Note on the number of semistar operations, XI

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Abstract

We study star operations and semistar operations on an almost pseudo-valuation domain.

The notion of a star operation is classical, and that of a Kronecker function ring which arises by a star operation is also classical. Let D be an integral domain, let K be its quotient field, and let $F(D)$ be the set of non-zero fractional ideals of D . A mapping $I \mapsto I^*$ from $F(D)$ to $F(D)$ is called a star operation on D , if it satisfies the following conditions: (1) $(x)^* = (x)$ for each $x \in K - \{0\}$; (2) $(xI)^* = xI^*$ for each $x \in K - \{0\}$ and $I \in F(D)$; (3) $I \subset I^*$ for each $I \in F(D)$; (4) $(I^*)^* = I^*$ for each $I \in F(D)$; (5) $I \subset J$ implies $I^* \subset J^*$ for each $I, J \in F(D)$. The Kronecker function ring of D with respect to a star operation $*$ on D was first defined by Prüfer ([P]) and further investigated by Krull ([K]). Let $F'(D)$ be the set of non-zero D -submodules of K . A mapping $I \mapsto I^*$ from $F'(D)$ to $F'(D)$ is called a semistar operation on D , if it satisfies the following conditions: (1) $(xI)^* = xI^*$ for each $x \in K - \{0\}$ and $I \in F'(D)$; (2) $I \subset I^*$ for each $I \in F'(D)$; (3) $(I^*)^* = I^*$ for each $I \in F'(D)$; (4) $I \subset J$ implies $I^* \subset J^*$ for each $I, J \in F'(D)$. We confer Fontana-Loper([FL]) and Halter-Koch([HK]) for notions of star operations, semistar operations, and their Kronecker function rings.

Let $\Sigma(D)$ (resp. $\Sigma'(D)$) be the set of star operations (resp. semistar operations) on D . In this paper, we are interested in cardinalities $|\Sigma(D)|$ and $|\Sigma'(D)|$.

Let D be an integrally closed domain. Then D has only a finite number of semistar operations if and only if D is a finite dimensional Prüfer domain with only a finite number of maximal ideals ([M4]).

Let V be a valuation domain with dimension n , let v be a valuation belonging to V , and let Γ be its value group. Let $M = P_n \supsetneq P_{n-1} \supsetneq \cdots \supsetneq P_1 \supsetneq (0)$ be the prime ideals of V , and let $\{0\} \subsetneq H_{n-1} \subsetneq \cdots \subsetneq H_1 \subsetneq \Gamma$ be the convex subgroups of Γ . Let m be a positive integer with $n+1 \leq m \leq 2n+1$. Then the following conditions are equivalent: (1) $|\Sigma'(V)| = m$; (2) The maximal ideal of V_{P_i} is principal for exactly $2n+1-m$ of i ; (3) Γ/H_i has a minimal positive element for exactly $2n+1-m$ of i ([M1]).

Received 30 November, 2006; revised 7 March, 2007.

2000 *Mathematics Subject Classification*. Primary 13A15.

Key Words and Phrases. star operation, semistar operation, pseudo-valuation domain, almost pseudo-valuation domain.

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In [M5], we studied star operations and semistar operations on a pseudo-valuation domain (or a PVD) D . We gave conditions for D to have only a finite number of semistar operations, and showed that conditions for $|\Sigma'(D)| < \infty$ reduce to conditions for fields.

In this paper, we concern with star operations and semistar operations on almost pseudo-valuation domains. We study almost pseudo-valuation domains with simple associated valuation rings, and will prove the following,

Theorem Let D be an almost pseudo-valuation domain, let P be the maximal ideal of D , let $V = (P : P)$, let M be the maximal ideal of V , and let $K = V/M$. Assume that the valuation ring V is discrete with rank one and $D/P = K$. Then we have

- (1) If K is a finite field, then $|\Sigma(D)| < \infty$.
- (2) If $P = M^2$, then $|\Sigma(D)| = 1$ and $|\Sigma'(D)| = 3$, and if $P = M^3$, then $|\Sigma(D)| = 3$ and $|\Sigma'(D)| = 6$.
- (3) If $P = M^4$ and K is an infinite field, then $|\Sigma(D)| = \infty$ and $|\Sigma'(D)| = \infty$.
- (4) If $P = M^n$ with $n \geq 5$ and K is an infinite field, then $|\Sigma'(D)| = \infty$.

The paper consists of six sections, Section 1 is a review on almost pseudo-valuation domains, Section 2 is the general case, Section 3 is the case where $P = M^2$ or $P = M^3$, Section 4 is the case where $P = M^4$, Section 5 is the case where $P = M^n$ with $n \geq 5$, and Section 6 is examples.

§1 Review

In this section, we review a result in [M2] on semistar operations on almost pseudo-valuation domains.

Let I be an ideal of a domain D . If $ab \in I$ and $b \notin I$ imply $a^n \in I$ for some $n > 0$ for each elements $a, b \in q(D)$, then I is called a strongly primary ideal of D , where $q(D)$ denotes the quotient field of D . If each prime ideal of D is strongly primary, then D is called an almost pseudo-valuation domain (or an APVD). Every PVD is an almost pseudo-valuation domain. We confer Badawi-Houston([BH]) for almost pseudo-valuation domains.

(1.1) Let D be an APVD, let P be a maximal ideal of D , let $V = (P : P)$, and let M be the maximal ideal of V .

- (1) $F'(D) = F(D) \cup \{q(D)\}$.
- (2) D is a local ring, that is, D has only one maximal ideal.
- (3) If D is not a valuation ring, then $V = P^{-1}$.
- (4) The set of non-maximal prime ideals of D coincides with the set of non-maximal prime ideals of V , and $\dim(V) = \dim(D)$.
- (5) The integral closure $\bar{D}$ of D is a PVD with maximal ideal M .
- (6) Let T be an overring of D . Then either $T \supset V$ or $T \subsetneq V$.
- (7) Let $\Sigma'_1 = \{ * \in \Sigma'(D) \mid D^* \supset V \}$. Then there exists a canonical bijection from $\Sigma'(V)$ onto Σ'_1 .
- (8) Let $\Sigma'_2 = \{ * \in \Sigma'(D) \mid D^* \subsetneq V \}$. Then $\Sigma'(D) = \Sigma'_1 \cup \Sigma'_2$.

(9) If $|\Sigma'(D)| < \infty$, then $\dim(D) < \infty$, $V = \bar{D}$, V is a finitely generated D -module, and V/M is a simple extension field of D/P with $[V/M : D/P] < \infty$.

(1.2) Let D be an APVD which is not a PVD, let P be the maximal ideal of D , and let $V = (P : P)$. Assume that $\dim(D) < \infty$, and let $\{T_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings T of D with $T \subsetneq V$. Let Σ'_1 be the set of semistar operations $*$ on D such that $D^* \supset V$, and let Σ'_2 be the set of semistar operations $*$ on D such that $D^* \subsetneq V$. Then we have

- (1) $\Sigma'(D) = \Sigma'_1 \cup \Sigma'_2$.
- (2) $|\Sigma'(V)| < \infty$.
- (3) $|\Sigma'_1| = |\Sigma'(V)|$.
- (4) There exists a canonical bijection from the disjoint union $\bigcup_\lambda \Sigma(T_\lambda)$ onto Σ'_2 .

§2 The general case

Throughout the paper but the final §6, let D be an APVD which is not a PVD, let P be the maximal ideal of D , let $V = (P : P)$, let M be the maximal ideal of V , let v be a valuation belonging to V , and set $V/M = K$. We assume that v is $\mathbf{Z}$ -valued and that $K = (D + M)/M$, and let $\{\alpha_i \mid i \in \mathcal{I}\} = \mathcal{K}$ be a system of complete representatives of V modulo M , where $\{0, 1\} \subset \{\alpha_i \mid i \in \mathcal{I}\} \subset D$.

Note: Let V be a $\mathbf{Z}$ -valued valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V . Let k be a subfield of K , and let $D = k + M$. Assume that $\dim(D) < \infty$, and K is a simple extension of k with finite degree. Then $|\Sigma(D)|$ need not be finite ([M3]).

There exists $\pi \in V$ such that $M = \pi V$. Then we have $v(\pi) = 1$.

We have $V = P : P = P^{-1}$, where $P : P$ denotes $\{x \in \mathfrak{q}(D) \mid xP \subset P\}$ and P^{-1} denotes $D : P$.

Let $I, J \in \mathfrak{F}(D)$. If there exists $x \in \mathfrak{q}(D)$ such that $xJ = I$, then I and J are said to be similar, and is denoted by $I \sim J$. For each $I \in \mathfrak{F}(D)$, set $\{J \in \mathfrak{F}(D) \mid J \sim I\} = \text{cl}(I)$.

Let Σ'_1 be the set of semistar operations $*$ on D such that $D^* \supset V$, and let Σ'_2 be the set of semistar operations on D such that $D^* \subsetneq V$. We can apply (1.1) and (1.2) for D .

(2.1) Let $x \in \mathfrak{q}(D) - \{0\}$, and let k be a positive integer with $k > v(x)$. Then x can be expressed uniquely as $x = \alpha_l \pi^l + \alpha_{l+1} \pi^{l+1} + \cdots + \alpha_{k-1} \pi^{k-1} + \pi^k a$, where $l = v(x)$ and each $\alpha_i \in \mathcal{K}$ with $\alpha_l \neq 0$ and $a \in V$.

(2.2) There exists a positive integer $n \geq 2$ uniquely so that $P = M^n$.

Proof. Set $\min\{v(x) \mid x \in P\} = n$. Since $PV = P$, we have $P = M^n$. If $n = 1$, then D is a valuation ring; a contradiction.

For every subset $A \subset \mathfrak{q}(D)$, we denote by (A) the D -submodule of $\mathfrak{q}(D)$ generated by A . If $P = M^n$, then we have $P = (\pi^n, \pi^{n+1}, \dots, \pi^{2n-2}, \pi^{2n-1})$, and $V = (1, \pi, \dots, \pi^{n-1})$.

- (2.3) (1) We have $P^v = P$, where P^v denotes $(P^{-1})^{-1}$.
 (2) The set $\text{Spec}(D)$ of prime ideals of D is $\{P, (0)\}$.

(2) follows from (1.1)(4).

The mapping $I \mapsto I^v$ from $F(D)$ to $F(D)$ is a star operation on D , and is called the v -operation. The identity mapping d from $F(D)$ to $F(D)$ is a star operation on D , and is called the d -operation.

We note that V is a divisorial fractional ideal of D , that is, $V^v = V$. Each star operation on D can be uniquely extended to a semistar operation on D . Also let D' be an overring of D . Then there exists a canonical mapping δ from $\Sigma'(D')$ to $\Sigma'(D)$, and is called the descent mapping. And δ is an injective mapping.

We have $|\Sigma'_1| = |\Sigma'(V)| = 2$.

If $P = M^2$, then we have

$$\{I \in F(D) \mid D \subset I \subset V\} = \{(1), (1, \pi)\}.$$

(2.4) **Example** Assume that $P = M^3$. Set $(1) = I_0, (1, \pi^2) = I_{0,2}, (1, \pi, \pi^2) = I_{0,1,2}$, and $(1, \pi + \alpha\pi^2) = I_{0,1}^\alpha$, where $\alpha \in \mathcal{K}$. Then we have
 $\{I \in F(D) \mid D \subset I \subset V\} = \{I_0, I_{0,2}, I_{0,1,2}\} \cup \{I_{0,1}^\alpha \mid \alpha \in \mathcal{K}\}.$

Proof. Because, $\{v(x) \mid x \in I - P\}$ is either $\{0\}$ or $\{0, 1\}$ or $\{0, 2\}$ or $\{0, 1, 2\}$.

(2.5) **Example** Assume that $P = M^4$. For every element $\alpha_i \in \mathcal{K}$, set

$$\begin{aligned} (1) &= I_0, \\ (1, \pi + \alpha_1\pi^2 + \alpha_2\pi^3) &= I_{0,1}^{\alpha_1, \alpha_2}, \\ (1, \pi^2 + \alpha_1\pi^3) &= I_{0,2}^{\alpha_1}, \\ (1, \pi^3) &= I_{0,3}, \\ (1, \pi + \alpha_1\pi^3, \pi^2 + \alpha_2\pi^3) &= I_{0,1,2}^{\alpha_1, \alpha_2}, \\ (1, \pi + \alpha_1\pi^2, \pi^3) &= I_{0,1,3}^{\alpha_1}, \\ (1, \pi^2, \pi^3) &= I_{0,2,3}, \\ (1, \pi, \pi^2, \pi^3) &= I_{0,1,2,3}. \end{aligned}$$

Then we have

$$\{I \in F(D) \mid D \subset I \subset V\} = \{I_0, I_{0,1}^{\alpha_1, \alpha_2}, I_{0,2}^{\alpha_1}, I_{0,3}, I_{0,1,2}^{\alpha_1, \alpha_2}, I_{0,1,3}^{\alpha_1}, I_{0,2,3}, I_{0,1,2,3} \mid \text{each } \alpha_i \in \mathcal{K}\}.$$

Proof. Let I be a fractional ideal of D such that $D \subset I \subset V$. Let $\tau = \{v(x) \mid x \in I - P\}$, say let $\tau = \{0, 1, 3\}$. Then I contains elements $a, b \in V$ of the form $a = \pi + \alpha_1\pi^2, b = \pi^3$, where $\alpha_1 \in \mathcal{K}$. We have $I \supset (1, a, b)$. Let $I \ni x = \beta_0 + \beta_1\pi + \beta_2\pi^2 + \beta_3\pi^3 + p$, where each $\beta_i \in \mathcal{K}$ and $p \in P$. We have $x = \beta_0 + \beta_1a + \beta_3b + \beta'\pi^2 + p'$ for some $\beta' \in \mathcal{K}$ and $p' \in P$. By the choice of τ , we have $\beta' = 0$. Hence $I = (1, a, b)$.

(2.6) **Example** Assume that $P = M^5$. For every element $\alpha_i \in \mathcal{K}$, set

$$\begin{aligned} (1) &= I_0, \\ (1, \pi + \alpha_1\pi^2 + \alpha_2\pi^3 + \alpha_3\pi^4) &= I_{0,1}^{\alpha_1, \alpha_2, \alpha_3}, \\ (1, \pi^2 + \alpha_1\pi^3 + \alpha_2\pi^4) &= I_{0,2}^{\alpha_1, \alpha_2}, \end{aligned}$$

$$\begin{aligned}
 (1, \pi^3 + \alpha_1 \pi^4) &= I_{0,3}^{\alpha_1}, \\
 (1, \pi^4) &= I_{0,4}, \\
 (1, \pi + \alpha_1 \pi^3 + \alpha_2 \pi^4, \pi^2 + \alpha_3 \pi^3 + \alpha_4 \pi^4) &= I_{0,1,2}^{\alpha_1, \alpha_2, \alpha_3, \alpha_4}, \\
 (1, \pi + \alpha_1 \pi^2 + \alpha_2 \pi^4, \pi^3 + \alpha_3 \pi^4) &= I_{0,1,3}^{\alpha_1, \alpha_2, \alpha_3}, \\
 (1, \pi + \alpha_1 \pi^2 + \alpha_2 \pi^3, \pi^4) &= I_{0,1,4}^{\alpha_1, \alpha_2}, \\
 (1, \pi^2 + \alpha_1 \pi^4, \pi^3 + \alpha_2 \pi^4) &= I_{0,2,3}^{\alpha_1, \alpha_2}, \\
 (1, \pi^2 + \alpha_1 \pi^3, \pi^4) &= I_{0,2,4}^{\alpha_1}, \\
 (1, \pi^3, \pi^4) &= I_{0,3,4}, \\
 (1, \pi + \alpha_1 \pi^4, \pi^2 + \alpha_2 \pi^4, \pi^3 + \alpha_3 \pi^4) &= I_{0,1,2,3}^{\alpha_1, \alpha_2, \alpha_3}, \\
 (1, \pi + \alpha_1 \pi^3, \pi^2 + \alpha_2 \pi^3, \pi^4) &= I_{0,1,2,4}^{\alpha_1, \alpha_2}, \\
 (1, \pi + \alpha_1 \pi^2, \pi^3, \pi^4) &= I_{0,1,3,4}^{\alpha_1}, \\
 (1, \pi^2, \pi^3, \pi^4) &= I_{0,2,3,4}, \\
 (1, \pi, \pi^2, \pi^3, \pi^4) &= I_{0,1,2,3,4}.
 \end{aligned}$$

Then we have

$$\{I \in F(D) \mid D \subset I \subset V\} = \{I_0, I_{0,1}^{\alpha_1, \alpha_2, \alpha_3}, I_{0,2}^{\alpha_1, \alpha_2}, I_{0,3}^{\alpha_1}, I_{0,4}, I_{0,1,2}^{\alpha_1, \alpha_2, \alpha_3, \alpha_4}, I_{0,1,3}^{\alpha_1, \alpha_2, \alpha_3}, I_{0,1,4}^{\alpha_1, \alpha_2}, I_{0,2,3}^{\alpha_1, \alpha_2}, I_{0,2,4}^{\alpha_1}, I_{0,3,4}, I_{0,1,2,3}^{\alpha_1, \alpha_2, \alpha_3}, I_{0,1,2,4}^{\alpha_1, \alpha_2}, I_{0,1,3,4}^{\alpha_1}, I_{0,2,3,4}, I_{0,1,2,3,4} \mid \text{each } \alpha_i \in \mathcal{K}\}.$$

Proof. Let I be a fractional ideal of D such that $D \subset I \subset V$. Let $\tau = \{v(x) \mid x \in I - P\}$, say let $\tau = \{0, 1, 3\}$. Then I contains elements $a, b \in V$ of the form $a = \pi + \alpha_1 \pi^2 + \alpha_2 \pi^4, b = \pi^3 + \alpha_3 \pi^4$, where $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}$. We have $I \supset (1, a, b)$. Let $I \ni x = \beta_0 + \beta_1 \pi + \beta_2 \pi^2 + \beta_3 \pi^3 + \beta_4 \pi^4 + p$, where each $\beta_i \in \mathcal{K}$ and $p \in P$. We have $x = \beta_0 + \beta_1 a + \beta_3 b + \beta'_1 \pi^2 + \beta'_2 \pi^4 + p'$ for some $\beta'_i \in \mathcal{K}$ and $p' \in P$. By the choice of τ , we have $\beta'_1 = \beta'_2 = 0$. Hence $I = (1, a, b)$.

Each subset τ of $\{0, 1, 2, 3, 4\}$ which contains 0 is called a type associated to D . We have the number 16 of associated types to D . The set of types has a canonical order so that $\{0\}$ is the minimal member and $\{0, 1, 2, 3, 4\}$ is the maximal member: $\{0\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 3\}, \{0, 2, 4\}, \{0, 3, 4\}, \{0, 1, 2, 3\}, \{0, 1, 2, 4\}, \{0, 1, 3, 4\}, \{0, 2, 3, 4\}, \{0, 1, 2, 3, 4\}$. Let τ be a type associated to D , say let $\tau = \{0, 1, 2\}$. Then the tuple $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ of elements in $\mathcal{K}$ is called a system of parameters of τ (or, a system of parameters associated to τ), and 4 is called the length of the system of parameters $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$. The pair $\langle 0, 1, 2; \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle = \sigma$ of τ and $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is called a data of τ (or, a data associated to τ). Set $1 = f_1^\sigma, \pi + \alpha_1 \pi^3 + \alpha_2 \pi^4 = f_2^\sigma, \pi^2 + \alpha_3 \pi^3 + \alpha_4 \pi^4 = f_3^\sigma$. Then the tuple $\langle f_1^\sigma, f_2^\sigma, f_3^\sigma \rangle$ is called an associated system of generators to σ . The fractional ideal $(f_1^\sigma, f_2^\sigma, f_3^\sigma) = I$ is called associated to σ , and is denoted by I^σ . The tuple $\langle f_1^\sigma, f_2^\sigma, f_3^\sigma \rangle$ is also called a canonical system of generators for I associated to σ .

Assume that $P = M^n$ for a positive integer $n \geq 2$. We confer the previous examples. Each subset τ of $\{0, 1, 2, \dots, n-1\}$ which contains 0 is called a type associated to D . We have the number 2^{n-1} of associated types to D . The set of types has a canonical order so that $\{0\} = \tau_1$ is the minimal member and $\{0, 1, \dots, n-1\} = \tau_{2^{n-1}}$ is the maximal member. Let $\tau = \{0, k_1, \dots, k_m\}$ be a type associated to D with $0 < k_1 < \dots < k_m$. We can define a system of parameters $\langle \alpha_1, \dots, \alpha_l \rangle$ of τ . It is a tuple of elements in $\mathcal{K}$. The pair $\langle 0, k_1, \dots, k_m; \alpha_1, \dots, \alpha_l \rangle = \sigma$ of τ and

$\langle \alpha_1, \dots, \alpha_l \rangle$ is called a data of τ (or, a data associated to τ). We can define an associated system $\langle f_1^\sigma, f_2^\sigma, \dots, f_{m+1}^\sigma \rangle$ of generators to σ . It is a tuple of elements in V . We denote also $f_i^\sigma = f_i(\sigma)$. The fractional ideal $(f_1^\sigma, f_2^\sigma, \dots, f_{m+1}^\sigma) = I$ is said to be associated to σ , and is denoted by I^σ (or, by $I(\sigma)$). The tuple $\langle f_1^\sigma, f_2^\sigma, \dots, f_{m+1}^\sigma \rangle$ is also called a canonical system of generators for I associated to σ .

(2.7) Proposition Assume that $P = M^n$ with $n \geq 2$. Then we have
 $\{I \in F(D) \mid D \subset I \subset V\} = \{I(\sigma_1), \dots, I(\sigma_{2^n-1}) \mid \text{each } \sigma_i \text{ is a data associated to the type } \tau_i \text{ for } 1 \leq i \leq 2^{n-1}\}.$

§3 The case where $P = M^2$ or $P = M^3$

(3.1) Proposition Assume that $P = M^2$. Then we have $|\Sigma(D)| = 1$ and $|\Sigma'(D)| = 3$.

Proof. Since $\pi^2(1, \pi) = P$, each $I \in F(D)$ is divisorial. It follows that $|\Sigma(D)| = 1$. Let T be an overring of D with $V \supsetneq T \supsetneq D$, and take $t \in T - D$. There may arise the following two cases: (1) $v(t) = 1$, and (2) $v(t) = 0$.

Case (1): We may assume that $t = \pi + p$ for some $p \in P$. Hence we have $T \ni \pi$, and hence $T = V$; a contradiction.

Case (2): We may assume that $t = 1 + \alpha\pi + p$ for some $\alpha \in \mathcal{K}$ and $p \in P$. Since $t \notin D$, we have $\alpha\pi \in T - D$. Case (1) implies that $T = V$; a contradiction.

We will apply (1.2). Since $|\Sigma'_1| = 2$, we have $|\Sigma'(D)| = 2 + |\Sigma(D)| = 3$.

Throughout the rest of the section, assume that $P = M^3$.

(3.2) We have

$$\{T \mid T \text{ is an overring of } D \text{ with } T \subsetneq V\} = \{D, D + M^2\}.$$

Proof. Let T be an overring of D with $V \supsetneq T \supsetneq D$, and take $t \in T - D$. The proof of (3.1) shows that, if $T \supsetneq D + M^2$ then $T = V$. There may arise the following three cases: (1) $v(t) = 2$, (2) $v(t) = 1$, and (3) $v(t) = 0$.

Case (1): We may assume that $t = \pi^2 + p$ for some $p \in P$. Hence we have $T \ni \pi^2$, and $T \supset D + M^2$.

Case (2): We may assume that $t = \pi + \alpha\pi^2 + p$ for some $\alpha \in \mathcal{K}$ and $p \in P$. Since $T \ni t^2$, we have $T \ni \pi^2$, and hence $T \supset D + M^2$.

Case (3): We may assume that $t = 1 + \alpha_1\pi + \alpha_2\pi^2 + p$ for some $\alpha_1, \alpha_2 \in \mathcal{K}$ and $p \in P$. Since $t \notin D$, we have $\alpha_1\pi + \alpha_2\pi^2 \in T - D$. Cases 1 and 2 imply that $T \supset D + M^2$.

(3.3) (1) $I_{0,2}$ and $I_{0,1}^\alpha$ are incomparable for each $\alpha \in \mathcal{K}$.

(2) $I_{0,1}^\alpha \subset I_{0,1}^\beta$ if and only if $\alpha = \beta$.

Proof. (2) Assume that $I_{0,1}^\alpha \subset I_{0,1}^\beta$. Then $\pi + \alpha\pi^2 \in (1, \pi + \beta\pi^2)$ implies $\pi + \alpha\pi^2 = (\pi + \beta\pi^2) + p$ for some $p \in P$. Hence $\alpha = \beta$.

- (3.4) (1) Each two in $\{I_0, I_{0,2}, I_{0,1,2}, I_{0,1}^\alpha\}$ are not similar for each $\alpha \in \mathcal{K}$.
 (2) $I_{0,1}^\alpha$ and $I_{0,1}^\beta$ are similar for each $\alpha, \beta \in \mathcal{K}$.

Proof. (2) Set $1 + \alpha\pi + \alpha^2\pi^2 = x$. Then we have $x(1, \pi) = (1, \pi + \alpha\pi^2)$.

(3.5) Let $*$ be a star operation on D . Then $(I_{0,2})^*$ is either $I_{0,2}$ or V , and $(I_{0,1}^0)^*$ is either $I_{0,1}^0$ or V .

Proof. Since V is a divisorial fractional ideal of D , we have $(I_{0,2})^* \subset V$ and $(I_{0,1}^0)^* \subset V$.

(3.6) (1) If we set $(I_{0,2})^{*1} = I_{0,2}$ and $(I_{0,1}^0)^{*1} = I_{0,1}^0$, then there is determined a unique mapping $*_1$ from $F(D)$ to $F(D)$.

(2) If we set $(I_{0,2})^{*2} = I_{0,2}$ and $(I_{0,1}^0)^{*2} = V$, then there is determined a unique mapping $*_2$ from $F(D)$ to $F(D)$.

(3) If we set $(I_{0,2})^{*3} = V$ and $(I_{0,1}^0)^{*3} = I_{0,1}^0$, then there is determined a unique mapping $*_3$ from $F(D)$ to $F(D)$.

(4) If we set $(I_{0,2})^{*4} = V$ and $(I_{0,1}^0)^{*4} = V$, then there is determined a unique mapping $*_4$ from $F(D)$ to $F(D)$.

For, each element $I \in F(D)$ is similar to one and only one in $\{I_0, I_{0,2}, I_{0,1,2}, I_{0,1}^0\}$.

- (3.7) (1) $*_1$ is a star operation on D , and $*_1 = d$.
 (2) $*_2$ is a star operation on D .
 (3) $*_3$ is not a star operation on D .
 (4) $*_4$ is a star operation on D , and $*_4 = v$.

Proof. (2) For each $x \in q(D) - \{0\}$, we have $(x)^{*2} = (x)$.

For each $x \in q(D) - \{0\}$ and each $I \in F(D)$, we have $(xI)^{*2} = xI^{*2}$.

For each $I \in F(D)$, we have $I \subset I^{*2}$.

For each $I \in F(D)$, we have $(I^{*2})^{*2} = I^{*2}$.

Let $I_1, I_2 \in F(D)$ with $I_1 \subset I_2$. To prove $I_1^{*2} \subset I_2^{*2}$, it is sufficient to show that, if $xI_{0,1}^0 \subset J$ then $xV \subset J$ for each $x \in q(D) - \{0\}$ and each $J \in \{I_0, I_{0,2}, I_{0,1,2}\}$.

(3) Set $\pi + \pi^2 = x$. Then we have $x(1, \pi^2) \subset (1, \pi + \pi^2)$ and $xV \not\subset (1, \pi + \pi^2)$.

(3.8) Proposition Assume that $P = M^3$. Then we have $|\Sigma(D)| = 3$ and $|\Sigma'(D)| = 6$.

Proof. It follows that $\Sigma(D) = \{d, v, *_2\}$, and that $|\Sigma(D)| = 3$. We can apply (3.1) for $D' = D + M^2$. Then we have

$$|\Sigma'_2| = |\Sigma(D)| + |\Sigma(D + M^2)| = 3 + 1 = 4.$$

Since $|\Sigma'(V)| = 2$, it follows that

$$|\Sigma'(D)| = |\Sigma'_1| + |\Sigma'_2| = 2 + 4 = 6.$$

§4 The case where $P = M^4$

In this section, we assume that $P = M^4$.

(4.1) Proposition If K is a finite field, then we have $|\Sigma(D)| < \infty$.

Proof. Let

$$X = \{I_0, I_{0,1}^{\alpha_1, \alpha_2}, I_{0,2}^{\alpha_1}, I_{0,3}, I_{0,1,2}^{\alpha_1, \alpha_2}, I_{0,1,3}^{\alpha_1}, I_{0,2,3}, I_{0,1,2,3} \mid \text{each } \alpha_i \in \mathcal{K}\}.$$

Then X is a finite set. Let $*$ be a star operation on D . Since V is a divisorial fractional ideal of D , we have $D \subset I^* \subset V$ for each $I \in X$. If we set $I^* = g_*(I)$, each element $*$ in $\Sigma(D)$ gives rise to an element $g_* \in X^X$, where X^X is the set of mappings from X to X . And the mapping $g : * \mapsto g_*$ from $\Sigma(D)$ to X^X is an injection.

(4.2) Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{K}$.

$$I_{0,1}^{\alpha_1, \alpha_2} \subset I_{0,1}^{\beta_1, \beta_2} \text{ if and only if } \alpha_1 = \beta_1, \alpha_2 = \beta_2.$$

$$I_{0,2}^{\alpha_1} \subset I_{0,2}^{\beta_1} \text{ if and only if } \alpha_1 = \beta_1.$$

$$I_{0,1,2}^{\alpha_1, \alpha_2} \subset I_{0,1,2}^{\beta_1, \beta_2} \text{ if and only if } \alpha_1 = \beta_1, \alpha_2 = \beta_2.$$

$$I_{0,1,3}^{\alpha_1} \subset I_{0,1,3}^{\beta_1} \text{ if and only if } \alpha_1 = \beta_1.$$

Proof. For instance, assume that $I_{0,1,2}^{\alpha_1, \alpha_2} \subset I_{0,1,2}^{\beta_1, \beta_2}$. Then we have

$$\pi + \alpha_1 \pi^3 = (\pi + \beta_1 \pi^3) + (\pi^2 + \beta_3 \pi^3) p_1 + p_2$$

for some $p_i \in P$. Hence $\alpha_1 = \beta_1$. We have $\pi^2 + \alpha_2 \pi^3 = (\pi^2 + \beta_2 \pi^3) + p_3$ for some $p_3 \in P$. Hence $\alpha_2 = \beta_2$.

(4.3) (1) $I_{0,1}^{\alpha_1, \alpha_2} \sim I_{0,1}^{\beta_1, \beta_2}$ if and only if $\beta_1^2 - \beta_2 \equiv \alpha_1^2 - \alpha_2 \pmod{P}$.

(2) $I_{0,2}^{\alpha} \sim I_{0,2}^{\beta}$ if and only if $\alpha = \beta$.

(3) $I_{0,1,2}^{\alpha_1, \alpha_2} \sim I_{0,1,2}^{\beta_1, \beta_2}$ for each $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{K}$.

(4) $I_{0,1,3}^{\alpha} \sim I_{0,1,3}^{\beta}$ for each $\alpha, \beta \in \mathcal{K}$.

Proof. (1) Assume that $\beta_1^2 - \beta_2 \equiv \alpha_1^2 - \alpha_2 \pmod{P}$. Set $1 + (\pi + \beta_1 \pi^2 + \beta_2 \pi^3)(\beta_1 - \alpha_1) = x$. Then we have $x I_{0,1}^{\alpha_1, \alpha_2} = I_{0,1}^{\beta_1, \beta_2}$.

(3) Set $1 - \alpha_2 \pi - \alpha_1 \pi^2 = x$. Then we have $x I_{0,1,2}^{\alpha_1, \alpha_2} = I_{0,1,2}^{0,0}$.

(4) Set $1 - \alpha \pi = x$. Then we have $x I_{0,1,3}^{\alpha} = I_{0,1,3}^0$.

(4.4) Each two in $\{I_0, I_{0,1}^{\alpha_1, \alpha_2}, I_{0,2}^{\alpha_3}, I_{0,3}, I_{0,1,2}^{\alpha_4, \alpha_5}, I_{0,1,3}^{\alpha_6}, I_{0,2,3}, I_{0,1,2,3}\}$ are not similar for each $\alpha_i \in \mathcal{K}$.

Proof. For instance, suppose that there exists $x \in q(D)$ so that $I_{0,1}^{\alpha_1, \alpha_2} = x I_{0,1,3}^{\alpha_6}$. We may assume that $x = 1 + \beta_1 \pi + \beta_2 \pi^2 + \beta_3 \pi^3 + p$ for some $\beta_i \in \mathcal{K}$ and $p \in P$. Then $x \pi^3 \in I_{0,1}^{\alpha_1, \alpha_2}$ implies a contradiction.

(4.5) Let $x \in q(D) - \{0\}$.

(1) $x I_{0,1}^{\alpha_1, \alpha_2} \subset I_0$ implies $xV \subset I_0$.

(2) $x I_{0,1,2}^{\alpha_1, \alpha_2} \subset I_0$ implies $xV \subset I_0$.

- (3) $xI_{0,1,3}^{\alpha_1} \subset I_0$ implies $xV \subset I_0$.
- (4) $xI_{0,1}^{\alpha_{(1)}, \alpha_{(2)}} \subset I_{0,1}^{\alpha_1, \alpha_2}$ with $\alpha_{(1)}^2 - \alpha_{(2)} \not\equiv \alpha_1^2 - \alpha_2 \pmod{P}$ implies $xV \subset I_{0,1}^{\alpha_1, \alpha_2}$.
- (5) $xI_{0,1}^{\alpha_1, \alpha_2} \subset I_{0,2}^\beta$ implies $xV \subset I_{0,2}^\beta$.
- (6) $xI_{0,1}^{\alpha_1, \alpha_2} \subset I_{0,3}$ implies $xV \subset I_{0,3}$.
- (7) $xI_{0,1}^{\alpha_1, \alpha_2} \subset I_{0,2,3}$ implies $xV \subset I_{0,2,3}$.
- (8) $xI_{0,1,2}^{\alpha_1, \alpha_2} \subset I_{0,1}^{\beta_1, \beta_2}$ implies $xV \subset I_{0,1}^{\beta_1, \beta_2}$.
- (9) $xI_{0,1,3}^{\alpha_1, \alpha_2} \subset I_{0,1}^{\beta_1, \beta_2}$ implies $xV \subset I_{0,1}^{\beta_1, \beta_2}$.
- (10) $xI_{0,1,2}^{\alpha_1, \alpha_2} \subset I_{0,2}^\beta$ implies $xV \subset I_{0,2}^\beta$.
- (11) $xI_{0,1,3}^{\alpha_1, \alpha_2} \subset I_{0,2}^\beta$ implies $xV \subset I_{0,2}^\beta$.
- (12) $xI_{0,1,2}^{\alpha_1, \alpha_2} \subset I_{0,3}$ implies $xV \subset I_{0,3}$.
- (13) $xI_{0,1,3}^{\alpha_1, \alpha_2} \subset I_{0,3}$ implies $xV \subset I_{0,3}$.
- (14) $xI_{0,1,2}^{\alpha_1, \alpha_2} \subset I_{0,2,3}$ implies $xV \subset I_{0,2,3}$.
- (15) $xI_{0,1,3}^{\alpha_1, \alpha_2} \subset I_{0,2,3}$ implies $xV \subset I_{0,2,3}$.

Proof. (4) We may assume that $v(x) = 0$. Then we may assume that $x = 1 + (\pi + \alpha_1\pi^2 + \alpha_2\pi^3)\alpha + p$ for some $\alpha \in \mathcal{K}$ and $p \in P$. Then $x(\pi + \alpha_{(1)}\pi^2 + \alpha_{(2)}\pi^3) \in I_{0,1}^{\alpha_1, \alpha_2}$ implies $\alpha_{(1)}^2 - \alpha_{(2)} \equiv \alpha_1^2 - \alpha_2 \pmod{P}$; a contradiction. It follows that $x \in M^4$, and hence $xV \subset I_{0,1}^{\alpha_1, \alpha_2}$.

(4.6) Fix a data $\langle 0, 1; \alpha_{(1)}, \alpha_{(2)} \rangle$, and set $(I_{0,1}^{\alpha_{(1)}, \alpha_{(2)}})^* = V$. For $I_{0,1}^{\alpha_1, \alpha_2}$ with $\alpha_1^2 - \alpha_2 \not\equiv \alpha_{(1)}^2 - \alpha_{(2)} \pmod{P}$, set $(I_{0,1}^{\alpha_1, \alpha_2})^* = I_{0,1}^{\alpha_1, \alpha_2}$. For each $\beta_1, \beta_2 \in \mathcal{K}$, set $(I_{0,2}^{\beta_1})^* = I_{0,2}^{\beta_1}$, $(I_{0,1,2}^{\beta_1, \beta_2})^* = V$, $(I_{0,1,3}^{\beta_1})^* = V$, and set $(I_{0,3})^* = I_{0,3}$, $(I_{0,2,3})^* = I_{0,2,3}$.

Then we have

- (1) There is determined a unique mapping $*$ from $F(D)$ to $F(D)$.
- (2) For each $x \in q(D) - \{0\}$, we have $(x)^* = (x)$.
- (3) For each $x \in q(D) - \{0\}$ and each $I \in F(D)$, we have $(xI)^* = xI^*$.
- (4) For each $I \in F(D)$, we have $I \subset I^*$.
- (5) For each $I \in F(D)$, we have $(I^*)^* = I^*$.
- (6) For each $I_1, I_2 \in F(D)$ with $I_1 \subset I_2$, we have $I_1^* \subset I_2^*$.

(6) follows from (4.5).

(4.7) **Proposition** Assume that $P = M^4$ and K is an infinite field. Then we have $|\Sigma(D)| = \infty$ and $|\Sigma'(D)| = \infty$.

Proof. Let $*_{\alpha_{(1)}, \alpha_{(2)}}$ be the star operation on D determined in (4.6). If $I_{0,1}^{\alpha_1, \alpha_2} \not\subset I_{0,1}^{\beta_1, \beta_2}$, we have $*_{\alpha_1, \alpha_2} \neq *_{\beta_1, \beta_2}$. It follows that $|\Sigma(D)| = \infty$.

§5 The case where $P = M^n$ with $n \geq 5$

(5.1) **Proposition** Assume that $P = M^5$ and K is a finite field. Then we have $|\Sigma(D)| < \infty$.

Proof. Let

$$X = \{I_0, I_{0,1}^{\alpha_1, \alpha_2, \alpha_3}, I_{0,2}^{\alpha_1, \alpha_2}, I_{0,3}^{\alpha_1}, I_{0,4}, I_{0,1,2}^{\alpha_1, \alpha_2, \alpha_3, \alpha_4}, I_{0,1,3}^{\alpha_1, \alpha_2, \alpha_3}, I_{0,1,4}^{\alpha_1, \alpha_2}, I_{0,2,3}^{\alpha_1, \alpha_2}, I_{0,2,4}^{\alpha_1}, I_{0,3,4}^{\alpha_1, \alpha_2, \alpha_3}, I_{0,1,2,3}^{\alpha_1, \alpha_2}, I_{0,1,2,4}^{\alpha_1}, I_{0,1,3,4}^{\alpha_1}, I_{0,2,3,4}, I_{0,1,2,3,4} \mid \text{each } \alpha_i \in \mathcal{K}\}.$$

The similar argument to the proof of (4.1) completes the proof.

(5.2) Assume that $P = M^5$. Let $\alpha_i, \beta_j \in \mathcal{K}$ for each i, j .

$$I_{0,1}^{\alpha_1, \alpha_2, \alpha_3} \subset I_{0,1}^{\beta_1, \beta_2, \beta_3} \text{ if and only if } \alpha_i = \beta_i \text{ for each } i.$$

$$I_{0,2}^{\alpha_1, \alpha_2} \subset I_{0,2}^{\beta_1, \beta_2} \text{ if and only if } \alpha_i = \beta_i \text{ for each } i.$$

$$I_{0,3}^{\alpha_1} \subset I_{0,3}^{\beta_1} \text{ if and only if } \alpha_1 = \beta_1.$$

$$I_{0,1,2}^{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \subset I_{0,1,2}^{\beta_1, \beta_2, \beta_3, \beta_4} \text{ if and only if } \alpha_i = \beta_i \text{ for each } i.$$

$$I_{0,1,3}^{\alpha_1, \alpha_2, \alpha_3} \subset I_{0,1,3}^{\beta_1, \beta_2, \beta_3} \text{ if and only if } \alpha_i = \beta_i \text{ for each } i.$$

$$I_{0,1,4}^{\alpha_1, \alpha_2} \subset I_{0,1,4}^{\beta_1, \beta_2} \text{ if and only if } \alpha_i = \beta_i \text{ for each } i.$$

$$I_{0,2,3}^{\alpha_1, \alpha_2} \subset I_{0,2,3}^{\beta_1, \beta_2} \text{ if and only if } \alpha_i = \beta_i \text{ for each } i.$$

$$I_{0,2,4}^{\alpha_1} \subset I_{0,2,4}^{\beta_1} \text{ if and only if } \alpha_1 = \beta_1.$$

$$I_{0,1,2,3}^{\alpha_1, \alpha_2, \alpha_3} \subset I_{0,1,2,3}^{\beta_1, \beta_2, \beta_3} \text{ if and only if } \alpha_i = \beta_i \text{ for each } i.$$

$$I_{0,1,2,4}^{\alpha_1, \alpha_2} \subset I_{0,1,2,4}^{\beta_1, \beta_2} \text{ if and only if } \alpha_i = \beta_i \text{ for each } i.$$

$$I_{0,1,3,4}^{\alpha_1} \subset I_{0,1,3,4}^{\beta_1} \text{ if and only if } \alpha_1 = \beta_1.$$

Proof. For instance, assume that $I_{0,1,2}^{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \subset I_{0,1,2}^{\beta_1, \beta_2, \beta_3, \beta_4}$. Then we have

$$\pi + \alpha_1 \pi^3 + \alpha_2 \pi^4 = (\pi + \beta_1 \pi^3 + \beta_2 \pi^4) + (\pi^2 + \beta_3 \pi^3 + \beta_4 \pi^4) p_1 + p_2$$

for some $p_i \in P$. Hence $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. We have

$$\pi^2 + \alpha_3 \pi^3 + \alpha_4 \pi^4 = (\pi^2 + \beta_3 \pi^3 + \beta_4 \pi^4) + p_3$$

for some $p_3 \in P$. Hence $\alpha_3 = \beta_3$ and $\alpha_4 = \beta_4$.

(5.3) Assume that $P = M^5$. Then each two in $\{I_0, I_{0,1}^{\alpha_1, \alpha_2, \alpha_3}, I_{0,2}^{\alpha_4, \alpha_5}, I_{0,3}^{\alpha_6}, I_{0,4}, I_{0,1,2}^{\alpha_7, \alpha_8, \alpha_9, \alpha_{10}}, I_{0,1,3}^{\alpha_{11}, \alpha_{12}, \alpha_{13}}, I_{0,1,4}^{\alpha_{14}, \alpha_{15}}, I_{0,2,3}^{\alpha_{16}, \alpha_{17}}, I_{0,2,4}^{\alpha_{18}}, I_{0,3,4}, I_{0,1,2,3}^{\alpha_{19}, \alpha_{20}, \alpha_{21}}, I_{0,1,2,4}^{\alpha_{22}, \alpha_{23}}, I_{0,1,3,4}^{\alpha_{24}}, I_{0,2,3,4}, I_{0,1,2,3,4}\}$ are not similar for each $\alpha_i \in \mathcal{K}$.

Proof. Because the each two have distinct types.

(5.4) **Proposition** Assume that $P = M^n$ with $n \geq 6$ and K is a finite field. Then we have $|\Sigma(D)| < \infty$.

The proof is similar to that of (5.1).

(5.5) **Proposition** Assume that $P = M^n$ with $n \geq 5$ and K is an infinite field. Then we have $|\Sigma'(D)| = \infty$.

Proof. Set $D + M^4 = D'$. Then D' is an APVD with maximal ideal M^4 . D' is not a PVD. We have $|\Sigma(D')| = \infty$ by (4.7). Hence $|\Sigma'(D')| = \infty$. Since the descent map δ from $\Sigma'(D')$ to $\Sigma'(D)$ is an injection, we have $|\Sigma'(D)| = \infty$.

The proof of our Theorem is complete.

§6 Examples

In this final section, D does not necessarily denote an APVD which is not a PVD. We will apply our Theorem to some APVD's.

(6.1) Proposition Let V be a rank one discrete valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V , and let $D = K + M^n$ for a positive integer $n \geq 2$. Then we have

- (1) If K is a finite field, then $|\Sigma(D)| < \infty$.
- (2) If $n = 2$, then $|\Sigma(D)| = 1$ and $|\Sigma'(D)| = 3$, and if $n = 3$, then $|\Sigma(D)| = 3$ and $|\Sigma'(D)| = 6$.
- (3) If $n = 4$ and K is an infinite field, then $|\Sigma(D)| = \infty$ and $|\Sigma'(D)| = \infty$.
- (4) If $n \geq 5$ and K is an infinite field, then $|\Sigma'(D)| = \infty$.

Proof. Then D is an APVD which is not a PVD, $P = M^n$ is the maximal ideal of D , $V = (P : P)$, and $(D + M)/M = V/M$.

(6.2) Let $V = K[[X]]$ be the formal power series ring of a variable X over a field K , let M be the maximal ideal of V , and let $D = K + M^n$ for a positive integer $n \geq 2$. Then we have the same (1) $\sim$ (4) of (6.1).

(6.3) Let V be a rank one discrete valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V , and let $D = k + M^n$ for a positive integer $n \geq 2$ and for a subfield k of K . Then, if $n \geq 4$ and K is an infinite field, then $|\Sigma'(D)| = \infty$.

Proof. Set $K + M^n = D'$. Then $|\Sigma'(D')| = \infty$ by (6.1). Since the descent map δ from $\Sigma'(D')$ to $\Sigma'(D)$ is an injection, we have $|\Sigma'(D)| = \infty$.

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