Note on the number of semistar operations, X

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To the memory of Professor Tai-ichi Kitamura the founder of the Journal

ABSTRACT. We determine the number of semistar operations on some 1-dimensional Prüfer domains.

Let D be an integral domain, let K be its quotient field, let $F(D)$ be the set of nonzero fractional ideals of D , and let $F'(D)$ be the set of nonzero D -submodules of K . A mapping $I \mapsto I^*$ from $F'(D)$ to $F'(D)$ is called a semistar operation on D , if it satisfies the following conditions: (1) $(xI)^* = xI^*$ for each $x \in K - \{0\}$ and $I \in F'(D)$; (2) $I \subset I^*$ for each $I \in F'(D)$; (3) $(I^*)^* = I^*$ for each $I \in F'(D)$; (4) $I \subset J$ implies $I^* \subset J^*$ for each $I, J \in F'(D)$. Let $\Sigma'(D)$ (resp. $\Sigma(D)$) be the set of semistar operations (resp. star operations) on D .

We want to know $|\Sigma'(D)|$ for Prüfer domains D . In this paper we will prove the following,

Theorem 1. *Let D be a 1-dimensional Prüfer domain with exactly two maximal ideals M and N , and let Γ (resp. Γ') be the value group of D_M (resp. D_N). We may assume that Γ (resp. Γ') is a subgroup of the additive group $\vec{\mathcal{R}}$ of the real numbers.*

- (1) *If both M and N are principal, then $|\Sigma(D)| = 1$ and $|\Sigma'(D)| = 7$.*
- (2) *If M is principal, and Γ' is not complete, then $|\Sigma(D)| = 2$ and $|\Sigma'(D)| = 14$.*
- (3) *If Γ is not complete, and N is principal, then $|\Sigma(D)| = 2$ and $|\Sigma'(D)| = 14$.*
- (4) *If neither Γ nor Γ' is complete, then $|\Sigma(D)| = 9$ and $|\Sigma'(D)| = 54$.*
- (5) *If M is principal, and $\Gamma' = \vec{\mathcal{R}}$, then $|\Sigma(D)| = 2$ and $|\Sigma'(D)| = 14$.*
- (6) *If $\Gamma = \vec{\mathcal{R}}$, and N is principal, then $|\Sigma(D)| = 2$ and $|\Sigma'(D)| = 14$.*
- (7) *If $\Gamma = \Gamma' = \vec{\mathcal{R}}$, then $|\Sigma(D)| = 4$ and $|\Sigma'(D)| = 30$.*
- (8) *If $\Gamma = \vec{\mathcal{R}}$, and Γ' is not complete, then $|\Sigma(D)| = 6$ and $|\Sigma'(D)| = 40$.*
- (9) *If Γ is not complete, and $\Gamma' = \vec{\mathcal{R}}$, then $|\Sigma(D)| = 6$ and $|\Sigma'(D)| = 40$.*

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Theorem 2. *Let D be a 1-dimensional Prüfer domain with exactly three maximal ideals. Then $|\Sigma'(D)| < \infty$.*

In Theorem 1, (1) $\sim$ (4) were proved in [M2]. In §2 we will prove (5). In §3 we will prove (7). In §4 we will prove (8). The proof for (6) is similar to that of (5). The proof for (9) is similar to (8). §1 is the general case of Theorem 1. In §5, we will prove Theorem 2.

[M1] is a self-contained, well got together and systematically written paper on the number of semistar operations on integral domains.

§1 The general case of Theorem 1.

Throughout §1 $\sim$ §4, let D be a 1-dimensional Prüfer domain with exactly two maximal ideals M and N , let $V = D_M$ (resp. $W = D_N$), let v (resp. w) be a valuation belonging to V (resp. W), and let Γ (resp. Γ') be the value group of v (resp. w). We assume that Γ (resp. Γ') is a subgroup of $\vec{\mathcal{R}}$. Let K be the quotient field of D . Let A (resp. B) be the D -submodule of K generated by the subset $\{\frac{1}{p} \mid p \in M - N\}$ (resp. $\{\frac{1}{q} \mid q \in N - M\}$) of K . By p we denote an element of $M - N$, and by q we denote an element of $N - M$.

(1.1) (1) Each element $x \in D - \{0\}$ can be expressed as 1 or p or q or pq uniquely up to associates, where $p \in M - N$ and $q \in N - M$.

(2) Each element $x \in K - \{0\}$ can be expressed as 1 or p or q or pq or $\frac{1}{p}$ or $\frac{1}{q}$ or $\frac{q}{p}$ or $\frac{p}{q}$ or $\frac{1}{pq}$ uniquely up to associates.

A fractional ideal $A_0 \in F(D)$ is called of type α , if there exists a subset $\{p_\lambda \mid \lambda \in \Lambda\}$ of $M - N$ so that A_0 is generated by the set $\{\frac{1}{p_\lambda} \mid \lambda\}$, $v(A_0)$ is bounded below, and there does not exist $\inf v(A_0)$. A fractional ideal $B_0 \in F(D)$ is called of type β , if there exists a subset $\{q_\sigma \mid \sigma \in \Sigma\}$ of $N - M$ so that B_0 is generated by the set $\{\frac{1}{q_\sigma} \mid \sigma\}$, $w(B_0)$ is bounded below, and there does not exist $\inf w(B_0)$.

(1.2) Let I be a nonzero fractional ideal of D . Then there may arise the following 9-cases:

- (1) There exists $\min v(I)$, and there exists $\min w(I)$.
- (2) There exists $\min v(I)$, and there exists $\inf w(I)$ with $\inf w(I) \notin w(I)$.
- (3) There exists $\min v(I)$, and there does not exist $\inf w(I)$.
- (4) There exists $\inf v(I)$ with $\inf v(I) \notin v(I)$, and there exists $\min w(I)$.
- (5) There exists $\inf v(I)$ with $\inf v(I) \notin v(I)$, and there exists $\inf w(I)$ with $\inf w(I) \notin w(I)$.
- (6) There exists $\inf v(I)$ with $\inf v(I) \notin v(I)$, and there does not exist $\inf w(I)$.

(7) There does not exist $\inf v(I)$, and there exists $\min w(I)$.

(8) There does not exist $\inf v(I)$, and there exists $\inf w(I)$ with $\inf w(I) \notin w(I)$.

(9) There does not exist $\inf v(I)$, and there does not exist $\inf w(I)$.

Case (1): There exist elements $a, b \in I$ such that $v(a) = \min v(I)$ and $w(b) = \min w(I)$. Let $(a, b) = (c)$ for an element $c \in I$. Then we have $I = (c)$.

Case (2): There exist p, q such that $v(p^{e_1}) = \min v(I)$, $w(q^{e_2}) = \inf w(I)$, where $e_1, e_2 \in \{-1, 0, 1\}$. Set $x = \frac{1}{p^{e_1}q^{e_2}}$. Then $xI = N$. We have $N^v = D$.

Case (3): Assume that $\min v(I) = 0$ and $w(I) \ni 0$. Let $\{q \in N - M \mid \frac{1}{q} \in I\} = \{q_\lambda \mid \lambda\}$. Then the fractional ideal $B_0 = (\frac{1}{q_\lambda} \mid \lambda)$ of D is of type β , and $I = B_0$. Let $\{q \mid w(\frac{1}{q}) \notin w(I)\} = \{q_\sigma \mid \sigma\}$. Then we have $I = \cap_\sigma (\frac{1}{q_\sigma})$.

Case (5): Assume that $0 = \inf v(I) = \inf w(I)$. Then we have $I = MN$ and $I^v = D$.

Let J be an ideal of D such that $I \subset J \subset D$. Then J is either I or M or N or D .

Case (6): Assume that $0 = \inf v(I)$ and $0 \in w(I)$. Let $\{q \in N - M \mid \frac{1}{q} \in I\} = \{q_\lambda \mid \lambda\}$. Then we have $I = M(\frac{1}{q_\lambda} \mid \lambda)$, and the fractional ideal $B_0 = (\frac{1}{q_\lambda} \mid \lambda)$ of D is of type β . Let $\{q \mid w(\frac{1}{q}) \notin w(I)\} = \{q_\sigma \mid \sigma\}$. Then we have $I^v = \cap_\sigma (\frac{1}{q_\sigma}) = B_0$.

Let J be a fractional ideal of D such that $I \subset J \subset B_0$. Then J is either I or B_0 .

Case (9): Assume that $0 \in v(I)$, $0 \in w(I)$. Let $\{p \in M - N \mid \frac{1}{p} \in I\} = \{p_\lambda \mid \lambda\}$, and let $\{q \in N - M \mid \frac{1}{q} \in I\} = \{q_\sigma \mid \sigma\}$. Then the fractional ideal $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$ of D is of type α , the fractional ideal $B_0 = (\frac{1}{q_\sigma} \mid \sigma)$ of D is of type β , and $I = A_0B_0$. Let $\{p \mid v(\frac{1}{p}) \notin v(I)\} = \{p_\alpha \mid \alpha\}$ and $\{q \mid w(\frac{1}{q}) \notin w(I)\} = \{q_\beta \mid \beta\}$. Then we have $I = \cap_{\alpha, \beta} (\frac{1}{p_\alpha q_\beta})$.

Let A_0 (resp. B_0) be a fractional ideal of D with type α (resp. type β). Then we have $A_0 = A_0^v$, $B_0 = B_0^v$ and $A_0B_0 = (A_0B_0)^v$.

(1.3) Let A_0, A'_0 (resp. B_0, B'_0) be fractional ideals of D with type α (resp. type β), and let $x \in K - \{0\}$.

If either $xM = M$ or $xN = N$ or $xMN = MN$ or $xA_0N = A_0N$ or $xB_0M = B_0M$, then $(x) = D$.

If $xA_0N = A'_0N$, then $xA_0 = A'_0$.

If $xB_0M = B'_0M$, then $xB_0 = B'_0$.

(1.4) Let $*$ be a star operation on D . Then we have that M^* is either M or D , N^* is either N or D , and $(MN)^*$ is either MN or M or N or D .

Either $(A_0N)^* = A_0N$ for each fractional ideal A_0 of D with type α , or $(A_0N)^* = A_0$ for each fractional ideal A_0 of D with type α .

Either $(B_0M)^* = B_0M$ for each fractional ideal B_0 of D with type β , or $(B_0M)^* = B_0$ for each fractional ideal B_0 of D with type β .

(1.5) We have $|\Sigma(D)| \leq 64$.

(1.6) Let $I \in F'(D) - F(D)$. There may arise the following 7-cases:

- (1) There exists $\min v(I)$, and $w(I) = \Gamma' \cup \{\infty\}$.
- (2) $v(I)$ is bounded below, there exists $\inf v(I)$ with $\inf v(I) \notin v(I)$, and $w(I) = \Gamma' \cup \{\infty\}$.
- (3) $v(I)$ is bounded below, there does not exist $\inf v(I)$, and $w(I) = \Gamma' \cup \{\infty\}$.
- (4) $v(I) = \Gamma \cup \{\infty\}$, and there exists $\min w(I)$.
- (5) $v(I) = \Gamma \cup \{\infty\}$, and there exists $\inf w(I)$ with $\inf w(I) \notin w(I)$.
- (6) $v(I) = \Gamma \cup \{\infty\}$, $w(I)$ is bounded below, and there does not exist $\inf w(I)$.
- (7) $v(I) = \Gamma \cup \{\infty\}$, and $w(I) = \Gamma' \cup \{\infty\}$.

Case (1): Assume that $\min v(I) = 0$. Then we have $I = B$.

Case (2): Assume that $0 = \inf v(I)$. Then we have $I = MB$, and $I^v = K$.

Case (3): Assume that $0 \in v(I)$. Let $\{p \in M - N \mid \frac{1}{p} \in I\} = \{p_\lambda \mid \lambda\}$, and let $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$. Then A_0 is a fractional ideal of D with type α , and $I = A_0B$.

Case (7): $I = K$.

Let $I^{d'} = I$ (resp. $I^{v'} = (I^{-1})^{-1}, I^e = K$) for each $I \in F'(D)$. Then d' (resp. v' , e) is a semistar operation on D , and is called the d' -operation (resp. v' -operation, e -operation) on D .

Let $I \subset J$ be an inclusion relation of elements of $F(D)$. A mapping $*$ from $F(D)$ to $F(D)$ is called monotone for $I \subset J$, if $I^* \subset J^*$. Let $I \subset J$ be an inclusion relation of elements of $F'(D)$. A mapping $*$ from $F'(D)$ to $F'(D)$ is called monotone for $I \subset J$, if $I^* \subset J^*$.

(1.7) Let $*$ be a semistar operation on D .

- (1) A^* is either A or K . B^* is either B or K .
- (2) $(AN)^*$ is either AN or A or K . $(BM)^*$ is either BM or B or K .
- (3) If $A^* = A$, then $(AB_0)^* = AB_0$ for each fractional ideal B_0 of D with type β . If $B^* = B$, then $(A_0B)^* = A_0B$ for each fractional ideal A_0 of D with type α .

(1.8) We have $|\Sigma'(D)| < \infty$.

§2 The case where M is principal and $\Gamma' = \vec{\mathcal{R}}$.

Then N is not principal.

(2.1) We have $\{I \in F(D) \mid I \subsetneq I^v\} = \{xN \mid x \in K - \{0\}\}$.

If $xN \subset N$ for an element $x \in K - \{0\}$, then $x \in D$.

(2.2) (1) Set $N^{*1} = N$. Then there is determined a star operation $*_1$ on D uniquely, and then $*_1 = d$.

(2) Set $N^{*2} = D$. Then there is determined a star operation $*_2$ on D uniquely, and then $*_2 = v$.

For the proof, we confer (1.3).

(2.3) We have $|\Sigma(D)| = 2$ and $\Sigma(D) = \{d, v\}$.

(2.4) (1) We have $F'(D) - F(D) = \{xA, xB, xAN, K \mid x \in K - \{0\}\}$.

(2) The sets $\{xA \mid x \in K - \{0\}\}$, $\{xB \mid x \in K - \{0\}\}$, $\{xAN \mid x \in K - \{0\}\}$ and $\{K\}$ are disjoint with each other.

(2.5) Let $*$ be a semistar operation on D . Then we have that A^* is either A or K , B^* is either B or K , and $(AN)^*$ is either AN or A or K .

(2.6) Let $x \in K - \{0\}$.

If $xN \subset A$, then $(x) \subset A$.

If $xN \subset B$, then $(x) \subset B$.

If $xAN \subset A$, then $xA \subset A$.

(2.7) Let $*_i$ be a star operation on D .

(1) Set $I_0^{*1} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*1} = A$, $B^{*1} = B$ and $(AN)^{*1} = AN$. Then there is determined a unique mapping $*_i^1$ from $F'(D)$ to $F'(D)$.

(2) Set $I_0^{*2} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*2} = A$, $B^{*2} = B$ and $(AN)^{*2} = A$. Then there is determined a unique mapping $*_i^2$ from $F'(D)$ to $F'(D)$.

(3) Set $I_0^{*3} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*3} = A$, $B^{*3} = K$ and $(AN)^{*3} = AN$. Then there is determined a unique mapping $*_i^3$ from $F'(D)$ to $F'(D)$.

(4) Set $I_0^{*4} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*4} = A$, $B^{*4} = K$ and $(AN)^{*4} = A$. Then there is determined a unique mapping $*_i^4$ from $F'(D)$ to $F'(D)$.

(5) Set $I_0^{*5} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*5} = K$, $B^{*5} = B$, and $(AN)^{*5} = K$. Then there is determined a unique mapping $*_i^5$ from $F'(D)$ to $F'(D)$.

(6) Set $I_0^{*^6} = I_0^{*^i}$ for each $I_0 \in F(D)$, and set $A^{*^6} = K$, $B^{*^6} = K$, and $(AN)^{*^6} = K$. Then there is determined a unique mapping $*_i^6$ from $F'(D)$ to $F'(D)$.

We set $K^{*^j} = K$ for each i, j , and set $(xI)^{*^j} = xI^{*^j}$ for each $x \in K - \{0\}$ and $I \in F'(D)$.

(2.8) Each $*_i^j$ satisfies the following conditions: $I \subset I^{*^j}$ for each $I \in F'(D)$, and $(I^{*^j})^{*^j} = I^{*^j}$ for each $I \in F'(D)$.

(2.9) $*_1^1$ is a semistar operation on D , and $*_1^1 = d'$.

$*_1^2$ is a semistar operation on D .

$*_2^2$ is a semistar operation on D .

$*_1^3$ is a semistar operation on D .

$*_1^4$ is a semistar operation on D .

$*_2^4$ is a semistar operation on D .

$*_1^5$ is a semistar operation on D .

$*_2^5$ is a semistar operation on D .

$*_1^6$ is a semistar operation on D .

$*_2^6$ is a semistar operation on D , and $*_2^6 = v'$.

(2.10) $*_2^1$ is not monotone for $N \subset AN$. $*_2^3$ is not monotone for $N \subset AN$.

(2.11) $|\Sigma'(D)| = 14$.

§3 The case where $\Gamma = \Gamma' = \vec{\mathcal{R}}$.

Then neither M nor N is principal.

(3.1) (1) We have $\{I \in F(D) \mid I \subsetneq I^v\} = \{xM, xN, xMN \mid x \in K - \{0\}\}$.

(2) The sets $\{xM \mid x \in K - \{0\}\}$, $\{xN \mid x \in K - \{0\}\}$ and $\{xMN \mid x \in K - \{0\}\}$ are disjoint with each other.

(3.2) Let $*$ be a star operation on D . Then we have that M^* is either M or D , N^* is either N or D , and $(MN)^*$ is either MN or M or N or D .

(3.3) (1) Set $M^{*1} = M$ and $N^{*1} = N$. Then there is determined a unique mapping $*_1$ from $F(D)$ to $F(D)$.

(2) Set $M^{*2} = M$ and $N^{*2} = D$. Then there is determined a unique mapping $*_2$ from $F(D)$ to $F(D)$.

(3) Set $M^{*3} = D$ and $N^{*3} = N$. Then there is determined a unique mapping $*_3$ from $F(D)$ to $F(D)$.

(4) Set $M^{*4} = D$ and $N^{*4} = D$. Then there is determined a unique mapping $*_4$ from $F(D)$ to $F(D)$.

For the proof, we confer (1.3). Set $(MN)^{*i} = (M^{*i}N^{*i})^{*i}$ for each i ; If I is a nonzero fractional ideal of D with $I = I^v$, set $I^{*i} = I$; And set $(xI)^{*i} = xI^{*i}$ for each $x \in K - \{0\}$ and each $I \in F(D)$.

(3.4) Let $x \in K - \{0\}$.

If $xM \subset D$, then $(x) \subset D$.

If $xM \subset N$, then $(x) \subset N$.

If $xN \subset D$, then $(x) \subset D$.

If $xN \subset M$, then $(x) \subset M$.

If $xMN \subset D$, then $(x) \subset D$.

If $xMN \subset M$, then $xM \subset M$.

If $xMN \subset M$, then $xN \subset D$.

If $xMN \subset N$, then $xM \subset D$.

If $xMN \subset N$, then $xN \subset N$.

(3.5) Each $*_i$ is a star operation on D . We have $*_1 = d$ and $*_4 = v$.

(3.6) $|\Sigma(D)| = 4$.

(3.7) (1) We have $F'(D) - F(D) = \{xA, xB, xAN, xBM, K \mid x \in K - \{0\}\}$.

(2) The sets $\{xA \mid x \in K - \{0\}\}, \{xB \mid x \in K - \{0\}\}, \{xAN \mid x \in K - \{0\}\}, \{xBM \mid x \in K - \{0\}\}, \{K\}$ are disjoint with each other.

(3.8) Let $*$ be a semistar operation on D . Then we have that A^* is either A or K , B^* is either B or K , $(AN)^*$ is either AN or A or K , and $(BM)^*$ is either BM or B or K .

(3.9) Let $x \in K - \{0\}$.

If $xAN \subset A$, then $xA \subset A$.

If $xBM \subset B$, then $xB \subset B$.

(3.10) Let $*_i$ be a star operation on D .

(1) Set $I_0^{*1} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*1} = A, B^{*1} = B, (AN)^{*1} = AN$ and $(BM)^{*1} = BM$. Then there is determined a unique mapping $*_i^1$ from $F'(D)$ to $F'(D)$.

(2) Set $I_0^{*2} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*2} = A, B^{*2} = B, (AN)^{*2} = AN$ and $(BM)^{*2} = B$. Then there is determined a unique mapping $*_i^2$ from $F'(D)$ to $F'(D)$.

(3) Set $I_0^{*3} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*3} = A, B^{*3} = B, (AN)^{*3} = A$ and $(BM)^{*3} = BM$. Then there is determined a unique mapping $*_i^3$ from $F'(D)$ to $F'(D)$.

(4) Set $I_0^{*4} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*4} = A, B^{*4} = B, (AN)^{*4} = A$ and $(BM)^{*4} = B$. Then there is determined a unique mapping $*_i^4$ from $F'(D)$ to $F'(D)$.

(5) Set $I_0^{*5} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*5} = A, B^{*5} = K, (AN)^{*5} = AN$, and $(BM)^{*5} = K$. Then there is determined a unique mapping $*_i^5$ from $F'(D)$ to $F'(D)$.

(6) Set $I_0^{*6} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*6} = A, B^{*6} = K, (AN)^{*6} = A$, and $(BM)^{*6} = K$. Then there is determined a unique mapping $*_i^6$ from $F'(D)$ to $F'(D)$.

(7) Set $I_0^{*7} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*7} = K, B^{*7} = B, (AN)^{*7} = K$, and $(BM)^{*7} = BM$. Then there is determined a unique mapping $*_i^7$ from $F'(D)$ to $F'(D)$.

(8) Set $I_0^{*8} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*8} = K, B^{*8} = B, (AN)^{*8} = K$ and $(BM)^{*8} = B$. Then there is determined a unique mapping $*_i^8$ from $F'(D)$ to $F'(D)$.

(9) Set $I_0^{*9} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*9} = K, B^{*9} = K, (AN)^{*9} = K$, and $(BM)^{*9} = K$. Then there is determined a unique mapping $*_i^9$ from $F'(D)$ to $F'(D)$.

Set $K^{*j} = K$ for each i, j , and set $(xI)^{*j} = xI^{*j}$ for each $x \in K - \{0\}$ and each $I \in F'(D)$.

(3.11) Each $*_i^j$ satisfies the following conditions: $I \subset I^{*j}$ for each $I \in F'(D)$, and $(I^{*j})^{*j} = I^{*j}$ for each $I \in F'(D)$.

(3.12) Let $x \in K - \{0\}$.

If $xM \subset AN$, then $(x) \subset AN$.

If $xM \subset BM$, then $(x) \subset B$.

If $xN \subset AN$, then $(x) \subset A$.

If $xN \subset BM$, then $(x) \subset BM$.

If $xMN \subset A$, then $(x) \subset A$.

If $xMN \subset B$, then $(x) \subset B$.

(3.13) (1) Each of $*_3^1, *_3^3, *_3^3, *_4^7, *_4^7$ is not monotone for $M \subset BM$.

(2) Each of $*_2^1, *_4^1, *_2^2, *_4^2, *_2^5, *_4^5$ is not monotone for $N \subset AN$.

(3.14) Let $I \in F(D)$ and $J \in F'(D)$ so that $I \subset J$. Then each member in $\{*_1^1, *_2^1, *_3^2, *_3^3, *_2^3, *_1^4, *_2^4, *_3^4, *_4^4, *_1^5, *_3^5, *_1^6, *_2^6, *_3^6, *_4^6, *_1^7, *_2^7, *_1^8, *_2^8, *_3^8, *_4^8, *_1^9, *_2^9, *_3^9, *_4^9\}$ is monotone for $I \subset J$.

(3.15) Each member in $\{*_1^1, *_2^1, *_3^2, *_3^3, *_1^4, *_2^4, *_3^4, *_4^4, *_1^5, *_3^5, *_1^6, *_2^6, *_3^6, *_4^6, *_1^7, *_2^7, *_1^8, *_2^8, *_3^8, *_4^8, *_1^9, *_2^9, *_3^9, *_4^9\}$ is a semistar operation on D . We have $*_1^1 = d'$ and $*_9^9 = v'$.

(3.16) We have $|\Sigma'(D)| = 30$.

§4 The case where $\Gamma = \vec{\mathcal{R}}$ and Γ' is not complete.

Then neither M nor N is principal.

(4.1) (1) We have $\{I \in F(D) \mid I \subsetneq I^\nu\} = \{xM, xN, xMN, xB_0M \mid x \in K - \{0\}, B_0 \text{ is a fractional ideal of } D \text{ with type } \beta\}$.

(2) The sets $\{xM \mid x \in K - \{0\}\}$, $\{xN \mid x \in K - \{0\}\}$, $\{xMN \mid x \in K - \{0\}\}$ and $\{xB_0M \mid x \in K - \{0\}, B_0 \text{ is a fractional ideal of } D \text{ with type } \beta\}$ are disjoint with each other.

(4.2) Let $*$ be a star operation on D . Then we have that M^* is either M or D , N^* is either N or D , and $(MN)^*$ is either MN or M or N or D .

Either $(B_0M)^* = B_0M$ for each fractional ideal B_0 with type β , or $(B_0M)^* = B_0$ for each fractional ideal B_0 with type β .

(4.3) Let B_0 be a fractional ideal of D with type β .

(1) Set $M^{*1} = M$, $N^{*1} = N$, and $(B_0M)^{*1} = B_0M$. Then there is determined a unique mapping $*_1$ from $F(D)$ to $F(D)$.

(2) Set $M^{*2} = M$, $N^{*2} = N$, and $(B_0M)^{*2} = B_0$. Then there is determined a unique mapping $*_2$ from $F(D)$ to $F(D)$.

(3) Set $M^{*3} = M$, $N^{*3} = D$ and $(B_0M)^{*3} = B_0M$. Then there is determined a unique mapping $*_3$ from $F(D)$ to $F(D)$.

(4) Set $M^{*4} = M$, $N^{*4} = D$ and $(B_0M)^{*4} = B_0$. Then there is determined a unique mapping $*_4$ from $F(D)$ to $F(D)$.

(5) Set $M^{*5} = D$, $N^{*5} = N$ and $(B_0M)^{*5} = B_0$. Then there is determined a unique mapping $*_5$ from $F(D)$ to $F(D)$.

(6) Set $M^{*6} = D$, $N^{*6} = D$ and $(B_0M)^{*6} = B_0$. Then there is determined a unique mapping $*_6$ from $F(D)$ to $F(D)$.

For the proof we confer (1.3). Set $(MN)^{*i} = (M^{*i}N^{*i})^{*i}$ for each i ; If I is a nonzero fractional ideal of D with $I = I^\nu$, set $I^{*i} = I$; And set $(xI)^{*i} = xI^{*i}$ for each $x \in K - \{0\}$ and each $I \in F(D)$.

(4.4) Let B_0 be a fractional ideal of D with type β , and let $x \in K - \{0\}$.

If $xM \subset D$, then $(x) \subset D$.

If $xM \subset N$, then $(x) \subset N$.

If $xM \subset B_0M$, then $(x) \subset B_0$.

If $xN \subset D$, then $(x) \subset D$.

If $xN \subset M$, then $(x) \subset M$.

If $xN \subset B_0M$, then $(x) \subset B_0M$.

If $xMN \subset D$, then $(x) \subset D$.

If $xMN \subset M$, then $xM \subset M$.

If $xMN \subset N$, then $xN \subset D$.

If $xMN \subset N$, then $xM \subset D$.

If $xMN \subset N$, then $xN \subset N$.

If $xMN \subset B_0M$, then $(x) \subset B_0$.

If $xB_0M \subset D$, then $xB_0 \subset D$.

If $xB_0M \subset N$, then $xB_0 \subset N$.

If $xB_0M \subset MN$, then $xB_0 \subset MN$.

If $xB_0M \subset B_0M$, then $xB_0 \subset B_0$.

(4.5) Each $*_i$ is a star operation on D . We have $*_1 = d$ and $*_6 = v$.

(4.6) $|\Sigma(D)| = 6$.

(4.7) (1) We have $F'(D) - F(D) = \{xA, xB, xAN, xBM, xAB_0, K \mid x \in K - \{0\}, B_0 \text{ is a fractional ideal of } D \text{ with type } \beta\}$.

(2) The sets $\{xA \mid x \in K - \{0\}\}$, $\{xB \mid x \in K - \{0\}\}$, $\{xAN \mid x \in K - \{0\}\}$, $\{xEM \mid x \in K - \{0\}\}$, $\{xAB_0 \mid x \in K - \{0\}\}$, B_0 is a fractional ideal of D with type β and $\{K\}$ are disjoint with each other.

(4.8) Let $*$ be a semistar operation on D . Then we have that A^* is either A or K , B^* is either B or K , $(AN)^*$ is either AN or A or K , and $(BM)^*$ is either BM or B or K .

If $A^* = A$, then $(AB_0)^* = AB_0$ for each fractional ideal B_0 of D with type β .

(4.9) If $xAN \subset A$, then $xA \subset A$.

If $xBM \subset B$, then $xB \subset B$.

(4.10) Let $*_i$ be a star operation on D .

(1) Set $I_0^{*^1} = I_0^{*^i}$ for each $I_0 \in F(D)$, and set $A^{*^1} = A, B^{*^1} = B, (AN)^{*^1} = AN$ and $(BM)^{*^1} = BM$. Then there is determined a unique mapping $*_i^1$ from $F'(D)$ to $F'(D)$.

(2) Set $I_0^{*^2} = I_0^{*^i}$ for each $I_0 \in F(D)$, and set $A^{*^2} = A, B^{*^2} = B, (AN)^{*^2} = AN$ and $(BM)^{*^2} = B$. Then there is determined a unique mapping $*_i^2$ from $F'(D)$ to $F'(D)$.

(3) Set $I_0^{*^3} = I_0^{*^i}$ for each $I_0 \in F(D)$, and set $A^{*^3} = A, B^{*^3} = B, (AN)^{*^3} = A$ and $(BM)^{*^3} = BM$. Then there is determined a unique mapping $*_i^3$ from $F'(D)$ to $F'(D)$.

(4) Set $I_0^{*^4} = I_0^{*^i}$ for each $I_0 \in F(D)$, and set $A^{*^4} = A, B^{*^4} = B, (AN)^{*^4} = A$ and $(BM)^{*^4} = B$. Then there is determined a unique mapping $*_i^4$ from $F'(D)$ to $F'(D)$.

(5) Set $I_0^{*^5} = I_0^{*^i}$ for each $I_0 \in F(D)$, and set $A^{*^5} = A, B^{*^5} = K, (AN)^{*^5} = AN$ and $(BM)^{*^5} = K$. Then there is determined a unique mapping $*_i^5$ from $F'(D)$ to $F'(D)$.

(6) Set $I_0^{*^6} = I_0^{*^i}$ for each $I_0 \in F(D)$, and set $A^{*^6} = A, B^{*^6} = K, (AN)^{*^6} = A$ and $(BM)^{*^6} = K$. Then there is determined a unique mapping $*_i^6$ from $F'(D)$ to $F'(D)$.

(7) Set $I_0^{*7} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*7} = K, B^{*7} = B, (AN)^{*7} = K$ and $(BM)^{*7} = BM$. Then there is determined a unique mapping $*_i^7$ from $F'(D)$ to $F'(D)$.

(8) Set $I_0^{*8} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*8} = K, B^{*8} = B, (AN)^{*8} = K$ and $(BM)^{*8} = B$. Then there is determined a unique mapping $*_i^8$ from $F'(D)$ to $F'(D)$.

(9) Set $I_0^{*9} = I_0^{*i}$ for each $I_0 \in F(D)$, and set $A^{*9} = K, B^{*9} = K, (AN)^{*9} = K$ and $(BM)^{*9} = K$. Then there is determined a unique mapping $*_i^9$ from $F'(D)$ to $F'(D)$.

If $A^{*j} = A$, set $(AB_0)^{*j} = AB_0$; If $A^{*j} = K$, set $(AB_0)^{*j} = K$; Set $K^{*j} = K$; And set $(xI)^{*j} = xI^{*j}$ for each $x \in K - \{0\}$ and $I \in F'(D)$.

(4.11) Each $*_i^j$ satisfies the following conditions: $I \subset I^{*j}$ for each $I \in F'(D)$, and $(I^{*j})^{*j} = I^{*j}$ for each $I \in F'(D)$.

(4.12) Let $x \in K - \{0\}$, and let B_0 be a fractional ideal of D with type β .

If $xMN \subset A$, then $(x) \subset A$.

If $xMN \subset B$, then $(x) \subset B$.

If $xM \subset AN$, then $(x) \subset AN$.

If $xN \subset BM$, then $(x) \subset BM$.

If $xMN \subset AN$, then $xN \subset AN$.

If $xMN \subset BM$, then $xM \subset BM$.

If $xMN \subset AB_0$, then $xM \subset AB_0$.

If $xMN \subset AB_0$, then $xN \subset AB_0$.

If $xMN \subset AB_0$, then $(x) \subset AB_0$.

If $xB_0M \subset A$, then $xB_0 \subset A$.

If $xB_0M \subset B$, then $xB_0 \subset B$.

If $xB_0M \subset AN$, then $xB_0 \subset AN$.

If $xB_0M \subset BM$, then $xB_0 \subset B$.

If $xB_0M \subset AB_0$, then $xB_0 \subset AB_0$.

(4.13) Let $I \in F(D)$ and $J \in F'(D)$ such that $I \subset J$. Then each member in $\{*_1^1, *_1^2, *_2^2, *_5^2, *_1^3, *_3^3, *_1^4, *_2^4, *_3^4, *_4^4, *_5^4, *_6^4, *_1^5, *_2^5, *_5^5, *_6^5, *_3^6, *_4^6, *_5^6, *_6^6, *_1^7, *_3^7, *_8^8, *_2^8, *_3^8, *_4^8, *_5^8, *_6^8, *_1^9, *_2^9, *_3^9, *_4^9, *_5^9, *_6^9\}$ is monotone for $I \subset J$.

(4.14) (1) Each of $*_2^1, *_5^1, *_2^3, *_4^3, *_5^3, *_6^3, *_2^7, *_4^7, *_5^7, *_6^7$ is not monotone for $B_0M \subset BM$.

(2) Each of $*_3^1, *_4^1, *_6^1, *_3^2, *_4^2, *_6^2, *_3^5, *_4^5, *_6^5$ is not monotone for $N \subset AN$.

(4.15) Let B_0 be a fractional ideal of D with type β , and let $x \in K - \{0\}$.

If $xAN \subset A$, then $xA \subset A$.

If $xAN \subset AB_0$, then $xA \subset AB_0$.

If $xBM \subset B$, then $xB \subset B$.

(4.16) Each member in $\{*_1^1, *_1^2, *_2^2, *_5^2, *_1^3, *_3^3, *_1^4, *_2^4, *_3^4, *_4^4, *_5^4, *_6^4, *_1^5, *_2^5, *_5^5, *_1^6, *_2^6, *_3^6, *_4^6, *_5^6, *_1^7, *_3^7, *_1^8, *_2^8, *_3^8, *_4^8, *_5^8, *_6^8, *_1^9, *_2^9, *_3^9, *_4^9, *_5^9, *_6^9\}$ is a semistar operation on D . We have $*_1^1 = d'$ and $*_6^9 = v'$.

(4.17) We have $|\Sigma'(D)| = 40$.

§5 The case where D has three maximal ideals.

Throughout the section, let D be a 1-dimensional Prüfer domain with exactly three maximal ideals M_1, M_2 and M_3 , let $V_1 = D_{M_1}$ (resp. $V_2 = D_{M_2}, V_3 = D_{M_3}$), let v_1 (resp. v_2, v_3) be a valuation belonging to V_1 (resp. V_2, V_3), and let Γ_1 (resp. Γ_2, Γ_3) be the value group of v_1 (resp. v_2, v_3). We assume that Γ_1 (resp. Γ_2, Γ_3) is a subgroup of $\tilde{\mathcal{R}}$. Let K be the quotient field of D . Let A (resp. B, C) be the D -submodule of K generated by the subset $\{\frac{1}{p} \mid p \in M_1 - (M_2 \cup M_3)\}$ (resp. $\{\frac{1}{q} \mid q \in M_2 - (M_1 \cup M_3), \{\frac{1}{r} \mid r \in M_3 - (M_1 \cup M_2)\}$) of K .

Throughout the section, p denotes an element of $M_1 - (M_2 \cup M_3)$, q denotes an element of $M_2 - (M_1 \cup M_3)$, and r denotes an element of $M_3 - (M_1 \cup M_2)$.

(5.1) Let n be a positive integer, and let D be a Prüfer domain with exactly n maximal ideals. Then D is a Bezout domain.

For the proof we confer [G, (7.4)].

(5.2) (1) Each element $x \in D - \{0\}$ can be expressed as one of $1, p, q, r, pq, pr, qr, pqr$ uniquely up to associates, where $p \in M_1 - (M_2 \cup M_3)$, $q \in M_2 - (M_1 \cup M_3)$ and $r \in M_3 - (M_1 \cup M_2)$.

(2) Each element $x \in K - \{0\}$ can be expressed as $p^{e_1}q^{e_2}r^{e_3}$ uniquely up to associates, where each $e_i \in \{-1, 0, 1\}$.

Proof. Let $x \in M_1 \cap M_2 - M_3$. Choose $a \in M_1 \cap M_3 - M_2$. There exists a positive integer n such that $v_1(a^n) > v_1(x)$. Let $p' = a^n + x$. Then we have $v_1(p') = v_1(x)$ and $p' \in M_1 - (M_2 \cup M_3)$. Similarly, there exists q' such that $v_2(q') = v_2(x)$. Then we have $x = p'q'u$ with unit u .

A fractional ideal $A_0 \in F(D)$ is called of type α , if there exists a subset $\{p_\lambda \mid \lambda\}$ of $M_1 - (M_2 \cup M_3)$ so that A_0 is generated by the set $\{\frac{1}{p_\lambda} \mid \lambda\}, v_1(A_0)$ is bounded below, and there does not exist $\inf v_1(A_0)$. A fractional ideal $B_0 \in F(D)$ is called of type β , if there exists a subset $\{q_\sigma \mid \sigma\}$ of $M_2 - (M_1 \cup M_3)$ so that B_0 is generated by the set $\{\frac{1}{q_\sigma} \mid \sigma\}, v_2(B_0)$ is bounded below, and there does not exist $\inf v_2(B_0)$. A fractional ideal $C_0 \in F(D)$ is called of type γ , if there exists a subset $\{r_\tau \mid \tau\}$ of $M_3 - (M_1 \cup M_2)$ so that C_0 is generated by the set $\{\frac{1}{r_\tau} \mid \tau\}, v_3(C_0)$ is bounded below, and there does not exist $\inf v_3(C_0)$.

Let I be a nonzero fractional ideal of D . Then either there exists $\min v_i(I)$, or there exists $\inf v_i(I)$ with $\inf v_i(I) \notin v_i(I)$, or there does not exist $\inf v_i(I)$.

(5.3) Let I be a nonzero fractional ideal of D . Then there may arise the following 27-cases:

- (1) There exists $\min v_1(I)$, there exists $\min v_2(I)$, and there exists $\min v_3(I)$.
- (2) There exists $\inf v_1(I)$ with $\inf v_1(I) \notin v_1(I)$, there exists $\min v_2(I)$, and there exists $\min v_3(I)$.
- (3) There does not exist $\inf v_1(I)$, there exists $\min v_2(I)$, and there exists $\min v_3(I)$.
- (4) There exists $\inf v_1(I)$ with $\inf v_1(I) \notin v_1(I)$, there exists $\inf v_2(I)$ with $\inf v_2(I) \notin v_2(I)$, and there exists $\min v_3(I)$.
- (5) There does not exist $\inf v_1(I)$, there exists $\inf v_2(I)$ with $\inf v_2(I) \notin v_2(I)$, and there exists $\min v_3(I)$.
- (6) There does not exist $\inf v_1(I)$, there does not exist $\inf v_2(I)$, and there exists $\min v_3(I)$.
- (7) There exists $\inf v_1(I)$ with $\inf v_1(I) \notin v_1(I)$, there exists $\inf v_2(I)$ with $\inf v_2(I) \notin v_2(I)$, and there exists $\inf v_3(I)$ with $\inf v_3(I) \notin v_3(I)$.
- (8) There does not exist $\inf v_1(I)$, there exists $\inf v_2(I)$ with $\inf v_2(I) \notin v_2(I)$, and there exists $\inf v_3(I)$ with $\inf v_3(I) \notin v_3(I)$.
- (9) There does not exist $\inf v_1(I)$, there does not exist $\inf v_2(I)$, and there exists $\inf v_3(I)$ with $\inf v_3(I) \notin v_3(I)$.
- (10) There does not exist $\inf v_1(I)$, there does not exist $\inf v_2(I)$, and there does not exist $\inf v_3(I)$.

... ..

Case (1): There exist elements $a, b, c \in I$ such that $v(a) = \min v_1(I)$, $v_2(b) = \min v_2(I)$, and $v_3(c) = \min v_3(I)$. Let $(a, b, c) = (d)$ for an element $d \in I$. Then we have $I = (d)$.

Case (2): We have $I = xM_1$ for some $x \in K - \{0\}$, and $M_1^v = D$.

Case (3): We have $I = x(\frac{1}{p_\lambda} \mid \lambda)$ for some $x \in K - \{0\}$, $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$ has type α , and $A_0^v = A_0$.

Case (4): We have $I = xM_1M_2$ for some $x \in K - \{0\}$, and $(M_1M_2)^v = D$.

Let J be an ideal of D with $M_1M_2 \subset J \subset D$. Then J is either M_1M_2 or M_1 or M_2 or D .

Case (5): We have $I = xM_2(\frac{1}{p_\lambda} \mid \lambda)$ for some $x \in K - \{0\}$, $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$ has type α , $A_0^v = A_0$, and $(A_0M_2)^v = A_0$.

Let J be a fractional ideal with $A_0M_2 \subset J \subset A_0$. Then J is either A_0M_2 or A_0 .

Case (6): We have $I = x(\frac{1}{p_\lambda} \mid \lambda)(\frac{1}{q_\sigma} \mid \sigma)$ for some $x \in K - \{0\}$, $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$ has type α , $B_0 = (\frac{1}{q_\sigma} \mid \sigma)$ has type β , and $(A_0B_0)^v = A_0B_0$.

Case (7): We have $I = xM_1M_2M_3$ for some $x \in K - \{0\}$, and $(M_1M_2M_3)^v = D$.

Let J be an ideal of D with $M_1M_2M_3 \subset J \subset D$. Then J is one of $M_1M_2M_3$, M_1M_2 , M_1M_3 , M_2M_3 , M_1 , M_2 , M_3 , D .

Case (8): We have $I = xM_2M_3(\frac{1}{p_\lambda} \mid \lambda)$ for some $x \in K - \{0\}$, $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$ has type α , and $(A_0M_2M_3)^v = A_0$.

Let J be a fractional ideal of D with $A_0M_2M_3 \subset J \subset A_0$. Then J is one of $A_0M_2M_3$, A_0M_2 , A_0M_3 , A_0 .

Case (9): We have $I = xM_3(\frac{1}{p_\lambda} \mid \lambda)(\frac{1}{q_\sigma} \mid \sigma)$ for some $x \in K - \{0\}$, $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$ has type α , $B_0 = (\frac{1}{q_\sigma} \mid \sigma)$ has type β , and $(A_0B_0M_3)^v = A_0B_0$.

Let J be a fractional ideal of D with $A_0B_0M_3 \subset J \subset A_0B_0$, then J is one of $A_0B_0M_3$, A_0B_0 .

Case (10): We have $I = x(\frac{1}{p_\lambda} \mid \lambda)(\frac{1}{q_\sigma} \mid \sigma)(\frac{1}{r_\tau} \mid \tau)$ for some $x \in K - \{0\}$, $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$ has type α , $B_0 = (\frac{1}{q_\sigma} \mid \sigma)$ has type β , and $C_0 = (\frac{1}{r_\tau} \mid \tau)$ has type γ , and $(A_0B_0C_0)^v = A_0B_0C_0$.

Proof. Case (2): There exist p, q, r such that $v_1(p^{e_1}) = \inf v_1(I)$, $v_2(q^{e_2}) = \min v_2(I)$, $v_3(r^{e_3}) = \min v_3(I)$, where each $e_i \in \{-1, 0, 1\}$. Let $x = \frac{1}{p^{e_1}q^{e_2}r^{e_3}}$. Then $I = xM_1$ and $M_1^v = D$.

Case (3): We may assume that $0 \in v_1(I)$, $0 = \min v_2(I)$, $0 = \min v_3(I)$. Let $\{p \mid \frac{1}{p} \in I\} = \{p_\lambda \mid \lambda\}$. Then $I = (\frac{1}{p_\lambda} \mid \lambda)$. Let $\{p \mid \frac{1}{p} \notin v_1(I)\} = \{p_\alpha \mid \alpha\}$. Then we have $I = \cap_\alpha (\frac{1}{p_\alpha})$.

Case (4): We may assume that $0 = \inf v_1(I)$, $0 = \inf v_2(I)$, $0 = \min v_3(I)$. Then $I = M_1M_2$. We may assume that $M_1M_2 \subsetneq J \subsetneq D$. Let $x \in J - M_1M_2$. Then either $v_1(x) = 0$ or $v_2(x) = 0$. If $v_1(x) = 0$, then $I = M_2$. If $v_2(x) = 0$, then $I = M_1$.

Case (5): We may assume that $0 \in v_1(I)$, $0 = \inf v_2(I)$, $0 = \min v_3(I)$. Let $\{p \mid \frac{1}{p} \in I\} = \{p_\lambda \mid \lambda\}$, and set $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$. Then $I = A_0M_2$. We may assume that $A_0M_2 \subsetneq J$. Let $x \in J - A_0M_2$. Then $v_2(x) = 0$, and hence $J = A_0$.

Case (6): We may assume that $0 \in v_1(I)$, $0 \in v_2(I)$, $0 = \min v_3(I)$. Let $\{p \mid \frac{1}{p} \in I\} = \{p_\lambda \mid \lambda\}$, $\{q \mid \frac{1}{q} \in I\} = \{q_\sigma \mid \sigma\}$, and set $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$, $B_0 = (\frac{1}{q_\sigma} \mid \sigma)$. Then $I = A_0B_0$. Let $\{p \mid v_1(\frac{1}{p}) \notin v_1(I)\} = \{p_\alpha \mid \alpha\}$, $\{q \mid v_2(\frac{1}{q}) \notin v_2(I)\} = \{q_\beta \mid \beta\}$. Then $I = \cap_{\alpha, \beta} (\frac{1}{p_\alpha q_\beta})$.

Case (7): We may assume that $0 = \inf v_1(I) = \inf v_2(I) = \inf v_3(I)$. Then $I = M_1M_2M_3$. We may assume that $M_1M_2M_3 \subsetneq J$. Let $x \in J - M_1M_2M_3$. Then $v_i(x) = 0$ for some i .

Case (8): We may assume that $0 \in v_1(I)$, $0 = \inf v_2(I) = \inf v_3(I)$. Let

$\{p \mid \frac{1}{p} \in I\} = \{p_\lambda \mid \lambda\}$, and set $A_0 = (\frac{1}{p_\lambda} \mid \lambda)$. Then $I = A_0 M_2 M_3$. We may assume that $A_0 M_2 M_3 \subsetneq J$. Let $x \in J - A_0 M_2 M_3$. Then either $v_2(x) = 0$ or $v_3(x) = 0$.

Case (9): We may assume that $0 \in v_1(I), 0 \in v_2(I), 0 = \inf v_3(I)$. Let $\{p \mid \frac{1}{p} \in I\} = \{p_\lambda \mid \lambda\}, \{q \mid \frac{1}{q} \in I\} = \{q_\sigma \mid \sigma\}$, and set $A_0 = (\frac{1}{p_\lambda} \mid \lambda), B_0 = (\frac{1}{q_\sigma} \mid \sigma)$. Then $I = A_0 B_0 M_3$. We may assume that $A_0 B_0 M_3 \subsetneq J$. Let $x \in J - A_0 B_0 M_3$. Then $v_1(x) = 0$, and hence $J = A_0 B_0$.

(5.4) Each nonzero fractional ideal of D has one of the following forms:
 $(x), xM_1, xM_2, xM_3, xA_0, xB_0, xC_0, xM_1M_2, xM_1M_3, xM_2M_3, xA_0M_2, xA_0M_3,$
 $xB_0M_1, xB_0M_3, xC_0M_1, xC_0M_2, xA_0B_0, xA_0C_0, xB_0C_0, xM_1M_2M_3, xA_0M_2M_3,$
 $xB_0M_1M_3, xC_0M_1M_2, xA_0B_0M_3, xA_0C_0M_2, xB_0C_0M_1, xA_0B_0C_0.$

Let I, J be nonzero fractional ideals of D . Then the sets $\{xI \mid x \in K - \{0\}\}$ and $\{xJ \mid x \in K - \{0\}\}$ are disjoint or coincides with each other.

(5.5) In the following, A_0, A'_0 are fractional ideals of D with type α , B_0, B'_0 are fractional ideals of D with type β , C_0, C'_0 are fractional ideals of D with type γ , and x is an element of $K - \{0\}$.

(1) If $xM_1 = M_1$ (resp. $xM_2 = M_2, xM_3 = M_3, xM_1M_2 = M_1M_2, xM_1M_3 = M_1M_3, xM_2M_3 = M_2M_3, xM_1M_2M_3 = M_1M_2M_3$), then $(x) = D$.

(2) If $xA_0M_2 = A'_0M_2$ (resp. $xA_0M_3 = A'_0M_3, xB_0M_1 = B'_0M_1, xB_0M_3 = B'_0M_3, xC_0M_1 = C'_0M_1, xC_0M_2 = C'_0M_2, xA_0M_2M_3 = A'_0M_2M_3, xB_0M_1M_3 = B'_0M_1M_3, xC_0M_1M_2 = C'_0M_1M_2, xA_0B_0M_3 = A'_0B'_0M_3, xA_0C_0M_2 = A'_0C'_0M_2, xB_0C_0M_1 = B'_0C'_0M_1$), then $xA_0 = A'_0$ (resp. $xA_0 = A'_0, xB_0 = B'_0, xB_0 = B'_0, xC_0 = C'_0, xC_0 = C'_0, xA_0 = A'_0, xB_0 = B'_0, xC_0 = C'_0, xA_0B_0 = A'_0B'_0, xA_0C_0 = A'_0C'_0, xB_0C_0 = B'_0C'_0$).

(5.6) Let $*$ be a star operation on D . In the following, A_0 (resp. B_0, C_0) is a fractional ideal with type α (resp. type β , type γ).

M_1^* is either M_1 or D ; $\dots$.

$A_0^* = A_0$; $\dots$.

$(M_1M_2)^*$ is one of M_1M_2, M_1, M_2, D ; $\dots$.

Either $(A_0M_2)^* = A_0M_2$, or $(A_0M_2)^* = A_0$; $\dots$.

We have $(A_0B_0)^* = A_0B_0$; $\dots$.

$(M_1M_2M_3)^*$ is one of $M_1M_2M_3, M_1M_2, M_1M_3, M_2M_3, M_1, M_2, M_3, D$.

$(A_0M_2M_3)^*$ is one of $A_0M_2M_3, A_0M_2, A_0M_3, A_0$; $\dots$.

$(A_0B_0M_3)^*$ is one of $A_0B_0M_3, A_0B_0$; $\dots$.

$(A_0B_0C_0)^* = A_0B_0C_0$.

(5.7) Let $*$ be a star operation on D .

Either $(A_0M_2)^* = A_0M_2$ for each fractional ideal A_0 of D with type α , or $(A_0M_2)^* = A_0$ for each fractional ideal A_0 of D with type α ; $\dots$.

Either $(A_0M_2M_3)^* = A_0M_2M_3$ for each fractional ideal A_0 with type α , or $(A_0M_2M_3)^* = A_0M_2$ for each fractional ideal A_0 with type α , or $(A_0M_2M_3)^* = A_0M_3$ for each fractional ideal A_0 with type α , or $(A_0M_2M_3)^* = A_0$ for each fractional ideal A_0 with type α ; $\dots$.

Either $(A_0B_0M_3)^* = A_0B_0M_3$ for each fractional ideal A_0 (resp. B_0) with type α (resp. type β), or $(A_0B_0M_3)^* = A_0B_0$ for each fractional ideal A_0 (resp. B_0) with type α (resp. type β); $\dots$.

(5.8) We have $|\Sigma(D)| < \infty$.

(5.9) Let $I \in F'(D) - F(D)$. There may arise the following 37-cases:

- (1) $v_1(I) = \Gamma_1 \cup \{\infty\}$, there exists $\min v_2(I)$, and there exists $\min v_3(I)$.
- (2) $v_1(I) = \Gamma_1 \cup \{\infty\}$, there exists $\inf v_2(I)$ with $\inf v_2(I) \notin v_2(I)$, and there exists $\min v_3(I)$.
- (3) $v_1(I) = \Gamma_1 \cup \{\infty\}$, $v_2(I)$ is bounded below, there does not exist $\inf v_2(I)$, and there exists $\min v_3(I)$.
- (4) $v_1(I) = \Gamma_1 \cup \{\infty\}$, $v_2(I) = \Gamma_2 \cup \{\infty\}$, and there exists $\min v_3(I)$.
- (5) $v_1(I) = \Gamma_1 \cup \{\infty\}$, there exists $\inf v_2(I)$ with $\inf v_2(I) \notin v_2(I)$, and there exists $\inf v_3(I)$ with $\inf v_3(I) \notin v_3(I)$.
- (6) $v_1(I) = \Gamma_1 \cup \{\infty\}$, $v_2(I)$ is bounded below, there does not exist $\inf v_2(I)$, and there exists $\inf v_3(I)$ with $\inf v_3(I) \notin v_3(I)$.
- (7) $v_1(I) = \Gamma_1 \cup \{\infty\}$, $v_2(I) = \Gamma_2 \cup \{\infty\}$, and there exists $\inf v_3(I)$ with $\inf v_3(I) \notin v_3(I)$.
- (8) $v_1(I) = \Gamma_1 \cup \{\infty\}$, $v_2(I)$ is bounded below, there does not exist $\inf v_2(I)$, and $v_3(I)$ is bounded below, there does not exist $\inf v_3(I)$.
- (9) $v_1(I) = \Gamma_1 \cup \{\infty\}$, $v_2(I) = \Gamma_2 \cup \{\infty\}$, and $v_3(I)$ is bounded below, there does not exist $\inf v_3(I)$.
- (10) $v_1(I) = \Gamma_1 \cup \{\infty\}$, $v_2(I) = \Gamma_2 \cup \{\infty\}$, and $v_3(I) = \Gamma_3 \cup \{\infty\}$.

$\dots\dots$

- Case (1): Then we have $I = xA$ for some $x \in K - \{0\}$.
Case (2): Then we have $I = xAM_2$ for some $x \in K - \{0\}$.
Case (3): Then we have $I = xAB_0$ for some $x \in K - \{0\}$.
Case (4): Then we have $I = xAB$ for some $x \in K - \{0\}$.
Case (5): Then we have $I = xAM_2M_3$ for some $x \in K - \{0\}$.
Case (6): Then we have $I = xAB_0M_3$ for some $x \in K - \{0\}$.
Case (7): Then we have $I = xABM_3$ for some $x \in K - \{0\}$.
Case (8): Then we have $I = xAB_0C_0$ for some $x \in K - \{0\}$.
Case (9): Then we have $I = xABC_0$ for some $x \in K - \{0\}$.
Case (10): Then we have $I = K$.

(5.10) Each member in $F'(D) - F(D)$ has one of the following forms: $xA, xB, xC, xAM_2, xAM_3, xBM_1, xBM_3, xCM_1, xCM_2, xAB_0, xAC_0, xA_0B, xBC_0, xA_0C, xB_0C, xAB, xAC, xBC, xAM_2M_3, xBM_1M_3, xCM_1M_2, xAB_0M_3, xAC_0M_2, xA_0BM_3, xBC_0M_1, xA_0CM_2, xB_0CM_1, xABM_3, xACM_2, xBCM_1,$

xAB_0C_0 , xA_0BC_0 , xA_0B_0C , xAB_0C , xA_0BC , K , where x is an element of $K - \{0\}$, A_0 (resp. B_0, C_0) is a fractional ideal with type α (resp. type β , type γ).

Let $I, J \in F'(D) - F(D)$. Then $\{xI \mid x \in K - \{0\}\}, \{xJ \mid x \in K - \{0\}\}$ are disjoint or coincides with each other.

(5.11) Let A_0, A'_0 be fractional ideals of D with type α , let B_0, B'_0 be fractional ideals of D with type β , let C_0, C'_0 be fractional ideals of D with type γ , and let $x \in K - \{0\}$.

- If $xAM_2 = AM_2$ (resp. $xAM_3 = AM_3, xAM_2M_3 = AM_2M_3$), then $xA = A$.
- If $xBM_1 = BM_1$ (resp. $xBM_3 = BM_3, xBM_1M_3 = BM_1M_3$), then $xB = B$.
- If $xCM_1 = CM_1$ (resp. $xCM_2 = CM_2, xCM_1M_2 = CM_1M_2$), then $xC = C$.
- If $xAB_0M_3 = AB'_0M_3$, then $xAB_0 = AB'_0$.
- If $xAC_0M_2 = AC'_0M_2$, then $xAC_0 = AC'_0$.
- If $xA_0BM_3 = A'_0BM_3$, then $xA_0B = A'_0B$.
- If $xBC_0M_1 = BC'_0M_1$, then $xBC_0 = BC'_0$.
- If $xA_0CM_2 = A'_0CM_2$, then $xA_0C = A'_0C$.
- If $xB_0CM_1 = B'_0CM_1$, then $xB_0C = B'_0C$.
- If $xABM_3 = ABM_3$, then $xAB = AB$.
- If $xACM_2 = ACM_2$, then $xAC = AC$.
- If $xBCM_1 = BCM_1$, then $xBC = BC$.

Let $I^{d'} = I$ (resp. $I^{v'} = (I^{-1})^{-1}, I^e = K$) for each $I \in F'(D)$. Then d' (resp. v' , e) is a semistar operation on D , and is called the d' -operation (resp. v' -operation, e -operation) on D .

(5.12) Let $*$ be a semistar operation on D .

(1) $(AB)^*$ is one of AB, K .

...

(2) A^* is one of A, AB, AC, K .

...

(3) $(ABM_3)^*$ is one of ABM_3, AB, K .

...

(4) $(AM_2)^*$ is one of $AM_2, A, AB, ACM_2, AC, K$.

...

(5) $(AM_2M_3)^*$ is one of $AM_2M_3, AM_2, AM_3, ABM_3, ACM_2, A, AB, AC, K$.

...

Proof. (1) Assume that $AB \subsetneq (AB)^*$, and let $x \in (AB)^* - AB$. Since $v_3(x) < 0$, we have $\frac{1}{r} \in (AB)^* - AB$ for some r . Then $AB \subset \frac{1}{r}AB \subset (AB)^*$. Hence $(AB)^* = \frac{1}{r}(AB)^*$, and hence $(AB)^* = K$.

(2) Assume that $A \subsetneq A^*$, and let $x \in A^* - A$. Then either $v_2(x) < 0$ or $v_3(x) < 0$. Assume that $v_2(x) < 0$. We have $\frac{1}{q} \in A^* - A$ for some q , and hence

$A \subset \frac{1}{q}A \subset A^*$. Then $A^* = \frac{1}{q}A^*$, and hence $A^* = (AB)^*$.

(3) Assume that $ABM_3 \subsetneq (ABM_3)^*$, and let $x \in (ABM_3)^* - ABM_3$. Since $v_3(x) \leq 0$, we have $AB \subset (ABM_3)^*$, and hence $(ABM_3)^* = (AB)^*$.

(4) Assume that $AM_2 \subsetneq (AM_2)^*$ and $A = A^*$. It follows that $(AM_2)^* = A$.

(5) Assume that $AM_2M_3 \subsetneq (AM_2M_3)^*$ and $A = A^*$, $(AM_2)^* = AM_2$, $(AM_3)^* = AM_3$. Then $(AM_2M_3)^* \subset AM_2$ and $(AM_2M_3)^* \subset AM_3$. Hence $(AM_2M_3)^* = AM_2M_3$.

(5.13) In the following, let A_0 (resp. B_0, C_0) be a fractional ideal with type α (resp. type β , type γ). Let $*$ be a semistar operation on D .

(1) If $(AB)^* = AB$, then $(ABC_0)^* = ABC_0$ for each fractional ideal C_0 with type γ .

...

(2) If $A^* = A$, then $(AB_0)^* = AB_0$, $(AC_0)^* = AC_0$, $(AB_0C_0)^* = AB_0C_0$ for each fractional ideal B_0 (resp. C_0) with type α (resp. type γ).

...

(3) If $A^* = AB$, then $(AB_0)^* = AB$, $(AC_0)^* = ABC_0$, $(AB_0C_0)^* = ABC_0$ for each fractional ideal B_0 (resp. C_0) with type α (resp. type γ).

...

(4) If $A^* = AC$, then $(AB_0)^* = AB_0C$, $(AC_0)^* = AC$, $(AB_0M_3)^* = AB_0C$, $(AB_0C_0)^* = AB_0C$ for each fractional ideal B_0 (resp. C_0) with type α (resp. β).

...

(5) Assume that $A^* = A$. Then either $(AB_0M_3)^* = AB_0M_3$ for each fractional ideal B_0 with type β , or $(AB_0M_3)^* = AB_0$ for each fractional ideal B_0 with type β .

...

(6) Assume that $A^* = AB$. Then either $(AB_0M_3)^* = ABM_3$ for each fractional ideal B_0 with type β , or $(AB_0M_3)^* = AB$ for each fractional ideal B_0 with type β .

...

Proof. (1) Suppose that $ABC_0 \subsetneq (ABC_0)^*$, and let $x \in (ABC_0)^* - ABC_0$. Then $v_3(x) \notin v_3(C_0)$, and $v_3(x) = v_3(\frac{1}{r})$ for some r . There exists r' with $v_3(\frac{1}{r'}) > v_3(\frac{1}{r})$ such that $v_3(\frac{1}{r}) \notin v_3(C_0)$. Then $ABC_0 \subset \frac{1}{r'}AB \subsetneq \frac{1}{r}AB \subset (ABC_0)^*$; a contradiction.

(2) We have $(AB_0)^* \subset AB$. Suppose that $AB_0 \subsetneq (AB_0)^*$. Let $x \in (AB_0)^* - AB_0$. We have $v_2(x) = v_2(\frac{1}{q})$ for some q . Since $v_2(x) \notin v_2(B_0)$, there exists

q' with $v_2(\frac{1}{q}) < v_2(\frac{1}{q'})$ such that $v_2(\frac{1}{q'}) \notin v_2(B_0)$. Then $AB_0 \subset \frac{1}{q'}A \subsetneq \frac{1}{q}A \subset (AB_0)^*$; a contradiction.

The proof for $(AB_0C_0)^* = AB_0C_0$ is similar.

(3) We have $(AB_0)^* = (AB)^* = AB$.

Next, since $(AB)^* = AB$, we have $(ABC_0)^* = ABC_0$. Then $(AB_0C_0)^* = (ABC_0)^* = ABC_0$.

(5.14) We have $|\Sigma'(D)| < \infty$.

Proof. If D has a fractional ideal with type α (resp. type β , type γ), then we fix a fractional ideal A_0 (resp. B_0, C_0) with type α (resp. type β , type γ). Let $\{I_1, \dots, I_n\}$ be a maximal subset of $F'(D) - F(D)$ so that each I_i has one of the following forms: $A, B, C, AM_2, AM_3, BM_1, BM_3, CM_1, CM_2, AB_0, AC_0, A_0B, BC_0, A_0C, B_0C, AB, AC, BC, AM_2M_3, BM_1M_3, CM_1M_2, AB_0M_3, AC_0M_2, A_0BM_3, BC_0M_1, A_0CM_2, B_0CM_1, ABM_3, ACM_2, BCM_1, AB_0C_0, A_0BC_0, A_0B_0C, ABC_0, AB_0C, A_0BC, K$. Then the set $\{I_1, \dots, I_n\}$ contains $\{A, B, C, AM_2, AM_3, BM_1, BM_3, CM_1, CM_2, AB, AC, BC, AM_2M_3, BM_1M_3, CM_1M_2, ABM_3, ACM_2, BCM_1, K\}$. Let $*, *'$ be semistar operations on D satisfying the conditions: We have that $*|_{F(D)} = *'|_{F(D)}$ is a star operation on D , and $I_i^* = I_i^{*'}$ for each i . It follows that $* = *'$. For instance, let B'_0 be a fractional ideal with type β , and assume $A^* = AB$. We must show that $(AB'_0M_3)^* = (AB'_0M_3)^{*'}$. Thus, if $(AB_0M_3)^* = ABM_3$, then $(AB'_0M_3)^* = ABM_3$. Since $(AB_0M_3)^* = (AB_0M_3)^{*'}$, we have $(AB'_0M_3)^{*' = ABM_3$. If $(AB_0M_3)^* = AB$, we have $(AB'_0M_3)^* = (AB'_0M_3)^{*'}$ similarly.

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