Kronecker function rings of semistar operations on semigroups, II

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Abstract

We study the Kronecker function ring of any semistar operation on a grading monoid.

Introduction

We know that various terms in ideal theory are defined analogously for commutative semigroups; those are ideal, integral, divisor, dimension, valuation, star operation, etc. Let G be a torsion-free abelian (additive) group and let S be its subsemigroup containing the zero element. Then S is called a *grading monoid* (or a *g-monoid*). A motivation and an outline of ideas for ideal theory of a grading monoid are as follows: Almost all of ideal theory of a commutative ring R concern properties of ideals of R with respect to the multiplication on R . Abandoning the addition on R we will extract the multiplication on R . Then we have an idea of an algebraic system S of a semigroup which is called a grading monoid.

We already have the Kronecker function ring theory of an e.a.b. semistar operation on a g-monoid ([M2]). In 2001, M. Fontana and K. Loper [FL] outlined a general approach to the theory of Kronecker function rings of an integral domain by semistar operations. In this paper, after them, we will define a Kronecker function ring $Kr(S, *)$ of any semistar operation $*$ on a g-monoid S and will study it. We refer to [G], [GP1, 2] and [M4] for the general theory of a commutative semigroup ring, and [M3] for the general theory of a grading monoid.

1. Preliminary results on semistar operations

Let S be a g-monoid with quotient group G . Let E be a non-empty subset of G such that $S + E \subset E$ with $s + E \subset S$ for some $s \in S$. Then E is called a fractional ideal of S . We denote the set of all fractional ideals of S by $F(S)$. A non-empty subset E of G is called an S -submodule of G if $S + E \subset E$. We denote the set of all S -submodules

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of G by $\bar{F}(S)$. The set of all finitely generated members in $F(S)$ is denoted by $f(S)$.

Definition (1.1) ([OMS]) A map $*$: $\bar{F}(S) \longrightarrow \bar{F}(S)$, $E \longmapsto E^*$, is called a semistar operation on S if, for all $x \in G$, and for all $E, F \in \bar{F}(S)$, the following conditions hold:

- (1) $(x+E)^* = x+E^*$; (2) $E \subset E^*$; (3) $E \subset F$ implies $E^* \subset F^*$; (4) $(E^*)^* = E^*$.

We denote the set of all the semistar operations on S by $\text{SStar}(S)$.

Let $E, F \in \bar{F}(S)$. Then we denote the set $\{x \in G \mid x + F \subset E\}$ by $(E : F)$.

Lemma (1.2) Let $*$ be a semistar operation on S , and let $E, F \in \bar{F}(S)$. Then we have $(E : F)^* \subset (E^* : F^*) = (E^* : F)$.

Proof. Since $(E : F) + F \subset E$, we have $(E : F)^* + F^* \subset E^*$. Hence $(E : F)^* \subset (E^* : F^*)$.

Let $\mathcal{S} = \{S_\lambda \mid \lambda \in \Lambda\}$ be a family of oversemigroups of S . Then the semistar operation $E \longmapsto \cap_\lambda (E + S_\lambda)$ on S is denoted by $*_{\mathcal{S}}$.

A mapping $E \longmapsto E^*$ of $F(S)$ to $F(S)$ is called a star operation on S if the following conditions hold for all $x \in G$ and for all $E, F \in F(S)$ ([M1]):

- (1) $(x)^* = (x)$; (2) $(x + E)^* = x + E^*$; (3) $E \subset E^*$; (4) If $E \subset F$, then $E^* \subset F^*$; (5) $(E^*)^* = E^*$.

Let $*$ be a star operation on S . If we set $E^{**} = E^*$ for all $E \in F(S)$, and $E^{**} = G$ for all $E \in \bar{F}(S) - F(S)$, then $*_e$ is a semistar operation on S and is called *the trivial extension of $*$ to a semistar operation*.

Let $*$ be a semistar operation on S . For each $E \in \bar{F}(S)$, set $E^{*f} = \cup\{F^* \mid F \in f(S) \text{ with } F \subset E\}$. Then $*_f$ is a semistar operation on S , and is called *the finite semistar operation associated to $*$* . A semistar operation $*$ is said to be *of finite type* if $* = *_f$. Since $(*_f)_f = *_f$, $*_f$ is of finite type.

For any subset E of G , the subset $(S : E)$ is also denoted by E^{-1} (We set $\emptyset^{-1} = G$). The mapping $E \longmapsto E^v = (E^{-1})^{-1}$ of $\bar{F}(S)$ to $\bar{F}(S)$ is a semistar operation on S and is called *the v -semistar operation* on S .

The finite semistar operation associated to the v -semistar operation is called *the t -semistar operation* on S .

If $\mathcal{W}$ is a family of valuation oversemigroups of S , then $*_{\mathcal{W}}$ is called a *w -semistar operation (associated to $\mathcal{W}$)*. If $\mathcal{W}$ is the family of all the valuation oversemigroups of S , then $*_{\mathcal{W}}$ is called *the b -semistar operation* on S .

Let $*_1, *_2$ be semistar operations on S . If $(*_1)_f = (*_2)_f$, then $*_1$ and $*_2$ are said to be equivalent, and is denoted by $*_1 \sim *_2$. By definition, $*_1$ and $*_2$ are equivalent if and only if $E^{*1} = E^{*2}$ for each $E \in f(S)$.

Definition (1.3). A semistar operation $*$ on S is said to be e.a.b. (endlich arithmetisch brauchbar) if, for all $A, B, C \in f(S)$, $(A + B)^* \subset (A + C)^*$ implies $B^* \subset C^*$, and is said to be a.b. (arithmetisch brauchbar) if, for all $A \in f(S)$ and for all $B, C \in \bar{F}(S)$, $(A + B)^* \subset (A + C)^*$ implies $B^* \subset C^*$.

Lemma (1.4). Let $*$ be a semistar operation on S . Then the following conditions are equivalent:

- (1) $*$ is e.a.b.
- (2) Let $A, B \in f(S)$ such that $A^* \subset (A + B)^*$. Then $0 \in B^*$.
- (3) Let $A, B \in f(S)$. Then $((A + B)^* : A) \subset B^*$.
- (4) Let $A, B, C \in f(S)$ such that $(A + B)^* = (A + C)^*$. Then $B^* = C^*$.

Proof. (1) $\implies$ (2): Then $(A + S)^* \subset (A + B)^*$, hence $S^* \subset B^*$.

(2) $\implies$ (3): Let $x \in ((A + B)^* : A)$, then $x + A \subset (A + B)^*$. Then we have $A \subset (A + B - x)^*$. Hence $0 \in (B - x)^*$, and hence $x \in B^*$.

(3) $\implies$ (4): Then $A + B \subset (A + C)^*$, hence $B \subset ((A + C)^* : A)$. It follows that $B^* \subset C^*$. Similarly, we have $C^* \subset B^*$.

(4) $\implies$ (1): If $(A + B)^* \subset (A + C)^*$, then

$$(A + C)^* = ((A + B)^*, (A + C)^*)^* = (A + B, A + C)^* = (A + (B, C))^*.$$

Therefore, $C^* = (B, C)^*$, thus $B^* \subset C^*$.

Proposition (1.5) Let T be an oversemigroup of S , and let $*$ be a semistar operation on S . Then we define $\alpha_T(*) : \bar{F}(T) \longrightarrow \bar{F}(T)$ by setting:

$$E^{\alpha_T(*)} = E^* \text{ for each } E \in \bar{F}(T) \subset \bar{F}(S).$$

- (1) $\alpha_T(*)$ is a semistar operation on T .
- (2) If $*$ is of finite type on S , then $\alpha_T(*)$ is of finite type on T .
- (3) If we set $*' = \alpha_S(*)$, then $*'|_{\bar{F}(S^*)}$, the restriction of $*'$ to $\bar{F}(S^*)$, is a star operation on S^* .
- (4) If $*$ is an e.a.b. (respectively, a.b.) semistar operation on S , then $*'$ is an e.a.b. (respectively, a.b.) semistar operation on S^* .

Proof. (1), (2) and (3) are easily shown.

(4) Let $E, F, G \in f(S^*)$ such that $(E + F)^{*'} \subset (E + G)^{*'}$. Note that $E = E_0 + S^*, F = F_0 + S^*, G = G_0 + S^*$, for some $E_0, F_0, G_0 \in f(S)$. Then,

$$(E_0 + F_0)^* = (E_0 + S + F_0 + S)^* = (E + F)^* = (E + F)^{*'} \subset (E + G)^{*'} = (E + G)^* = (E_0 + S^* + G_0 + S^*)^* = (E_0 + G_0)^*.$$

Since $*$ is e.a.b, we deduce that $F_0^* \subset G_0^*$, and hence $F^{*'} \subset G^{*'}$. Similar argument shows the a.b. statement.

Proposition (1.6) Let T be an oversemigroup of S , and let $*$ be a semistar operation on T . We define $\delta_S(*) : \bar{F}(S) \rightarrow \bar{F}(S)$ by setting:

$$E^{\delta_S(*)} = (E + T)^* \text{ for all } E \in \bar{F}(S).$$

- (1) $\delta_S(*)$ is a semistar operation on S .
- (2) If $*$ is an e.a.b. (respectively, a.b.) semistar operation on T , then $\delta_S(*)$ is an e.a.b. (respectively, a.b.) semistar operation on S .

Proof. (1) is straightforward.

(2) Let $E \in f(S)$ and let $F, G \in f(S)$ (respectively, $F, G \in \bar{F}(S)$) such that $(E + F)^{\delta_S(*)} \subset (E + G)^{\delta_S(*)}$. Then, $(E + T + F + T)^* \subset (E + T + G + T)^*$. The conclusion follows from the hypothesis on $*$.

Let T be an oversemigroup of S . Then by Propositions (1.5) and (1.6), we have canonical maps:

$$\alpha: \text{SStar}(S) \longrightarrow \text{SStar}(T) \text{ and } \delta: \text{SStar}(T) \longrightarrow \text{SStar}(S).$$

Proposition (1.7) Let T be an oversemigroup of S . Then

- (1) $\alpha(\delta(*)) = *$ for each $* \in \text{SStar}(T)$.
- (2) The following conditions are equivalent:
 - (i) δ is bijective.
 - (ii) α is bijective.
 - (iii) $S = T$.

Proof. (1) is straightforward.

(2) (i) $\implies$ (iii): Since δ is surjective, there is $* \in \text{SStar}(T)$ such that $\delta_S(*)$ coincides with the d-semistar operation on S . Therefore, $S = S^d = S^{\delta_S(*)} = (S+T)^* = T^*$. It follows that $S = T^* \supset T$, and hence $S = T$.

2. Background on Kronecker function rings

Let D be an integral domain with quotient field $\text{q}(D) = k$ and let S be a g -monoid with quotient group G . Then the semigroup ring of S over D is denoted by $D[X; S]$.

Note that $S \subset D[X; S]$. Let $f = \sum_{i=1}^n a_i X^{t_i}$ be a non-zero element of $k[X; G]$, where $a_i \neq 0$ for each i and $t_i \neq t_j$ for each $i \neq j$. Then the fractional ideal $(t_1, \dots, t_n)$ of S is called the e -content of f , and is denoted by $e_S(f)$ or simply by $e(f)$. The subset $\{t_1, t_2, \dots, t_n\}$ of G is called the e -support of f , and is denoted by $\text{Supp}_e(f)$ or simply by $\text{Supp}(f)$. We refer to [M4] for semigroup rings.

Proposition (2.1) ([M2, Proposition 4]) Let $*$ be an e.a.b. semistar operation on S , let k be a field, and set

$$S_*^k = \{f/g \mid f, g \in k[X; S] - \{0\} \text{ with } e(f)^* \subset e(g)^*\} \cup \{0\}.$$

(1) S_*^k is a well-defined extension domain of $k[X; S]$ with $\text{q}(S_*^k) = \text{q}(k[X; S])$ such that $S_*^k \cap G = S^*$.

(2) S_*^k is a Bezout domain.

(3) If $F \in \text{f}(S)$, then $FS_*^k \cap G = F^*$ and $FS_*^k = F^* S_*^k$.

S_*^k is called the Kronecker function ring of S with respect to $*$ and k (or, simply the Kronecker function ring of S with respect to $*$), and is also denoted by $\text{Kr}(S, *, k)$ (or, simply by $\text{Kr}(S, *)$).

Proposition (2.2) Let $*$ be an e.a.b. semistar operation on S .

(1) $\text{Kr}(S, *) = \text{Kr}(S, *_f)$.

(2) Let $\alpha(*)$ be the ascent of $*$ to S^* . Then $\text{Kr}(S, *) = \text{Kr}(S^*, \alpha(*))$.

Proof. (1) is immediate from the definition.

(2) Set $T = S^*$. Suppose that $f, g \in k[X; S] - \{0\}$. Then,

$(e_T(f))^{\alpha(*)} \subset (e_T(g))^{\alpha(*)}$ iff $(e_T(f))^* \subset (e_T(g))^*$ iff $(e_S(f) + T)^* \subset (e_S(g) + T)^*$
 iff $(e_S(f) + S)^* \subset (e_S(g) + S)^*$ iff $(e_S(f))^* \subset (e_S(g))^*$ iff $f/g \in \text{Kr}(S, *)$.

Proposition (2.3) Let $*_1$ and $*_2$ be e.a.b. semistar operations on S .

- (1) If $*_1 \leq *_2$, then $\text{Kr}(S, *_1) \subset \text{Kr}(S, *_2)$.
- (2) $*_1 \sim *_2$ if and only if $\text{Kr}(S, *_1) = \text{Kr}(S, *_2)$.

Proof. (1) is immediate from the definition.

(2) The sufficiency: For each $F \in f(S)$, we have
 $F^{*1} = F\text{Kr}(S, *_1) \cap G = F\text{Kr}(S, *_2) \cap G = F^{*2}$.

Proposition (2.4) Let $S = \{S_\lambda \mid \lambda \in \Lambda\}$ be a family of oversemigroups of S , and let $*_S$ be the semistar operation associated to S .

- (1) If $*_{\{S_\lambda\}}$ is an e.a.b. (respectively, an a.b.) semistar operation for each λ , then $*_S$ is an e.a.b. (respectively, an a.b.) semistar operation on S .
- (2) If $*_{\{S_\lambda\}}$ is e.a.b. for each λ , then $\text{Kr}(S, *_S) = \bigcap_\lambda \text{Kr}(S, *_{\{S_\lambda\}})$.

Proof. (1) Let $E, F, G \in f(S)$ such that $(E+F)^{*s} \subset (E+G)^{*s}$. Then $E+F+S_\lambda \subset E+G+S_\lambda$ for each λ . Since $*_{\{S_\lambda\}}$ is e.a.b., we have $F+S_\lambda \subset G+S_\lambda$. Then $F^{*s} = \bigcap_\lambda (F+S_\lambda) \subset \bigcap_\lambda (G+S_\lambda) = G^{*s}$.

The proof for the a.b. statement is similar.

(2) For $f, g \in k[X; S] - \{0\}$, $e(f)^{*s} \subset e(g)^{*s}$ iff $e(f)^{*_{\{S_\lambda\}}} \subset e(g)^{*_{\{S_\lambda\}}}$ for each λ .

Proposition (2.5) Let $\mathcal{W} = \{V_\lambda \mid \lambda \in \Lambda\}$ be a family of valuation oversemigroups of S , and let W_λ be the trivial valuation extension ring of V_λ to $q(k[X; S])$. Then the w -semistar operation $*_{\mathcal{W}}$ is a.b. on S , and $\text{Kr}(S, *_{\mathcal{W}}) = \bigcap_\lambda \text{Kr}(S, *_{\{V_\lambda\}}) = \bigcap_\lambda W_\lambda$.

Proof. Easy consequence of (2.4) (2).

3. Some semistar operations associated to an semistar operation

Definition (3.1) Let $*$ be a semistar operation on S . An element $x \in G$ is called $*$ -integral over S if $x \in (F^* : F^*)$ for some $F \in f(S)$. The set $S^{[*]} = \bigcup \{(F^* : F^*) \mid F \in f(S)\}$ is called the *semistar integral closure of S with respect to $*$* or, simply the *$*$ -integral closure of S* . If $S = S^{[*]}$, then S is called *$*$ -integrally closed*.

Lemma (3.2) $S^{[*]} = \bigcup \{(F^* : F^*)^{*f} \mid F \in f(S)\}$.

Proof. Let $x \in (F^* : F^*)^{*f}$. There is $H \in f(S)$ with $H \subset (F^* : F^*)$ such that $x \in H^*$. Since $H^* + F^* \subset F^*$, we have $x + F^* \subset F^*$, and $x \in (F^* : F^*)$. Therefore, $(F^* : F^*)^{*f} = (F^* : F^*)$.

Proposition (3.3) Let $*$ be a semistar operation on S . We define an operation $[*]$ on S by setting:

$$\begin{aligned}
 H^{[*]} &= \bigcup \{((F^* : F^*) + H)^{*f} \mid F \in f(S)\}, \text{ for each } H \in f(S), \\
 E^{[*]} &= \bigcup \{H^{[*]} \mid H \in f(S) \text{ with } H \subset E\}, \text{ for each } E \in \bar{F}(S).
 \end{aligned}$$

Then the operation $[\ast]$ is a semistar operation of finite type on S .

Proof. (i) To prove $x + E^{[\ast]} = (x + E)^{[\ast]}$, it suffices to show that $x + E^{[\ast]} \subset (x + E)^{[\ast]}$. Let $a \in x + E^{[\ast]}$. We have $a - x \in F_1^{[\ast]}$ for some $F_1 \in \mathfrak{f}(S)$ with $F_1 \subset E$. Then $a - x \in ((F_2^\ast : F_2^\ast) + F_1)^{\ast f}$ for some $F_2 \in \mathfrak{f}(S)$. Then $a \in ((F_2^\ast : F_2^\ast) + x + F_1)^{\ast f} \subset (x + F_1)^{[\ast]} \subset (x + E)^{[\ast]}$.

(ii) Let $x \in E$, and set $(x) = H$. Then we have $x \in (H^\ast : H^\ast) + H \subset H^{[\ast]} \subset E^{[\ast]}$. Hence $E \subset E^{[\ast]}$.

(iii) Assume that $E_1 \subset E_2$. By the definition, we have $E_1^{[\ast]} \subset E_2^{[\ast]}$.

(iv) Let $y \in (E^{[\ast]})^{[\ast]}$. We have $y \in F^{[\ast]}$ for some $F \in \mathfrak{f}(S)$ with $F \subset E^{[\ast]}$. Since F is finitely generated, we have $F \subset H^{[\ast]}$ for some $H \in \mathfrak{f}(S)$ with $H \subset E$.

If $F = (x_1, \dots, x_n)$, we have, for each i , $x_i \in ((F_i^\ast : F_i^\ast) + H)^{\ast f}$ for some $F_i \in \mathfrak{f}(S)$. Then $F \subset ((F_1^\ast : F_1^\ast) + H, \dots, (F_n^\ast : F_n^\ast) + H)^{\ast f}$. Let $G_1 = \sum_i F_i$, then $F \subset ((G_1^\ast : G_1^\ast) + H)^{\ast f}$.

On the other hand, $y \in ((G_2^\ast : G_2^\ast) + F)^{\ast f}$ for some $G_2 \in \mathfrak{f}(S)$. Then,

$y \in ((G_2^\ast : G_2^\ast) + ((G_1^\ast : G_1^\ast) + H)^{\ast f})^{\ast f} = ((G_2^\ast : G_2^\ast) + (G_1^\ast : G_1^\ast) + H)^{\ast f} \subset (((G_1 + G_2)^\ast : (G_1 + G_2)^\ast) + H)^{\ast f} \subset H^{[\ast]} \subset E^{[\ast]}$. Therefore, $(E^{[\ast]})^{[\ast]} = E^{[\ast]}$.

Proposition (3.4) Let $\ast$ be a semistar operation on S . Then we have

- (1) $S^{[\ast]}$ is an oversemigroup of S .
- (2) $S^{[\ast]}$ is integrally closed.

Proof. (1) Let $a, b \in S^{[\ast]}$. Then $a \in (F_1^\ast : F_2^\ast)$ and $b \in (F_2^\ast : F_2^\ast)$ for some $F_1, F_2 \in \mathfrak{f}(S)$. Then $a + b \in ((F_1 + F_2)^\ast : (F_1 + F_2)^\ast) \in S^{[\ast]}$.

(2) Let $x \in G$ be integral over $S^{[\ast]}$. We have $nx = a \in S^{[\ast]}$ for some positive integer n . Since $a \in S^{[\ast]}$, we have $a \in (F^\ast : F^\ast)$ for some $F \in \mathfrak{f}(S)$. Set $H = (F, F + x, \dots, F + (n-1)x)$. Then $x + H \subset H^\ast$. It follows that $x \in (H^\ast : H^\ast) \subset S^{[\ast]}$.

Definition (3.5) Let $\ast$ be a semistar operation on S . We define the map $\ast_a : \bar{\mathfrak{F}}(S) \longrightarrow \bar{\mathfrak{F}}(S)$ by setting

$$F^{\ast_a} = \cup\{((F + H)^\ast : H^\ast) \mid H \in \mathfrak{f}(S)\}, \text{ for each } F \in \mathfrak{f}(S),$$

$$E^{\ast_a} = \cup\{F^{\ast_a} \mid F \in \mathfrak{f}(S) \text{ with } F \subset E\}, \text{ for each } E \in \bar{\mathfrak{F}}(S).$$

Proposition (3.6) Let $\ast$ be a semistar operation on S .

- (1) $\ast_a$ is a semistar operation of finite type.
- (2) $\ast_a$ is e.a.b.
- (3) $\ast_f \leq [\ast] \leq \ast_a$.
- (4) $[\ast] = [\ast_f] = [\ast]_f$.
- (5) $\ast_a = (\ast_f)_a = (\ast_a)_f$.
- (6) $\ast_a = \ast_f$ if and only if $\ast_f$ is an e.a.b. semistar operation.
- (7) $\ast_1 \leq \ast_2$ implies $(\ast_1)_a \leq (\ast_2)_a$.
- (8) $\ast_1 \leq \ast_2$ implies $[\ast_1] \leq [\ast_2]$.
- (9) $(\ast_a)_a = \ast_a$.
- (10) $[\ast]_a = [\ast_a] = \ast_a$.
- (11) $\ast_f \leq [\ast] \leq [[\ast]] \leq \ast_a$.
- (12) $S^{[\ast]} = S^{\ast_a} = S^{[[\ast]]}$.

Proof. (1) (i) To prove $(x + E)^{*\alpha} = x + E^{*\alpha}$, it suffices to show that $x + E^{*\alpha} \subset (x + E)^{*\alpha}$. Let $y \in x + E^{*\alpha}$. Since $y - x \in E^{*\alpha}$, there is $F \in \mathbf{f}(S)$ with $F \subset E$ such that $y - x \in F^{*\alpha}$. Hence $y - x \in ((F + H)^* : H^*)$ for some $H \in \mathbf{f}(S)$. Then $y \in ((x + F + H)^* : H^*) \subset (x + F)^{*\alpha} \subset (x + E)^{*\alpha}$. It follows that $x + E^{*\alpha} \subset (x + E)^{*\alpha}$.

(ii) Let $x \in E$, and set $(x) = H$. Then $x \in ((H + H)^* : H^*) \subset H^{*\alpha} \subset E^{*\alpha}$. Hence $E \subset E^{*\alpha}$.

(iii) Let $E_1 \subset E_2$. By definition, we have $E_1^{*\alpha} \subset E_2^{*\alpha}$.

(iv) Let $y \in (E^{*\alpha})^{*\alpha}$. Then $y \in F^{*\alpha}$ for some $F \in \mathbf{f}(S)$ with $F \subset E^{*\alpha}$. Since F is finitely generated, we have $F \subset H^{*\alpha}$ for some $H \in \mathbf{f}(S)$ with $H \subset E$. If $F = (x_1, \dots, x_n)$, we have, for each i , $x_i + F_i^* \subset (H + F_i)^*$ for some $F_i \in \mathbf{f}(S)$. Let $G_1 = \sum_i F_i$, then $F + G_1 \subset (H + G_1)^*$. On the other hand, $y + G_2 \subset (F + G_2)^*$ for some $G_2 \in \mathbf{f}(S)$. Then $y + G_1 + G_2 \subset (F + G_1 + G_2)^* \subset (H + G_1 + G_2)^*$. Hence $y \in H^{*\alpha} \subset E^{*\alpha}$. Hence $(E^{*\alpha})^{*\alpha} = E^{*\alpha}$.

(2) Let $I, J \in \mathbf{f}(S)$, we will show that $((I + J)^{*\alpha} : I) \subset J^{*\alpha}$. Let $z \in ((I + J)^{*\alpha} : I)$, then $z + I \subset (I + J)^{*\alpha}$. If $I = (x_1, \dots, x_n)$, we have, for each i , $z + x_i + F_i^* \subset (I + J + F_i)^*$ for some $F_i \in \mathbf{f}(S)$. Let $G = \sum_i F_i$, then $z + I + G \subset (I + J + G)^*$. It follows that $z \in ((J + I + G)^* : (I + G)^*) \subset J^{*\alpha}$.

(3) Let $x \in E^{*f}$. We have $x \in H^*$ for some $H \in \mathbf{f}(S)$ with $H \subset E$. Then we have $x \in ((S^* : S^*) + H)^{*f} \subset H^{[*]} \subset E^{[*]}$.

Hence $E^{*f} \subset E^{[*]}$, and $*_f \leq [*]$.

Next, let $x \in E^{[*]}$. We have $x \in H^{[*]}$ for some $H \in \mathbf{f}(S)$ with $H \subset E$. There is $F \in \mathbf{f}(S)$ such that $x \in ((F^* : F^*) + H)^{*f}$. Then $x + F \subset ((F^* : F^*) + H)^{*f} + F^* \subset (F + H)^*$. Hence $x \in H^{*\alpha} \subset E^{*\alpha}$. Therefore $E^{[*]} \subset E^{*\alpha}$, and $[*] \leq *_{\alpha}$.

(4), (5), (7) and (8) are obvious from the definitions.

(6) The necessity follows from (2).

The sufficiency: Let $F \in \mathbf{f}(S)$, and let $x \in F^{*\alpha}$. Then $x + H \subset (F + H)^*$ for some $H \in \mathbf{f}(S)$. Since $*_f$ is e.a.b., we have $x \in F^{*f}$. Hence $F^{*\alpha} \subset F^{*f}$, and $*_{\alpha} \leq *_f$. The conclusion follows from (3).

(9) Since $*_{\alpha} = (*_{\alpha})_f$ is e.a.b. by (2), we have $(*_{\alpha})_a = (*_{\alpha})_f$ by (6). Hence $(*_{\alpha})_a = *_{\alpha}$.

(10) By (3),(7), we have $(*_f)_a \leq [*]_a \leq (*_{\alpha})_a$ (resp. $(*_a)_f \leq [*]_a \leq (*_{\alpha})_a$). By (5),(9), we have $*_{\alpha} \leq [*]_a \leq *_{\alpha}$ (resp., $*_{\alpha} \leq [*]_a \leq *_{\alpha}$). Hence $[*]_a = *_{\alpha}$ (resp. $*_{\alpha} = [*]_a$).

(11) By (3),(8), we have $[*_f] \leq [[*]] \leq [*]_a$. Then (4),(10) imply the assertion.

(12) We have $S^{*\alpha} = \cup\{((S + H)^* : H^*) \mid H \in \mathbf{f}(S)\} = \cup\{(H^* : H^*) \mid H \in \mathbf{f}(S)\} = S^{[*]}$. Then (11) completes the proof.

4. The Kronecker function ring of any semistar operation

Lemma (4.1) (Dedekind-Mertens Lemma for semigroups)([GP1]) Let $f, g \in k[X; S] - \{0\}$. Then there is a positive integer m such that

$$(m + 1)e(g) + e(f) = me(g) + e(fg).$$

Lemma (4.2) Let $*$ be a semistar operation on S . Let $f, g, f', g' \in k[X; S] - \{0\}$ with $f/g = f'/g'$ such that $(e(f) + e(h))^* \subset (e(g) + e(h))^*$ for some $h \in k[X; S] - \{0\}$.

Then there is $h' \in k[X; S] - \{0\}$ such that $(e(f') + e(h'))^* \subset (e(g') + e(h'))^*$.

Proof. Then we have $fg' = f'g$. By (4.1), there is a positive integer m such that
 $(m+1)e(g) + e(f') = me(g) + e(f'g)$,
 $(m+1)e(f) + e(g') = me(f) + e(fg')$.
Then it follows that $\{(m+1)e(g) + e(f')\} + me(f) = \{(m+1)e(f) + e(g')\} + me(g)$.
Now, there are a finite set of elements $s_1, s_2, \dots, s_n$ of S with
 $s_i \neq s_j$ for each $i \neq j$ such that $(m+1)(e(g) + e(h)) + m(e(f) + e(h)) =$
 $(s_1, s_2, \dots, s_n)$. We set $h' = X^{s_1} + X^{s_2} + \dots + X^{s_n} \in k[X; S] - \{0\}$.
Then we have $e(h') = (m+1)(e(g) + e(h)) + m(e(f) + e(h))$ and therefore
 $e(f') + e(h') = \{(m+1)e(g) + e(f') + me(f)\} + (2m+1)e(h)$
 $= \{(m+1)e(f) + e(g') + me(g)\} + (2m+1)e(h)$
 $= (e(f) + e(h)) + m(e(f) + e(h)) + m(e(g) + e(h)) + e(g')$
 $\subset (e(g) + e(h))^* + m(e(f) + e(h)) + m(e(g) + e(h)) + e(g')$
 $\subset (e(g') + e(h'))^*$, as wanted.

Set $\text{Kr}(S, *) = \{f/g \mid f, g \in k[X; S] - \{0\} \text{ such that } (e(f) + e(h))^* \subset (e(g) + e(h))^* \text{ for some } h \in k[X; S] - \{0\} \cup \{0\}\}$. (4.2) shows that $\text{Kr}(S, *)$ is a well-defined subset of $q(k[X; S])$. If $*$ is e.a.b., this coincides with $\text{Kr}(S, *)$ in (2.1).

Proposition (4.3) $\text{Kr}(S, *)$ is an integral domain with quotient field $q(k[X; S])$.

Proof. Let $f/g, f'/g \in \text{Kr}(S, *) - \{0\}$. Then there are $h, h' \in k[X; S] - \{0\}$ such that

$(e(f) + e(h))^* \subset (e(g) + e(h))^*, (e(f') + e(h'))^* \subset (e(g) + e(h'))^*$. There is $j \in k[X; S] - \{0\}$ such that $e(j) = e(h) + e(h')$. Then we have
 $(e(f) + e(j))^* \subset (e(g) + e(j))^*, (e(f') + e(j))^* \subset (e(g) + e(j))^*$.

We may assume that $f + f' \neq 0$. Then it follows that

$(e(f + f') + e(j))^* \subset (e(g) + e(j))^*$. Hence $f/g + f'/g \in \text{Kr}(S, *)$.

Next, we have $(m+2)e(g) = me(g) + e(g^2)$ for some m . There is $j' \in k[X; S] - \{0\}$ such that $e(j') = (m+2)e(g) + 2e(j)$. Then we have

$$\begin{aligned} & e(ff') + e(j') \\ & \subset \{e(f) + e(f')\} + \{(m+2)e(g) + 2e(j)\} \\ & = \{e(f) + e(j)\} + \{e(f') + e(j)\} + (m+2)e(g) \\ & \subset 2(e(g) + e(j))^* + (m+2)e(g) \\ & = 2(e(g) + e(j))^* + \{me(g) + e(g^2)\} \\ & \subset (e(g^2) + e(j'))^*. \end{aligned}$$

Therefore $(e(ff') + e(j'))^* \subset (e(g^2) + e(j'))^*$. Hence $(ff')/(gg') \in \text{Kr}(S, *)$.

Proposition (4.4) $\text{Kr}(S, *)$ is a Bezout domain.

Proof. Set $R = \text{Kr}(S, *)$, and let $f \in k[X; S] - \{0\}$ with $\text{Supp}(f) = \{s_1, \dots, s_n\}$. Then we have $fR = (X^{s_1}, \dots, X^{s_n})R$.

Let ξ and η be non-zero elements of R . We set $\xi = f/g$ and $\eta = f'/g$ with $f, f', g \in k[X; S] - \{0\}$, and let $\text{Supp}(f) = \{s_1, \dots, s_n\}$, let $\text{Supp}(f') = \{t_1, \dots, t_m\}$ and let $\text{Supp}(f) \cup \text{Supp}(f') = \{u_1, \dots, u_l\}$ with $u_i \neq u_j$ for each $i \neq j$. Then we have

$$\begin{aligned}
(\xi, \eta)R &= \left(\frac{X^{s_1}}{g}, \dots, \frac{X^{s_n}}{g}, \eta\right)R \\
&= \left(\frac{X^{s_1}}{g}, \dots, \frac{X^{s_n}}{g}, \frac{X^{t_1}}{g}, \dots, \frac{X^{t_m}}{g}\right)R \\
&= \left(\frac{X^{u_1}}{g}, \dots, \frac{X^{u_l}}{g}\right)R = \left(\frac{\sum_i X^{u_i}}{g}\right)R.
\end{aligned}$$

Therefore $(\xi, \eta)R$ is a principal ideal of R .

Lemma (4.5) If $*_1 \leq *_2$, then $\text{Kr}(S, *_1) \subset \text{Kr}(S, *_2)$.

Proof. Let $f, g \in k[X; S] - \{0\}$ such that $(e(f) + e(h))^*{}^1 \subset (e(g) + e(h))^*{}^1$ for some $h \in k[X; S] - \{0\}$. Then we have $(e(f) + e(h))^*{}^2 \subset (e(g) + e(h))^*{}^2$.

Proposition (4.6) Let $*$ be a semistar operation on S . Then we have $\text{Kr}(S, *) = \text{Kr}(S, [*]) = \text{Kr}(S, *_a)$.

Proof. From the definitions, we have $\text{Kr}(S, *_f) = \text{Kr}(S, *)$.

Since $*_f \leq [*] \leq *_a$ by (3.6)(3), we have $\text{Kr}(S, *) \subset \text{Kr}(S, [*]) \subset \text{Kr}(S, *_a)$. Let $\xi \in \text{Kr}(S, *_a) - \{0\}$. Then $\xi = f/g$ with $f, g \in k[X; S] - \{0\}$ such that $(e(f) + e(h))^*{}^a \subset (e(g) + e(h))^*{}^a$ for some $h \in k[X; S] - \{0\}$. Let $e(f) + e(h) = (a_1, \dots, a_n)$. Then, for each i , $a_i + F_i \subset (e(f) + e(g) + F_i)^*$ for some $F_i \in \mathfrak{f}(S)$. Set $F = \sum_i F_i$, then $(a_1, \dots, a_n) + F \subset (e(g) + e(h) + F)^*$. Therefore $(e(f) + e(h) + F)^* \subset (e(g) + e(h) + F)^*$. It follows that $f/g \in \text{Kr}(S, *)$.

Proposition (4.7) Let $*$ be a semistar operation on S . Then, for each $E \in \bar{\mathfrak{F}}(S)$, we have $E^{*a} = \cup\{\text{FKr}(S, *) \cap G \mid F \in \mathfrak{f}(S) \text{ with } F \subset E\}$.

Proof. $E^{*a} = \cup\{F^{*a} \mid F \in \mathfrak{f}(S) \text{ with } F \subset E\} = \cup\{\text{FKr}(S, *_a) \cap G \mid F \in \mathfrak{f}(S) \text{ with } F \subset E\} = \cup\{\text{FKr}(S, *) \cap G \mid F \in \mathfrak{f}(S) \text{ with } F \subset E\}$.

Proposition (4.8) Let $*$ be a semistar operation on S . Set

$$T = S^{*a} \text{ and } *_T = \alpha_T(*_a) = \alpha(*_a).$$

Then, T is an integrally closed oversemigroup of S , and $*_T$ is an e.a.b. semistar operation on T such that $T^{*T} = T$ and $\text{Kr}(S, *) = \text{Kr}(T, *_T)$.

Proof. Since $T = S^{[*]}$, T is integrally closed by (3.4)(2). By (3.6)(2), $*_a$ is e.a.b. and, by (1.5)(4), $*_T$ is e.a.b. Since $(E^{*a})^{*a} = E^{*a}$ for each $E \in \bar{\mathfrak{F}}(S)$, we have $T^{*T} = T^{\alpha(*_a)} = T^{*a} = (S^{*a})^{*a} = S^{*a} = T$. By (2.2) and (4.6), we have $\text{Kr}(S, *) = \text{Kr}(S, *_a) = \text{Kr}(T, *_T)$.

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