \rangle)} \sum_{i,j=1}^n \gamma^{ij} \frac{\partial^2 f_\alpha}{\partial x_i \partial x_j} + \frac{n-2}{2} \frac{\rho'(\langle x \rangle)}{\rho(\langle x \rangle)^2} \sum_{j=1}^n \frac{x_j}{\langle x \rangle} \frac{\partial f_\alpha}{\partial x_j} \\ (13) \quad &= \frac{n-2}{2} \frac{\rho'(\langle x \rangle)}{\rho(\langle x \rangle)^2} \sum_{j=1}^n \frac{x_j}{\langle x \rangle} \frac{\partial f_\alpha}{\partial x_j} \\ &= \frac{n-2}{2} \frac{\rho'(\langle x \rangle)}{\langle x \rangle \rho(\langle x \rangle)^2} \nu(t) \sum_{i=1}^n r_{\alpha i}(t) x_i \end{aligned}$$

and

$$\begin{aligned} 2g(\nabla_g \log \varphi, \nabla_g f_i) &= \frac{2}{\rho(\langle x \rangle)} \sum_{j,k=1}^n \gamma^{jk} \frac{\partial \log \varphi}{\partial x_j} \frac{\partial f_i}{\partial x_k} \sum_{i=1}^n \frac{\partial \log \varphi}{\partial x_i} \frac{\partial f_\alpha}{\partial x_i} \\ (14) \quad &= \frac{2}{\rho(\langle x \rangle)} \sum_{i=1}^n \frac{\partial \log \varphi}{\partial x_i} \nu(t) r_{\alpha i}(t) \end{aligned}$$

for $\alpha = 1, 2, \dots, n$, by (1), (2) and (4). The formula (3) implies

$$\begin{aligned}
 f'_0[(\Delta_g x_\alpha) \circ f] &= f'_0(t) \frac{n-2}{2} \frac{\rho'(\langle f \rangle)}{\rho(\langle f \rangle)^2} \frac{f_\alpha}{\langle f \rangle} \\
 &= f'_0(t) \frac{n-2}{2} \frac{\rho'(\langle f \rangle)}{\rho(\langle f \rangle)^2} \frac{\sum_{i=1}^n \nu(t) r_{\alpha i}(t) x_i}{\nu(t) \langle x \rangle} \\
 &= f'_0(t) \frac{n-2}{2} \frac{\rho'(\nu(t) \langle x \rangle)}{\langle x \rangle \rho(\nu(t) \langle x \rangle)^2} \sum_{i=1}^n r_{\alpha i}(t) x_i
 \end{aligned}
 \tag{15}$$

for $\alpha = 1, 2, \dots, n$. On the other hand, differentiating (6) by $\langle x \rangle$, we have

$$\frac{f'_0(t) \rho'(\nu(t) \langle x \rangle)}{\rho(\nu(t) \langle x \rangle)^2} = \frac{\nu(t) \rho'(\langle x \rangle)}{\rho(\langle x \rangle)^2}.
 \tag{16}$$

Substituting (4), (13), (14), (15) and (16) into (E-2'):

$$\frac{\partial f_\alpha}{\partial t} = \Delta_g f_\alpha + 2g(\nabla_g \log \varphi, \nabla_g f_\alpha) - f'_0[(\Delta_g x_\alpha) \circ f],$$

we have

$$\begin{aligned}
 \sum_{j=1}^n (\nu'(t) r_{ij}(t) + \nu(t) r'_{ij}(t)) x_j &= \frac{2\nu(t)}{\rho(\langle x \rangle)} \sum_{j,k=1}^n \gamma^{jk} \frac{\partial \log \varphi}{\partial x_j} r_{ik}(t), \\
 \frac{1}{2} \sum_{j=1}^n \left(\frac{\nu'(t)}{\nu(t)} r_{ij}(t) + r'_{ij}(t) \right) x_j &= \sum_{k=1}^n r_{ik}(t) \frac{1}{\rho(\langle x \rangle)} \sum_{j=1}^n \gamma^{kj} \frac{\partial \log \varphi}{\partial x_j}, \\
 \frac{1}{2} \left[\frac{\nu'(t)}{\nu(t)} R(t) + R'(t) \right] x &= R(t) \nabla_g \log \varphi,
 \end{aligned}$$

and then

$$\nabla_g \log \varphi = \frac{1}{2} \left[\frac{\nu'(t)}{\nu(t)} I + R(t)^{-1} R'(t) \right] x.$$

Now multiply $\left(\frac{2\nu(t)}{\rho(\langle x \rangle)} R(t) \gamma^{-1} \right)^{-1} = \frac{\rho(\langle x \rangle)}{2\nu(t)} \gamma R(t)^{-1}$ on both sides. We have

$$\nabla_x \log \varphi = \frac{\rho(\langle x \rangle)}{2} \left[\frac{\nu'(t)}{\nu(t)} \gamma + S(t) \right] x,$$

where $S(t) = \gamma R(t)^{-1} R'(t)$. The matrix $S(t)$ is skew symmetric. In fact, $S + {}^t S = {}^t R \gamma R' + {}^t R' \gamma R = ({}^t R \gamma R)' = \gamma' = 0$, because $\gamma = {}^t R \gamma R$ follows from $R \in O_\gamma(n)$. Thus we have the equation (8) and

$$\Delta_g \log \varphi = \sum_{i=1}^n \frac{1}{\rho(\langle x \rangle)^{\frac{n}{2}}} \frac{\partial}{\partial x_i} \left(\rho(\langle x \rangle)^{\frac{n}{2}} \left[\frac{1}{2} \left[\frac{\nu'(t)}{\nu(t)} x_i + (\gamma^{-1} S(t) x)_i \right] \right) \right)$$

$$= \frac{n}{4} \frac{\nu'(t)}{\nu(t)} \left(\frac{\langle x \rangle \rho'(\langle x \rangle)}{\rho(\langle x \rangle)} + 2 \right) + (x, S(t)x) + \text{tr}(\gamma^{-1}S(t)).$$

Since S is skew-symmetric, $(x, S(t)x) = 0$ and $\text{tr}(\gamma^{-1}S(t)) = 0$. Therefore we have the equation (9)

$$\Delta_g \log \varphi = \frac{n}{4} \frac{\nu'(t)}{\nu(t)} \left(\frac{\langle x \rangle \rho'(\langle x \rangle)}{\rho(\langle x \rangle)} + 2 \right),$$

and (10)

$$\begin{aligned} g(\nabla_g \log \varphi, \nabla_g \log \varphi) &= \rho(\langle x \rangle) \frac{1}{4} \gamma \left(\frac{\nu'(t)}{\nu(t)} x + \gamma^{-1} S(t)x, \frac{\nu'(t)}{\nu(t)} x + \gamma^{-1} S(t)x \right) \\ &= \frac{\rho(\langle x \rangle)}{4} \left\{ \left(\frac{\nu'(t)}{\nu(t)} \right)^2 \langle x \rangle^2 + (x, {}^t S(t) \gamma^{-1} S(t)x) \right\}. \end{aligned}$$

We have the equations (9) and (10).

Next we proceed to prove the statement (2). Differentiating the equation

$$\frac{\partial \log \varphi}{\partial x_j} = \frac{\rho(\langle x \rangle)}{2} \left(\frac{\nu'(t)}{\nu(t)} y_j + \sum_{k=1}^n s_{jk}(t) x_k \right), \quad j = 1, 2, \dots, n,$$

by x_i ($i \neq j$), we have

$$\frac{\partial}{\partial x_i} \frac{\partial \log \varphi}{\partial x_j} = \frac{\rho'(\langle x \rangle)}{2 \langle x \rangle} \left(\frac{\nu'(t)}{\nu(t)} y_i y_j + \sum_{l=1}^n s_{jl}(t) y_i x_l \right) + \frac{\rho(\langle x \rangle)}{2} \left(\frac{\nu'(t)}{\nu(t)} \gamma_{ji} + s_{ji}(t) \right),$$

where $y = \gamma x$ and $s_{jk}(t)$ is the (j, k) element of the matrix $S(t)$. Also

$$\frac{\partial}{\partial x_j} \frac{\partial \log \varphi}{\partial x_i} = \frac{\rho'(\langle x \rangle)}{2 \langle x \rangle} \left(\frac{\nu'(t)}{\nu(t)} y_j y_i + \sum_{l=1}^n s_{il}(t) y_j x_l \right) + \frac{\rho(\langle x \rangle)}{2} \left(\frac{\nu'(t)}{\nu(t)} \gamma_{ij} + s_{ij}(t) \right)$$

holds. Since $\frac{\partial}{\partial x_i} \frac{\partial \log \varphi}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial \log \varphi}{\partial x_i}$,

$$\frac{\rho'(\langle x \rangle)}{\langle x \rangle} \sum_{l=1}^n s_{jl}(t) y_i x_l + \rho(\langle x \rangle) s_{ji}(t) = \frac{\rho'(\langle x \rangle)}{\langle x \rangle} \sum_{l=1}^n s_{il}(t) y_j x_l + \rho(\langle x \rangle) s_{ij}(t)$$

holds for all $i, j = 1, 2, \dots, n$ with $i \neq j$. Then we have

$$2s_{ij}(t) = \frac{\rho'(\langle x \rangle)}{2 \langle x \rangle \rho(\langle x \rangle)} \left(y_i \sum_{l=1}^n s_{jl}(t) x_l - y_j \sum_{l=1}^n s_{il}(t) x_l \right)$$

for all $i, j = 1, 2, \dots, n$ with $i \neq j$, and hence

$$(17) \quad s_{ij}(t) = \frac{\rho'(\langle x \rangle)}{2 \langle x \rangle \rho(\langle x \rangle)} (y_i z_j - z_i y_j),$$

where we put $z = Sx$.

First we consider the case that $n \geq 3$. Then for each fixed t and each triple indices i, j, k with $1 \leq i < j < l \leq n$, the equation (17) implies

$$s_{ij}(t)y_l + s_{jl}(t)y_i + s_{li}(t)y_j = 0$$

for all (y_i, y_j, y_l) in an open subset of $\mathbb{R}^3$. That implies $(s_{ij}(t), s_{jl}(t), s_{li}(t)) = 0$ for each $1 \leq i < j < l \leq n$ because γ is non-degenerate. Therefore $S(t) = 0$ and hence $R'(t) = 0$ for all t .

Next, in the case that $n = 2$, the equation (17) implies

$$s_{12}(t) = -\frac{r\rho'(r)}{2\rho(r)}s_{12}(t).$$

If $s_{12}(t) \neq 0$ for some t , then $-\frac{r\rho'(r)}{2\rho(r)} = 1$ and hence $\rho(r) = pr^{-2}$ for all r . Thus we have shown that $R'(t) = 0$ for all t unless $n = 2$ and $\rho(r) = pr^{-2}$. Consequently, the equations (11) and (12) immediately follow from (8) and (10) because $S(t) = 0$. $\square$

4. Some special cases.

Before the proof of Theorem 1, we deal with the case that ρ has the form $\rho(r) = pr^q$ in this section. The following Theorem 2 corresponds to the cases 1-a and 1-b of Theorem 1. To state the theorem, we introduce the two dimensional polar coordinate with respect to γ . Let U be an orthogonal matrix such that $\gamma = {}^tU \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} U$ ($\alpha > 0, \beta \neq 0$). If we put $\eta = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{|\beta|} \end{pmatrix} U$, then ${}^t\eta\eta = \begin{pmatrix} \alpha & 0 \\ 0 & |\beta| \end{pmatrix}$ and hence

$$\langle x \rangle^2 = \begin{cases} (\eta x)_1^2 + (\eta x)_2^2, & \det \gamma > 0, \\ (\eta x)_1^2 - (\eta x)_2^2, & \det \gamma < 0. \end{cases}$$

The polar coordinate with respect to γ is defined by

$$r = \langle x \rangle, \text{ and } \theta = \begin{cases} \arctan \frac{(\eta x)_2}{(\eta x)_1}, & \det \gamma > 0, \\ \operatorname{arctanh} \frac{(\eta x)_2}{(\eta x)_1}, & \det \gamma < 0. \end{cases}$$

Note that

$$(18) \quad \nabla_x \theta = \frac{\sqrt{|\det \gamma|}}{\langle x \rangle^2} Jx$$

holds in any case. In fact, if $\det \gamma > 0$,

$$\frac{\partial \theta}{\partial x_1} = \cos^2 \theta \frac{\partial}{\partial x_1} \frac{(\eta x)_2}{(\eta x)_1} = -\frac{(\eta x)_1^2}{(\eta x)_1^2 + (\eta x)_2^2} \frac{\det \eta}{(\eta x)_1^2} x_2 = -\frac{\sqrt{\det \gamma}}{\langle x \rangle^2} x_2,$$

$$\frac{\partial \theta}{\partial x_2} = \cos^2 \theta \frac{\partial}{\partial x_2} \frac{(\eta x)_2}{(\eta x)_1} = \frac{(\eta x)_1^2}{(\eta x)_1^2 + (\eta x)_2^2} \frac{\det \eta}{(\eta x)_1^2} x_1 = \frac{\sqrt{\det \gamma}}{\langle x \rangle^2} x_1,$$

and if $\det \gamma < 0$,

$$\frac{\partial \theta}{\partial x_1} = \cosh^2 \theta \frac{\partial}{\partial x_1} \frac{(\eta x)_2}{(\eta x)_1} = -\frac{(\eta x)_1^2}{(\eta x)_1^2 - (\eta x)_2^2} \frac{\det \eta}{(\eta x)_1^2} x_2 = -\frac{\sqrt{|\det \gamma|}}{\langle x \rangle^2} x_2,$$

$$\frac{\partial \theta}{\partial x_2} = \cosh^2 \theta \frac{\partial}{\partial x_2} \frac{(\eta x)_2}{(\eta x)_1} = \frac{(\eta x)_1^2}{(\eta x)_1^2 - (\eta x)_2^2} \frac{\det \eta}{(\eta x)_1^2} x_1 = \frac{\sqrt{|\det \gamma|}}{\langle x \rangle^2} x_1.$$

Now we state the theorem.

Theorem 2. *Let $n = 2$ and $\rho(r) = \frac{p}{r^2}$ ($p > 0$).*

(1) *Let (f, φ) be a caloric morphism such that f is of form (a). Then*

$$\begin{aligned} f(t, x) &= (t + d, ce^{at} R_0 \exp bt \gamma^{-1} J x), \\ \varphi(t, r, \theta) &= Cr^{\frac{1}{2}ap} \exp \frac{p}{2} \left(\frac{b}{\sqrt{|\det \gamma|}} \theta + \frac{1}{2} (a^2 + \frac{b^2}{\det \gamma}) t \right), \end{aligned}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(2) *Let (f, φ) be a caloric morphism such that f is of form (b). Then*

$$\begin{aligned} f(t, x) &= (t + d, \frac{ce^{at}}{\langle x \rangle^2} R_0 \exp bt \gamma^{-1} J x), \\ \varphi(t, r, \theta) &= Cr^{-\frac{1}{2}ap} \exp \frac{p}{2} \left(\frac{b}{\sqrt{|\det \gamma|}} \theta + \frac{1}{2} (a^2 + \frac{b^2}{\det \gamma}) t \right). \end{aligned}$$

In both cases, $a, b, c, d, C \in \mathbb{R}$ ($c, C > 0$), R_0 is an $O_\gamma(2)$ matrix and (r, θ) is the polar coordinate of $\mathbb{R}^2$ with respect to γ . $a = 0$ if and only if $\nu'(t) = 0$ for some t , where ν is the function appeared in the forms (a) and (b).

Proof. First of all, $f'_0(t) = 1$ follows from the equality (6) in Lemma 1. Therefore $f_0(t) = t + d$ with $d \in \mathbb{R}$.

Next, by Lemma 4 (1), $\log \varphi$ satisfies the equation

$$\nabla_g \log \varphi = \frac{1}{2} \left[\frac{\nu'(t)}{\nu(t)} I + \gamma^{-1} S(t) \right] x.$$

Since $S(t)$ is skew symmetric and $n = 2$, $S(t) = s(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = s(t) J$ where we put $s(t) = s_{21}(t)$ for simplicity. By (18), we have

$$\frac{ps_{21}(t)}{2\sqrt{|\det \gamma|}} \nabla_x \theta = \frac{p}{2} \langle x \rangle^q S(t) x.$$

Therefore, there exists a C^∞ -function $\psi(t)$ such that

$$(19) \quad \log \varphi(t, r, \theta) = \frac{p}{2} \frac{\nu'(t)}{\nu(t)} \log r + \frac{ps_{21}(t)}{2\sqrt{|\det \gamma|}} \theta + \psi(t).$$

On the other hand, φ satisfies the equation (E-1): $H_g \varphi = 0$. Since $\varphi > 0$, (E-1) is equivalent to the following:

$$(20) \quad \frac{\partial \log \varphi}{\partial t} - \Delta_g \log \varphi - g(\nabla_g \log \varphi, \nabla_g \log \varphi) = 0.$$

By (19), we have

$$\frac{\partial \log \varphi}{\partial t} = \frac{p}{2} \left(\frac{\nu'(t)}{\nu(t)} \right)' \log r + \frac{ps'_{21}(t)}{2\sqrt{|\det \gamma|}} \theta + \psi'(t).$$

By Lemma 4,

$$\Delta_g \log \varphi = \frac{n}{4} \frac{\nu'(t)}{\nu(t)} \left(\frac{\langle x \rangle \rho'(\langle x \rangle)}{\rho(\langle x \rangle)} + 2 \right) = \frac{n}{4} \frac{\nu'(t)}{\nu(t)} (-2 + 2) = 0,$$

$$g(\nabla_g \log \varphi, \nabla_g \log \varphi) = \frac{p \langle x \rangle^{-2}}{4} \left\{ \left(\frac{\nu'(t)}{\nu(t)} \right)^2 \langle x \rangle^2 + s(t)^2 (x, {}^t J \gamma^{-1} J x) \right\}.$$

Since

$${}^t J \gamma^{-1} J = \frac{1}{\det \gamma} \gamma$$

holds, we have

$$\begin{aligned} g(\nabla_g \log \varphi, \nabla_g \log \varphi) &= \frac{p \langle x \rangle^{-2}}{4} \left\{ \left(\frac{\nu'(t)}{\nu(t)} \right)^2 \langle x \rangle^2 + \frac{s(t)^2}{\det \gamma} (x, \gamma x) \right\} \\ &= \frac{p}{4} \left\{ \left(\frac{\nu'(t)}{\nu(t)} \right)^2 + \frac{s(t)^2}{\det \gamma} \right\}. \end{aligned}$$

Substitute these calculations into (20). Then we have the equation

$$(21) \quad \frac{p}{2} \left(\frac{\nu'(t)}{\nu(t)} \right)' \log r + \frac{ps'(t)}{2\sqrt{|\det \gamma|}} \theta + \psi'(t) - \frac{p}{4} \left\{ \left(\frac{\nu'(t)}{\nu(t)} \right)^2 + \frac{s(t)^2}{\det \gamma} \right\} = 0.$$

Therefore we obtain a system of differential equations

$$\begin{cases} \left(\frac{\nu'(t)}{\nu(t)} \right)' = 0, \\ s'(t) = 0, \\ \psi'(t) = \frac{p}{4} \left[\left(\frac{\nu'(t)}{\nu(t)} \right)^2 - \frac{s(t)^2}{\det \gamma} \right] \end{cases}$$

by (21). The solution of this system is

$$(22) \quad \begin{cases} \nu(t) = ce^{at}, \\ s(t) = b, \\ \psi(t) = \frac{p}{4}(a^2 - \frac{b^2}{\det \gamma})t + C_0, \end{cases}$$

where $a, b, c, C_0 \in \mathbb{R}$, $a \neq 0$ and $c > 0$. Clearly, $a = 0$ if and only if $\nu'(t) = 0$ for all t . Substituting (22) into (19), we have

$$\log \varphi(t, r, \theta) = \frac{1}{2}ap \log r + \frac{p}{2\sqrt{|\det \gamma|}}b\theta + \frac{p}{4}(a^2 + \frac{b^2}{\det \gamma})t + C,$$

and

$$(23) \quad S(t) = bJ.$$

Therefore

$$\varphi(t, r, \theta) = Cr^{\frac{1}{2}ap} \exp \left(\frac{p}{2\sqrt{|\det \gamma|}}b\theta + \frac{p}{4}(a^2 + \frac{b^2}{\det \gamma})t \right).$$

Now choose a number $t_0 \in \mathbb{R}$ so that $\{t = t_0\} \cap D \neq \emptyset$. Since $S(t) = \gamma R(t)^{-1}R'(t)$, $R(t)$ satisfies the differential equation

$$\gamma R(t)^{-1}R'(t) = bJ$$

by (23). The solution of this equation is

$$\begin{aligned} R(t) &= R(t_0) \exp \gamma^{-1}J(t - t_0) \\ &= R_0 \exp bt\gamma^{-1}J, \end{aligned}$$

where $R_0 = R(t_0) \exp b\gamma^{-1}Jt_0$.

Thus

$$\begin{aligned} f(t, x) &= (t + d, ce^{at}R_0 \exp bt\gamma^{-1}Jx), \\ \varphi(t, r, \theta) &= Cr^{\frac{1}{2}ap} \exp \frac{p}{2} \left(\frac{b}{\sqrt{|\det \gamma|}}\theta + \frac{1}{2}(a^2 + \frac{b^2}{\det \gamma})t \right) \end{aligned}$$

for all $(t, x) \in D$. We have shown the first statement (1). The proof of the second statement is reduced to the first statement by the composition with an inversion. In fact, $\rho(r) = pr^{-2}$ satisfies $\rho(\frac{1}{r}) = r^4\rho(r)$. Then the inversion $(j, 1)$, where

$$j(t, x) = (t, \frac{x}{\langle x \rangle^2}),$$

is a caloric morphism by Lemma 3. The composition $(f \circ j, \varphi \circ j)$ of (f, φ) and $(j, 1)$ is

$$(f_0(t), \nu(t)R(t)x), \quad \varphi(t, \frac{x}{\langle x \rangle^2}),$$

which is reduced to the case that $f \circ j$ is of form (a).

Thus we have shown Theorem 2. $\square$

The next theorem corresponds to the cases 2-a and 2-b of Theorem 1.

Theorem 3. Let $n \geq 3$ and ρ be of form $\rho(r) = \frac{p}{r^2}$ ($p > 0$).

(1) Let (f, φ) be a caloric morphism such that f is of form (a). Then

$$f(t, x) = (t + d, ce^{at} R_0 x), \quad \varphi(t, x) = C \langle x \rangle^{\frac{1}{2}ap} \exp\left(\frac{1}{4}pa^2 t\right).$$

(2) Let (f, φ) be a caloric morphism such that f is of form (b). Then

$$f(t, x) = (t + d, \frac{ce^{at}}{\langle x \rangle^2} R_0 x), \quad \varphi(t, x) = C \langle x \rangle^{-\frac{1}{2}ap} \exp\left(\frac{1}{4}pa^2 t\right).$$

In both cases, $a, c, d, C \in \mathbb{R}$ ($c, C > 0$), $R_0 \in O_\gamma(n)$. $a = 0$ if and only if $\nu'(t) = 0$ for some t , where ν is the function appeared in the forms (a) and (b).

Proof. First of all, $f'_0(t) = 1$ follows from the equality (6) in Lemma 1. Therefore $f_0(t) = t + d$ with $d \in \mathbb{R}$.

By Lemma 4 (2), R is constant and $\log \varphi$ satisfies the equation

$$\frac{\partial \log \varphi}{\partial x_j} = \frac{p}{2 \langle x \rangle^2} \frac{\nu'(t)}{\nu(t)} (\gamma x)_j, \quad j = 1, \dots, n,$$

because $n \geq 3$. It follows that φ is a function of $\langle x \rangle$, i.e.

$$\varphi(t, x) = \varphi(t, \langle x \rangle),$$

and that

$$\frac{\partial \log \varphi}{\partial r} = \frac{p \nu'(t)}{2 \nu(t)} \frac{1}{r},$$

and hence

$$(24) \quad \log \varphi(t, r) = \frac{p \nu'(t)}{2 \nu(t)} \log r + \psi(t).$$

By (E-1) and (20),

$$\frac{\partial \log \varphi}{\partial t} - \Delta_g \log \varphi - g(\nabla_g \log \varphi, \nabla_g \log \varphi) = 0.$$

From Lemma 4 and (24), it follows that

$$\frac{\partial \log \varphi}{\partial t} = \frac{p}{2} \left(\frac{\nu'(t)}{\nu(t)} \right)' \log r + \psi'(t),$$

$$\Delta_g \log \varphi = \frac{n(q+2)}{2} \frac{\nu'(t)}{\nu(t)} = 0,$$

$$g(\nabla_g \log \varphi, \nabla_g \log \varphi) = \frac{p}{4} \langle x \rangle^{q+2} \left(\frac{\nu'(t)}{\nu(t)} \right)^2 = \frac{p}{4} \left(\frac{\nu'(t)}{\nu(t)} \right)^2.$$

Hence, we have the equation

$$\frac{p}{2} \left(\frac{\nu'(t)}{\nu(t)} \right)' \log r + \psi'(t) - \frac{p}{4} \left(\frac{\nu'(t)}{\nu(t)} \right)^2 = 0.$$

Therefore we obtain a system of differential equations

$$\begin{cases} \left(\frac{\nu'}{\nu} \right)' = 0, \\ \psi' = \frac{p}{4} \left(\frac{\nu'}{\nu} \right)^2. \end{cases}$$

The solution is

$$(25) \quad \begin{cases} \nu(t) = ce^{at}, \\ \psi(t) = \frac{pa^2}{4}t + C_0, \end{cases}$$

where $a, c, C_0 \in \mathbb{R}$, $a \neq 0$ and $c > 0$. Clearly, $a = 0$ if and only if $\nu'(t) = 0$ for some t . Substituting (25) into (24), we have

$$\log \varphi(t, r) = \frac{ap}{2} \log r + \frac{pa^2}{4}t + C_0.$$

Thus we have

$$f(t, x) = (t + d, ce^{at}Rx), \quad \varphi(t, x) = C\langle x \rangle^{\frac{1}{2}ap} \exp\left(\frac{1}{4}pa^2t\right)$$

for all $(t, x) \in D$. We have shown the first statement (1). As in Theorem 2, the second statement is reduced to the first statement by composing the inversion $(j, 1)$. $\square$

The next theorem corresponds to the case 3 of Theorem 1. We note that the case of $d/a = -1$, $b = c = 0$, $p = 1$ and $q = 0$ is the Appell transformation.

Theorem 4. *Let ρ be of form $\rho(r) = pr^q$ ($p > 0$, $q \in \mathbb{R}$, $q \neq -2$) and let (f, φ) be a caloric morphism such that f is of form (a). Then*

$$\begin{aligned} f(t, x) &= \left(\frac{ct + d}{at + b}, \frac{R_0x}{|at + b|^{2/(q+2)}} \right), \\ \varphi(t, x) &= \frac{C}{|at + b|^{n/2}} \exp \left[- \frac{pa\langle x \rangle^{q+2}}{(q+2)^2(at + b)} \right], \end{aligned}$$

where $a, b, c, d, C \in \mathbb{R}$ ($bc - ad = 1$, $C > 0$) and $R_0 \in O_\gamma(n)$. $a = 0$ if and only if $\nu'(t) = 0$ for some t , where ν is the function appeared in the form (a).

Proof. Let $q \neq -2$. Then by Lemma 4 (2), R is constant and

$$\frac{\partial \log \varphi}{\partial x_j} = \frac{p\langle x \rangle^q}{2} \frac{\nu'(t)}{\nu(t)} (\gamma x)_j, \quad j = 1, \dots, n$$

hold. As in the proof of Theorem 3, φ is a function of $\langle x \rangle$, $\varphi(t, x) = \varphi(t, \langle x \rangle)$ and hence there exists a C^∞ -function $\psi(t)$ such that

$$(26) \quad \log \varphi(t, r) = \frac{p}{2(q+2)} \frac{\nu'(t)}{\nu(t)} r^{q+2} + \psi(t), \quad r = \langle x \rangle.$$

By (E-1) and (20), we obtain

$$\begin{aligned} \frac{\partial \log \varphi}{\partial t} - \Delta_g \log \varphi - g(\nabla_g \log \varphi, \nabla_g \log \varphi) &= 0, \\ \frac{\partial \log \varphi}{\partial t} &= \frac{p}{2(q+2)} \left(\frac{\nu'(t)}{\nu(t)} \right)' r^{q+2} + \psi'(t), \\ \Delta_g \log \varphi &= \frac{n(q+2)}{4} \frac{\nu'(t)}{\nu(t)}, \\ g(\nabla_g \log \varphi, \nabla_g \log \varphi) &= \frac{p}{4} r^{q+2} \left(\frac{\nu'(t)}{\nu(t)} \right)^2. \end{aligned}$$

Hence, we have the equation

$$\frac{p}{2(q+2)} \left[\left(\frac{\nu'(t)}{\nu(t)} \right)' - \frac{q+2}{2} \left(\frac{\nu'(t)}{\nu(t)} \right)^2 \right] r^{q+2} + \psi' - \frac{n(q+2)}{4} \left(\frac{\nu'(t)}{\nu(t)} \right)' = 0.$$

Therefore we obtain a system of differential equations

$$\begin{cases} \left(\frac{\nu'(t)}{\nu(t)} \right)' - \frac{q+2}{2} \left(\frac{\nu'(t)}{\nu(t)} \right)^2 = 0, \\ \psi' - \frac{n(q+2)}{4} \left(\frac{\nu'(t)}{\nu(t)} \right)' = 0. \end{cases}$$

The solution is

$$(27) \quad \begin{cases} \nu(t) = \frac{1}{|at + b|^{2/(q+2)}}, \\ \psi(t) = \log C \nu^{n(q+2)/4} = \log C |at + b|^{-n/2}, \end{cases}$$

where $a, b \in \mathbb{R}$ and $C > 0$. Clearly, $a = 0$ if and only if $\nu'(t) = 0$ for some t . Substituting (27) into (26), we have

$$\log \varphi(t, r) = -\frac{pa}{(q+2)^2(at+b)} r^{q+2} + \log C |at + b|^{-n/2}.$$

On the other hand, (6) in Lemma 1 implies

$$f'_0(t) = \nu^{q+2}(t) = \frac{1}{|at + b|^2}.$$

Therefore $f_0(t) = \frac{ct + d}{at + b}$ where $a, b, c, d \in \mathbb{R}$ and $bc - ad = 1$. Consequently,

$$\begin{aligned} f(t, x) &= \left(\frac{ct + d}{at + b}, \frac{R_0 x}{|at + b|^{2/(q+2)}} \right), \\ \varphi(t, x) &= \frac{C}{|at + b|^{n/2}} \exp \left[-\frac{pa \langle x \rangle^{q+2}}{(q+2)^2(at+b)} \right] \end{aligned}$$

for all $(t, x) \in D$. Thus we have shown Theorem 4. $\square$

5. Proof of Theorem 1.

Now we can prove our main theorem.

Proof of Theorem 1. First we deal with the case that the function $\nu(t)$ is not constant. Let I_0 be the open interval consisting of all t such that $(t, x) \in D$ for some $x \in \mathbb{R}^n$. Let I_1 be a connected component of the open set $\{t \in I_0; \nu'(t) \neq 0\}$ and let I_2 be the open interval consisting of all $\langle x \rangle$ such that $(t, x) \in D$ for some $t \in I_1$. Then $\rho(r) = pr^q$ on I_2 by Lemma 2. Case (a) has already been treated in the theorems, Theorem 2, 3 and 4. In Case (b), we have $q = -2$ by Lemma 1 and Lemma 2 (2). This case has already been treated in Theorem 2 and Theorem 3. Since we assume that ν is not constant, if $q = -2$,

$$\nu'(t) = cae^{at} \quad (a \neq 0)$$

on I_1 from Theorem 2 and Theorem 3, and if $q \neq -2$,

$$\nu'(t) = \frac{-2a}{(q+2)|at+b|^{2/(q+2)}} \quad (a \neq 0)$$

on I_1 from Theorem 4. In both cases, $\nu'(t) \neq 0$ on the closure of I_1 in I_0 . Hence, $I_1 = I_0$ because I_0 is connected. Therefore $(t, \langle x \rangle) \in I_1 \times I_2$ for all $(t, x) \in D$ and $\rho(r) = pr^q$ for all r . Thus we have shown that (f, φ) and ρ are completely determined as the cases 1-a, 1-b, 2-a, 2-b, 3 in Theorem 1, if the function $\nu(t)$ is not constant.

Next we deal with the case that ν is constant. Because of the preceding argument, we may exclude the case that $\rho(r)$ has the form $\rho(r) = pr^q$. We first consider the case (a) as before. By Lemma 4, $R'(t) = 0$ and

$$\frac{\partial \log \varphi}{\partial x_j} = 0$$

because $\nu'(t) = 0$. Therefore R is a constant matrix R_0 and φ depends only on t . Since φ satisfies (E-1): $H_g \varphi = \frac{\partial \varphi}{\partial t} = 0$, φ is a positive constant C . On the other hand, (6) in Lemma 1 implies $\rho(\nu r) = f'_0(t)\rho(r)$. Therefore $f'_0(t) = \rho(\nu r)/\rho(r)$ and hence $f'_0 = \lambda$ with some positive constant λ . Thus we have

$$f(t, x) = (\lambda t + d, \nu R_0 x), \quad \varphi(t, x) = C.$$

This is Case 4-a.

Finally, we consider the case (b). Since ν is constant, f'_0 is equal to a constant λ such that

$$\rho\left(\frac{\nu}{r}\right) = \frac{\lambda r^4}{\nu^2} \rho(r)$$

by (7) in Lemma 1. Then, the inversion $(h, 1)$, where

$$h(t, x) = \left(\lambda t, \frac{\nu x}{\langle x \rangle^2}\right),$$

is a caloric morphism by Lemma 3. The composition of $(h, 1)$ and (f, φ) is

$$(\lambda f_0(t), Rx), \quad \varphi(t, x),$$

which is reduced to the case that $f \circ h$ is of form (a). Thus we have completed the proof of Theorem 1. $\square$

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