Comparison of Computational Efficiency of Hole Relaxation and Kamenkovich's Method in Solving Vorticity Equations in Multi-Connected Regions

by

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Abstract

The computational efficiencies of the hole relaxation and Kamenkovich's method in solving vorticity equations in multi-connected regions are compared. Kamenkovich's method is somewhat modified. Although some extra region is required to store the homogeneous components of solution in Kamenkovich's method, it has some superiorities over the hole relaxation, especially in the initial-value problems : the computer time required to calculate one iterative step is shorter and the convergence of iteration is faster in Kamenkovich's method than in the hole relaxation. In particular, the difference in the convergence speed is large in the cyclic-channel cases.

1. Introduction

There are two methods of solving the vorticity equation by means of the relaxation in a multi-connected region such as a closed basin with islands or a cyclic channel : the hole relaxation and Kamenkovich's method. The latter was used in the earlier ocean general circulation models (e.g., Bryan, 1969), but recently most of the ocean models use the former (Takano, 1974 and Semtner, 1986 a,b). There is only a small amount of information published, however, on the direct comparison of computational efficiency of the two methods. Murata (1983) made it in an initial-value problem, and he concluded that the hole relaxation is more efficient than Kamenkovich's method. But his formulation of Kamenkovich's method is different from both Bryan (1969)'s formulation and the present one, and seems to be computationally less efficient. Furthermore, it is said that the solutions by the two methods

may be different from each other in some cases (Takano, personal communication).

In this article, we confirm the unity of the solutions obtained by the two methods and compare their computational efficiency. In Section 2 the methodology is briefly reviewed. Kamenkovich's method is modified from the original fashion (Kamenkovich, 1962 and Bryan, 1969). The modification contains the change of variable which is expanded to a linear combination of non-homogeneous and homogeneous components (from ψ to ψ_t) and the change of the formula of the condition for islands (from line integral to area integral). In this way, the direct comparison of the two methods can be made on the identical bases. The convergence speeds of the solutions in steady-state boundary-value problems are discussed in Section 3, and synthetic computational efficiencies in initial-value problems are compared in Section 4. Section 5 is devoted to summary and discussion.

2. Methodology

First, we review the methodology of the two methods in the case of initial-value problem. We deal with only one-layer, barotropic flat-bottom oceans for the sake of simplicity, but the essence of the results applies to the cases of more sophisticated ocean models.

The equation of motion and its associated vorticity equation are written as follows, respectively :

$$\mathbf{u}_t \equiv \mathbf{k} \times \nabla \psi_t = -\frac{1}{\rho_0} \nabla P + \mathbf{F}, \quad (1)$$

$$\nabla^2 \psi_t = \mathbf{k} \cdot \nabla \times \mathbf{F}, \quad (2)$$

where $\mathbf{u}$, ψ , ρ_0 and p are velocity vector, streamfunction, density and pressure, respectively. Subscript t denotes the Eulerian time derivative and $\mathbf{F}$ is the sum of all other terms in the equation of motion. $\mathbf{k}$ is the vertical unit vector. Equation (2) is a Poisson-type equation of ψ_t and first we solve it for ψ_t . Then, ψ at the next time step is obtained from ψ_t by using a time integration formula.

We assume $\psi = \psi_t = 0$ at the external boundary in the cases of closed oceans and $\psi = \psi_t = 0$ at one side in the case of cyclic channels.

If there are islands, the no-normal-flow condition at their boundaries requires $\psi_t = \text{const.}$ at each boundary. But the boundary values of ψ_t must be determined automatically in terms of internal physics, and must not be given externally. In order to determine them, we introduce a condition that pressure is a single-valued function around each island,

$$\oint_{A'} \nabla P \cdot d\mathbf{s} = 0, \quad (3)$$

where A' is an arbitrary closed curve containing the island A and $d\mathbf{s}$ is a length element along A' (Fig. 1). Substituting the pressure gradient term of the equation of motion (1) into Eq.(3), we obtain

$$\oint_{A'} (\mathbf{k} \times \nabla \psi_t - \mathbf{F}) \cdot d\mathbf{s} = 0, \quad (4)$$

or translating the line integral into the area integral by Stokes' theorem

Fig. 1 A schematic figure of a multi-connected region. 'A' denotes an island and 'A'' an arbitrary curve containing the island A.

$$\oint_{A'} (\nabla^2 \psi_t - \mathbf{k} \cdot \nabla \times \mathbf{F}) dS = 0, \quad (5)$$

where dS is an area element of the region surrounded by A' . The process so far derived is identical to both methods.

In the hole relaxation (hereafter referred to as 'H-method') (Takano, 1974), ψ_t is first calculated along with the finite difference formula of Eq.(2) in the internal region, and then, each internal boundary value of ψ_t is determined so as to satisfy the finite difference analog of Eq. (5). That is, the conditions determining the internal boundary values of ψ_t , Eq.(5), must be used in every iteration step at each time step.

On the other hand, Bryan (1969) applied Kamenkovich's method (hereafter referred to as 'K-method') to the initial-value problem for the first time. He expressed ψ , as in Kamenkovich's original fashion, as a linear combination of a non-homogeneous component $\tilde{\psi}_0$ and a set of time-independent homogeneous components $\tilde{\psi}_i$'s :

$$\psi = \tilde{\psi}_0 + \sum_{i=1}^N \tilde{c}_i \tilde{\psi}_i,$$

where N is the number of islands and $\tilde{c}_i$'s are a set of time-dependent coefficients. The non-homogeneous component $\tilde{\psi}_0$ is solved with a boundary condition that $\tilde{\psi}_0 = 0$ at all boundaries. The i 'th homogeneous component $\tilde{\psi}_i$, on the other hand, must satisfy the boundary con-

ditions that $\tilde{\psi}_i=1$ at the boundary of the i 'th island and that $\tilde{\psi}_i=0$ at all other boundaries.

But, here, we expand ψ_t , instead of ψ , as a linear combination of a non-homogeneous component ψ_{0t} of Eq.(2) and a set of time-independent homogeneous components ψ_{it} 's :

$$\psi_t = \psi_{0t} + \sum_{i=1}^N c_i \psi_{it} \quad (6)$$

where c_i 's are also a set of time-dependent coefficients. This formulation is straightforward for Eq.(2) and simplifies the governing equation for the non-homogeneous component ψ_{0t} compared with the one for $\tilde{\psi}_0$. The boundary conditions for ψ_{0t} and ψ_{it} 's are the same as those for $\tilde{\psi}_0$ and $\tilde{\psi}_i$'s. Note that the homogeneous components ψ_{it} 's ($\psi_{it} \equiv \tilde{\psi}_i$) are time-independent so that it is sufficient for them to be solved once at the outset.

In each relaxing iteration step at each time step, only ψ_{0t} should be calculated with the boundary condition stated above. After its convergence with sufficient (prescribed) accuracy, the conditions for islands (Eq.(4) or (5)) are used to determine the set of coefficients c_i , and then the linear combination Eq.(6) is calculated. It must be stressed that the conditions for islands are used only once at each time step, not many times as in H-method. This may save the computer time.

In the original Kamenkovich's paper (Kamenkovich, 1962) and in Bryan (1969), the line integration formula Eq.(4) is used as the conditions for islands, but we use the area integration formula Eq.(5) also in K-method, because of its simplicity in calculation. This change in the formula of the conditions for islands, together with the change of the variable which is expanded to a linear combination of non-homogeneous and homogeneous components (from ψ to ψ_t), may improve the computational efficiency of K-method. Furthermore, in this way, we can make comparison of the two methods on the identical bases.

Murata (1983), in his comparison of the two methods, also expanded ψ as a linear combination of the non-homogeneous and homogeneous components. But according to his formulation of K-method, the homogeneous components are

defined as those obtained by neglecting only the wind stress term, so that they are functions of time as well as the non-homogeneous component. Therefore, all the components of ψ must be integrated in each time step. Furthermore, his formulation of K-method can never be used when the non-linear advection term is included in the momentum equation.

In the Stommel-type steady-state boundary-value problems, which will be treated in the next section, the governing equations and the conditions for islands are written as follows :

the equation of motion

$$R\mathbf{k} \times \nabla \psi - f \nabla \psi = -\frac{1}{\rho_0} \nabla P + \frac{\boldsymbol{\tau}}{H}, \quad (1')$$

the vorticity equation

$$\nabla^2 \psi - \frac{\beta}{R} \frac{\partial \psi}{\partial x} = \frac{1}{RH} \text{curl}_z \boldsymbol{\tau}, \quad (2')$$

and conditions for islands

$$\oint_{A'} (R\mathbf{k} \times \nabla \psi - f \nabla \psi - \frac{\boldsymbol{\tau}}{H}) \cdot d\mathbf{s} = 0 \quad (4')$$

or

$$\int_{A'} (\nabla^2 \psi - \frac{\beta}{R} \frac{\partial \psi}{\partial x} - \frac{1}{RH} \text{curl}_z \boldsymbol{\tau}) dS = 0. \quad (5')$$

In these equations, R , f , $\boldsymbol{\tau}$, H and β denote the coefficient of the bottom friction, Coriolis parameter, wind stress vector, depth of the ocean and the latitudinal variation rate of f , respectively. $\nabla^2 \psi$ term in Eq.(2)' arises from the bottom friction term. In K-method, Eq.(6) is rewritten as :

$$\psi = \psi_0 + \sum_{i=1}^N c_i \psi_i. \quad (6')$$

The boundary conditions for ψ_0 and ψ_i are the same as those for ψ_{0t} and ψ_{it} in the initial-value problems.

3. Steady-state boundary-value problems

In order to compare the convergence speed of the solutions by the two methods, we apply them to the Stommel-type steady-state boundary-value problem in various situations. In this problem, we take as the condition for

convergence of solutions that the value of the domain-averaged kinetic energy becomes an identical one within a limit of accuracy of double precision (relatively 10^{-15}). In K-method, all of ψ_0 and ψ_i 's are calculated simultaneously, that is, they are calculated once in each iteration step.

Figure 2 shows the solutions of the cases of a closed basin with and without island(s). The basin is 10,000 km $\times$ 6,000 km $\times$ 200 m, and exists on the β -plane. Values for parameters are chosen as follows : $R = 0.02$, $\beta = 10^{-13}$ cm $^{-1}$ ·s $^{-1}$, $\tau = (-\cos(2\pi y/L), 0)$ and $f = \beta y$, where $L = 6,000$ km and y is meridional extent. The grid size is 100 km. In each case the solution by means of K-method coincides with that by H-method to the order of computer accuracy (double precision, relatively $\sim 10^{-15}$), so that only one set of solutions is shown here.

Figure 3 shows the relationship between the number of islands and that of iterations required for convergence of the solutions shown in Fig.2. In each case, the most suitable value is

chosen for the over-relaxation coefficient. We can see in Fig.3 that as the number of islands increases, the number of iterations required for convergence increases in H-method, but that it decreases in K-method.

Figure 4 shows relationship between the size of the island and the number of iterations needed for convergence in the case with one square island. We can also see that as the size becomes large, the number of iterations required generally increases in H-method, but that it decreases in K-method. The only exception in H-method is the case with the largest island size. Is this case the large decrease in the ocean area makes the convergence faster.

In Fig.5 we show processes of convergence of solution in a case of zonally cyclic channel. In this case the governing equations are the same as for the previous cases. Both the width and the length of the channel are 1,000 km. The southern boundary is set at the equator and the meridionally sine-type westerly wind stress is given. ψ is set to be zero at the northern bound-

Fig. 2 Examples of solution of transport streamfunction in the cases of a closed basin with and without island(s). Solutions obtained by both methods are identical to each other.

Fig. 3 Relationship between number of islands and number of iterations required for convergence of the solutions shown in Fig. 2.

Fig. 4 Relationship between size of island and number of iterations required for convergence in the case with one square island.

ary. Is this case the analytical solution is given and we can see both of the numerical solutions approach to it as iteration goes on. But the convergence of solution is much more rapid in K-method than in H-method. K-method needs 230 times but H-method needs 820 times for the convergence.

The results so far stated may be interpreted as follows (Fig. 6). In general, when the boundary-value problems are solved by means of the relaxation method, problems with boundary values changing in each iteration step need a larger number of iterations than do those with fixed boundary values. For example, the boundary-value problems of the second kind, in which spatial gradient of ψ is given at the boundaries, need much more iterations than do those of the first kind. In H-method ψ at the internal boundaries, though spatially constant, changes its value in the iterative process (Fig. 6). Therefore, we may expect the convergence of the solution to become slower as the number of islands increases or as the size of the island becomes larger, as long as the ocean area does not much decrease. On the other hand, in K-method all of ψ_0 and ψ_i 's have their fixed values (0 or 1) at all internal and external boundaries. Thus, we can expect the convergence of these solutions to become faster as the number of islands increases or as the size of island becomes larger, because then part of the regions with fixed values of ψ_0 and ψ_i 's increases in the whole domain.

In the case of cyclic channel, ψ is fixed only at the northern boundary in H-method, but ψ_0 and ψ_1 are fixed at both the northern and southern boundaries in K-method (Fig. 6). This difference leads to the large difference in the convergence speed of the solutions (Fig. 5).

4. Initial-value problems

Now let us look at the cases of initial-value problems. We consider three cases shown in Fig. 7.

Case I : a closed basin (6,000 km $\times$ 3,000 km $\times$ 500 m) with an island (1,500 km $\times$ 1,600 km). Wind stress is of one-gyre type with an amplitude of 1

Fig. 5 Process of convergence of solution in a case of zonally-cyclic channel, for the two methods.

dyne cm^{-2} .

Case II : a zonally-cyclic channel (2,500 km length and 1,000 km width with 500 m depth). Sine-type westerly wind stress is given. The amplitude is also 1 dyne cm^{-2} .

Case III : a zonally-cyclic channel with an island. The channel is the same as in Case II, but an island (200 km $\times$ 500 km) is set at the center of the channel. Wind stress is also the same as in Case II.

In each case, solutions are obtained by both methods. The governing equation is similar to the Munk-type, but a nonlinear advective term is included. The grid interval is 100 km and the eddy viscosity coefficient is chosen as $10^8 \text{ cm}^2 \text{ s}^{-1}$. Figure 8 shows the time change of the total transport in each case. In Case I, the total transport is of about 30 Sv after day-20 with small oscillations superposed. In Case II, the total transport is of about 600 Sv at day-200 and is still increasing. But it stops increasing, in Case III, around day-100 with some 200 Sv.

In each case we compare the two methods in two points : the mean number of iteration (M) over the duration between initial onset and the time indicated by the arrow in Fig.8, and the accuracy of solutions (A) at that time.

In these cases the condition for convergence of solutions is as follows. Let R_{max} be the maximum relative residual defined as

$$R_{max} = \frac{|\psi_t^n - \psi_t^{n-1}|_{max}}{|\psi_t^n|_{max}},$$

where superscript n or $n-1$ means the number of iterations. Given an arbitrary small number ϵ , the convergence condition is

$$R_{max} < \epsilon.$$

In K-method, ψ_t is replaced by ψ_{0t} in the definition of R_{max} . Accuracy A is defined as

$$A = \frac{\sqrt{(\psi - \psi_T)^2}}{|\psi_T|_{max}},$$

where ψ_T is the 'true' value of ψ obtained by

Fig. 6 A schematic figure showing the characteristics of boundary values of transport streamfunction ψ in the two methods.

sufficient iterations (500 times) at every time step. Values of ψ_T are identical in both methods on the order of accuracy of double precision.

Figure 9 shows the mean number of iteration M and the accuracy of solution ($-\log_{10}(A)$) plotted against $-\log_{10}(\epsilon)$ for Case I (closed basin with an island). The values of M in K-method are smaller by several to 15 percent than those in H-method. But the accuracy is almost the same in both.

Figures 10 and 11 are the same as Fig.9 but for Cases II and III, respectively. In Case II (Fig.10), the values of M in K-method are nearly equal to or less than half of those in H-method. This may be expected from the result of the similar situation in the boundary-value problem (Fig.5) stated in the previous section. Furthermore, the accuracy of solution ($-\log_{10}(A)$) is higher in K-method than in H-method by more than one order, for any value of ϵ . A

possible reason for this is as follows: the convergence of solution is faster in K-method than in H-method so that when R_{max} , which is the relative difference in ψ_t or ψ_{0t} between two adjacent iterative steps, is the same order ($\sim \epsilon$), the absolute difference $\sqrt{(\psi_t - \psi_{Tt})^2}$ and in turn $\sqrt{(\psi - \psi_T)^2}$ may be smaller in K-method than in H-method. An accumulation of such errors in ψ through previous time steps also affects the result.

In Case III (Fig.11), situations are similar to those in Case II, although M slightly decreases in both methods and the difference in the accuracy of solution slightly decreases.

Finally, in order to see the synthetic computational efficiency, computer time (in seconds) needed to calculate 400 time steps for Case I and 800 time steps for Cases II and III is plotted against the mean number of iterations for various values of ϵ in Fig.12. The declina-

Fig. 7 Three cases treated in the initial-value problem. Case I : a closed basin with an island, Case II : a zonally-cyclic channel and Case III : a zonally-cyclic channel with an island.

tion of each line is a measure of the computer time required to calculate one iterative step. In all of the three cases, the declination is greater in H-method than in K-method. This is because, as stated in Section 2, the condition for an island (Eq.(5)) must be used in each iterative step in H-method, but is used only once in one time step in K-method. Furthermore, comparing Case III with Case II, we see that the declination increases with the increasing number of island(s) in H-method but scarcely changes in K-method. That is, the computer time scarcely depends directly on the number of islands (N) in K-method, although it depends indirectly on N through the change in the number of iterations required for convergence, which generally decreases with increasing N in K-method. In H-method, however, the total

amount of computer time increases in Case III because of the existence of an island, in spite of the decrease in the mean number of iterations.

5. Summary and discussion

As so far stated, Kamenkovich's method is superior to the hole relaxation in several aspects. We summarize the superiorities.

First of all, the result in the boundary-value problems shown in Figs.3 and 4 should be stressed. Convergence speed of the iteration tends to be faster with the number of islands and with their size in K-method. But in H-method the speed becomes slower in general in the same circumstances, as long as the ocean area does not much decrease. This property of K-method is suitable to simulate the ocean

Fig. 8 Time change of total transport in the three cases shown in Fig.7. The arrows indicate the time used to compare the two methods.

Fig. 9 The mean number of iterations M and the accuracy of solution ($-\log_{10}(A)$) plotted against $-\log_{10}(\epsilon)$ for Case I.

Fig. 10 Same as Fig.9 but for Case II.

Fig. 11 Same as Fig.9 but for Case III. The broken lines in the left figure show those for Case II.

Fig. 12 Computer time needed to calculate 400 time steps for Case I and 800 time steps for Case II and III, plotted against mean number of iterations (M) for various values of ϵ . The points along each line (from lower left to upper right) correspond to ϵ values of 10^{-4} , 10^{-6} , 10^{-8} , 10^{-10} and 10^{-12} , respectively. The broken line is for Case I and the solid line for Cases II and III. The points with a triangle are for H-method and those with a circle for K-method.

circulation with realistic topography, in which there are a lot of islands.

Secondly, in the initial-value problems, the conditions for islands (*i.e.*, pressure is a single-valued function around an island) are used only once in each time step in K-method. But in H-method, the conditions are used the same number of times as the number of iterations in each time step. This means that the computation time required to calculate each iterative step is shorter in K-method than in H-method. In this respect, K-method saves a lot of computation time, especially in cases with many islands.

Finally, in the cyclic channel cases, the convergence speed of iteration is much higher (more than twice) in K-method than in H-

method (Figs.5 and 12). A possible reason for this is that all components of ψ or ψ_t are fixed at both side boundaries in K-method, while ψ or ψ_t is fixed at only one side boundary in H-method. Furthermore, accuracy of solution is higher in K-method (Figs.10 and 11). These mean, for example, that K-method is suitable for simulating the world ocean circulation including the Antarctic Circumpolar Current as a cyclic channel flow.

The only defect of K-method is that it requires some extra memory region to store the time-independent homogeneous components of solution ψ_{it} 's and their related inversion matrix used to obtain the coefficients c_i 's in Eq.(6). The number of homogeneous components is the same as the number of islands, and the number of components of the matrix is square of the number of islands. But in most of the cases dealing with sophisticated ocean circulation models, the extra memory size required in K-method may be trivial compared with the whole required memory size.

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多重連結領域において渦度方程式を解く場合の、
穴緩和法と Kamenkovich 法の計算効率性の比較

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多重連結領域において渦度方程式を解く場合の、穴緩和法と Kamenkovich 法の計算効率性が比較される。Kamenkovich 法には若干の修正が加えられる。Kamenkovich 法では、解の斉次成分を記憶させるために、余分の領域が必要であるが、穴緩和法と比較すると、特に初期値問題においては、いくつかの優れた点がある。すなわち、Kamenkovich 法では穴緩和法に比べて、反復計算の 1 ステップに要する計算時間は短く、さらに、解の収束も速い。特に、サイクリックなチャンネルの場合には、解の収束速度の差は大きい。