Indicial Response of the Electric Resistance Thermometer Composed of Three Concentric Cylinders

by

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Abstract

To account for the experimental characteristic of the Platinum electric resistance thermometer that it does not follow a first order system in usual meteorological conditions, the indicial response of a model for the electric resistance thermometer is obtained by analytically solving its thermal diffusion equation. The model is so simplified as to be composed of three infinite concentric cylinders, *i.e.*, protector sheath, filling material and electric resistance element. The theoretical indicial response is evaluated for various model thermometers with different properties in the filling material. The major results are as follows: (1) over a certain Fourier number the indicial response can be approximated with one term of the exponential function $C \exp(-t/\lambda)$ including two parameters, coefficient $C(>1)$ and exponent λ ; (2) when the Biot number is small, the response speed hardly depends upon the inner structure. It becomes, however, sensitive to the thickness and thermal properties of the filling material as the Biot number increases; (3) the thermal properties of protector sheath and electric resistance element have little effect on the response speed, so long as the filling material is thermally insulating.

1. Introduction

The governing equation of a thermometer is, in the strict sense, the thermal diffusion equation, *i.e.*, first order derivative as to the time variable and second order as to the spatial variables. However, in ordinary meteorological use the inner structure of the thermometer is neglected for simplicity and so are the spatial variables, and thereby the thermometer is approximated to be described by a linear first order differential equation with respect to time (see *e.g.*, Middleton and Spilhaus, 1953 ; Meteorological Office,

1956). Such an approximation to the linear first order system, *i.e.*, Newtonian heating (or cooling), means that the dynamic response of the thermometer depends upon only one quantity, time constant (e-folding time). For some cases, however, the linear first order system is invalid in describing the thermometer's response.

When a thermometer is subjected to large radiative heating, the Newtonian heating does not obey the linear first order differential equation because of the non-linear heating term. However, we do not treat the non-linear first order system like this, but

we investigate the linear yet non-first order system due to the inner structure itself. The criteria of the approximation to the first order system are calculated by Heisler (1947) for simple structures of the one-constituent system such as sphere, cylinder or plate, but it has not yet been calculated for complex structures of multi-constituent systems.

The electric resistance thermometer, in which a metal electric resistance wire is ordinarily put in a cylindrical metal protector sheath, is neither a simple structure nor a one-constituent system. Notwithstanding these facts the approximation of the first order system is usually applied to the electric resistance thermometer *a priori* (see e.g., Middleton and Spilhaus, 1953; Meteorological Office, 1956). The experiment carried out by Tokuue *et al.* (1984), however, indicated that the indicial response of the electric resistance thermometer does not coincide with that of the first order system in usual meteorological conditions.

In this paper we theoretically account for this experimental result by using much simplified model thermometers. We assume that the electric resistance thermometer is composed of three concentric cylinders of infinite length, *i.e.*, protector sheath, filling material and electric resistance element, and obtain its indicial response by analytically solving its thermal diffusion equation. In addition, we set various thermometers in order to investigate the effects of the protector sheath, filling material and electric resistance element on the response speed of the thermometer, and compare them with the experiment by Tokuue *et al.* (1984).

2. Dynamic characteristics

Tokuue *et al.* (1984) measured the indicial response of the Platinum electric resistance thermometer by decreasing the ambient temperature stepwise, in which experiment two ambient materials, air and water, were used. Fig. 1 shows the experimental result of the response of the thermometer. The disper-

sion of data is greater in water than in air, because the thermometer was stirred by hand in water, which treatment caused the relative water speed to vary widely in each trial. On the other hand, air was blown at constant speed parallel to the long axis of the thermometer in the air. Considering the accuracy of measurements, we pick out the thermometer which Tokuue *et al.* (1984) measured repeatedly. Then the e-folding times are summarised in the following way.

Characteristic-1. *The Platinum electric resistance thermometer of protector sheath 10 mm × 150mm (diameter × length) has the e-folding times of 140. sec in air (wind speed 6 m/s) and 33 sec in water (water speed unknown), respectively.*

If the approximation to a first order system is valid, the dimensionless temperature ratio Θ can be represented by the equation, $\ln \Theta = -t/\tau$, where τ is the e-folding time. Nevertheless, the equation does not agree well with the experiment as seen in Fig. 1. We then calculate the best-fit lines for the experimental data using the weight Θ to lessen the dispersion caused by the logarithmic coordinate. The result is summarised as:

Characteristic-2. *When $t/\tau > 1/5$, the best-fit lines in air and in water, respectively, are*

$$\ln \Theta = -1.11t/\tau + \ln 1.12, \sigma = 0.043, \text{ in air,}$$

$$\ln \Theta = -1.10t/\tau + \ln 1.11, \sigma = 0.040, \text{ in water,}$$

where σ represents the standard error.

In Characteristic-2 the e-folding time of Fig. 1 is multiplied by 0.97 so that $\ln \Theta = -1$ at $t = \tau$, and the same correction is made in Characteristic-1. That is, we redetermine τ from the best-fit line. When $t/\tau < 1/5$, the data are too scarce and cease to get the best-fit lines, but we can derive a qualitative feature from Fig. 1; the response is by far slower than that of a first order system. (There scarcely seems temperature decrease in the raw record. (see Tokuue *et al.*, (1984))

3. Mathematical analysis

The actual Platinum electric resistance thermometer has a very complicated inner

Fig. 1 Indicial response of the Platinum electric resistance thermometer. Open and solid circles are the experimental values in air and water, respectively. The solid line represents $\ln \Theta = -t/\tau$. (after Tokuue *et al.*, 1984)

structure ; for instance, the electric resistance element, which consists of a coil sandwiched with a number of electrically insulating sheets like mica, is framed by a metal and plugged into a stainless steel cylindrical protector sheath. We, however, consider intuitively that the response characteristics are determined, to a large extent, by filling material and, to a lesser extent, by the protector sheath and inner materials including a resistance element. Therefore, in mathematical models we treat the inner materials as one material, *i.e.*, electric resistance element and assume that the electric resistance thermometer consists of three concentric cylinders of electric resistance element, filling material, and protector sheath.

Let suffixes 1, 2, 3 refer to electric resistance element, filling material, protector sheath, respectively, and let the thermal diffusivity be κ , thermal conductivity K , outer radius of each material a, b, c ($=r_1, r_2, r_3$), the initial and ambient temperatures T_i and T_f , respectively. The boundary condition at $r=c$ is the radiation type, where the mean thermal emissivity is represented by h . First we investigate the end effect of a finite cylinder. Consider the indicial response of a solid cylinder of radius c and length $2l$. The boundary conditions at both ends are of two types: (1) one end is radiative and the other insulating ; and (2) both ends are radiative. The former type (1), however, becomes equivalent to the latter type when the length

is doubled, and hence we treat here only the

latter type. The solution is given by

$$\Theta(z, r, t) = R(r, t) \sum_m \frac{2(K^2 g_m^2 + h^2 l^2) \sin g_m}{g_m(K^2 g_m^2 + Khl + h^2 l^2)} \cos\left(\frac{g_m z}{l}\right) \exp\left(-\frac{\kappa g_m^2}{l^2} t\right),$$

with the eigenvalue equation

$$\frac{K}{hl} = \frac{\cot g_m}{g_m},$$

where $R(r, t)$ is the solution of a cylinder of infinite length (see *e.g.*, Carslaw and Jaeger, 1959). The end effect appears in coefficients and exponents. Since we evaluate the end effect not locally but globally, we average the coefficients with respect to the whole length. When the length is much larger than the radius, $l \gg c$, the exponent term can be neglected compared with that of the infinite cylinder. On the other hand, the coefficient is almost 1 when the Biot number (Bi) of axial direction is very small, $K/hl \ll 1$, while it approaches $8/\pi^2$ ($=0.81$) when Bi increases to infinity. For a solid cylinder of $l \gg c$ therefore the end effect can be represented by

$$\Theta(r, t) = \begin{cases} R(r, t) & K/hl \ll 1 \\ 0.81R(r, t) & K/hl \gg 1 \end{cases}$$

It is not easy to analytically evaluate the model thermometer, *i.e.*, a composite finite cylinder, and thereby we assume that the end effect of the model is as negligible as that of

a solid cylinder when $l \gg c$ and Bi of axial direction is small. The latter condition is replaced with the condition that Bi of radial direction is small, because Bi of radial direction is much larger than that of axial direction, $K/hc \gg K/hl$, for $l \gg c$. Since the electric resistance thermometer Tokue *et al.* (1984) used can be regarded as $l \gg c$ (see Characteristic-1), and Bi 's of radial direction in their experiments can be inferred to be small as will be shown later, we will neglect the end effect of the model. The spatial variable hence becomes only r , leading to the governing equations:

$$\frac{\partial T_j}{\partial t} = \kappa_j \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) T_j, \quad j=1, 2, 3. \quad (1)$$

$$T_j = T_i, \quad \text{when } t=0, \quad j=1, 2, 3. \quad (2)$$

$$T_j = T_{j+1}, \quad \text{at } r=r_j, \quad j=1, 2. \quad (3)$$

$$K_j \frac{\partial T_j}{\partial r} = K_{j+1} \frac{\partial T_{j+1}}{\partial r}, \quad \text{at } r=r_j, \quad j=1, 2. \quad (4)$$

$$K_3 \frac{\partial T_3}{\partial r} = h(T_f - T_3), \quad \text{at } r=r_3 \quad (5)$$

The solutions are (see Appendix)

$$\Theta_1 = -2 \sum_{n=1}^{\infty} \frac{\omega P(a, a, \alpha_n) P(a, a, \mu \alpha_n) J_0(\nu \alpha_n r)}{DN(\alpha_n)} \frac{\exp(-\kappa_2 \alpha_n^2 t)}{\alpha_n} \quad (6)$$

$$\Theta_2 = -2 \sum_{n=1}^{\infty} \frac{P(b, b, \nu \alpha_n) V(r, a, \alpha_n r)}{DN(\alpha_n)} \frac{\exp(-\kappa_2 \alpha_n^2 t)}{\alpha_n} \quad (7)$$

$$\Theta_3 = -2 \sum_{n=1}^{\infty} \frac{U(r, \alpha_n)}{DN(\alpha_n)} \frac{\exp(-\kappa_2 \alpha_n^2 t)}{\alpha_n} \quad (8)$$

with the eigenvalue equation

$$\frac{K_3}{hc} \mu c \alpha_n = \frac{V(\alpha_n) P(c, b, \mu \alpha_n) - \sigma W(\alpha_n) O(c, b, \mu \alpha_n)}{V(\alpha_n) P(c, b, \mu \alpha_n) + \sigma W(\alpha_n) O(c, b, \mu \alpha_n)}, \quad (9)$$

where

$$\mu = \sqrt{\kappa_2/\kappa_3}, \quad \sigma = K_2/K_3/\mu, \quad \nu = \sqrt{\kappa_2/\kappa_1}, \quad \omega = K_2/K_1/\nu, \quad (10)$$

$$O(x, y, z) = J_0(xz)Y_0(yz) - Y_0(xz)J_0(yz), \quad (11)$$

$$P(x, y, z) = J_0(xz)Y_1(yz) - Y_0(xz)J_1(yz), \quad (12)$$

$$Q(x, y, z) = J_1(xz)Y_1(yz) - Y_1(xz)J_1(yz), \quad (13)$$

$$S(z) = (1 - \nu\omega)J_1(\nu za)P(b, a, z) + (\omega - \nu)J_0(\nu za)O(b, a, z), \quad (14)$$

$$T(z) = (1 - \nu\omega)J_1(\nu za)Q(b, a, z) - (\omega - \nu)J_0(\nu za)P(a, b, z), \quad (15)$$

$$V(x, y, z) = \omega J_0(\nu yz)P(x, y, z) - J_1(\nu yz)O(x, y, z), \quad V(z) = V(b, a, z), \quad (16)$$

$$W(x, y, z) = \omega J_0(\nu yz)Q(x, y, z) - J_1(\nu yz)P(x, y, z), \quad W(z) = W(b, a, z), \quad (17)$$

$$U(r, z) = V(z)P(r, b, \mu z) - \sigma W(z)O(r, b, \mu z), \quad (18)$$

$$\begin{aligned} DN(z) = & \left[\{aS(z) - bW(z)\}P(c, b, \mu z) - \sigma \{aT(z) + bV(z)\}O(c, b, \mu z) \right. \\ & \left. - \mu \{cQ(c, b, \mu z) - bO(b, c, \mu z)\}V(z) + \sigma \mu \{bP(c, b, \mu z) - cP(b, c, \mu z)\}W(z) \right] \\ & - \frac{K_3}{hc} \mu cz \left[\{aS(z) - bW(z)\}Q(c, b, \mu z) + \sigma \{aT(z) + bV(z)\}P(b, c, \mu z) \right. \\ & \left. - \mu \{cP(c, b, \mu z) - bP(b, c, \mu z)\}V(z) + \sigma \mu \{bQ(c, b, \mu z) - cO(c, b, \mu z)\}W(z) \right], \quad (19) \end{aligned}$$

J_0 and J_1 are the Bessel functions of the first kind of order 0 and 1, respectively, and Y_0 and Y_1 those of the second kind of order 0 and 1, respectively.

All the solutions of Eqs.(6)-(8) can be formally represented by

$$\Theta = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{t}{\lambda_n}\right), \quad (20)$$

where C_n is a function of r and $\lambda_n = (\kappa_2 \alpha_n^2)^{-1}$. Θ is then approximated by a finite sum of exponential functions according to the time intervals as

$$\Theta = C_1 \exp\left(-\frac{t}{\lambda_1}\right), \quad t_1 < t < \infty,$$

$$\Theta = C_1 \exp\left(-\frac{t}{\lambda_1}\right) + C_2 \exp\left(-\frac{t}{\lambda_2}\right), \quad t_2 < t < t_1,$$

$$\Theta = C_1 \exp\left(-\frac{t}{\lambda_1}\right) + \cdots + C_n \exp\left(-\frac{t}{\lambda_n}\right), \quad t_n < t < t_{n-1},$$

where t_n is the time beyond which the solution is represented by a sum of n eigenfunctions with reasonable precision. If the e-folding time τ is larger than t_1 as in Fig. 1, then τ takes a simple form:

$$\tau = (1 + \ln C_1) \lambda_1. \quad (21)$$

4. Numerical calculation of the models' responses

4.1 Parameter setting

Making the radius of each cylinder non-dimensional form with respect to the outer radius of the protector sheath, " c ", the eigenvalue becomes $c\alpha_n$ and Eqs.(6)-(8) come to include 6 parameters, a/c , b/c , μ , σ , ν , and

ω . Since the details of the electric resistance thermometer Tokuue *et al.* (1984) used is not described, we set the following thermometer as a prototype thermometer (I): The outer radius and thickness of the protector sheath are 5 mm and 1 mm, respectively, and the thickness of the filling material is 0.5 mm, leading to $a/c = 0.7$ and $b/c = 0.8$. To decrease the number of parameters, we set the electric resistance element to be composed of the same material as the protector sheath, i.e., $\omega = \sigma$, $\nu = \mu$. This setting is, as will be shown later, based upon the fact that the response speed hardly depends upon the thermal properties of the protector sheath and resistance element, but mainly upon that of the filling material, when the filling material is not so good a thermal conductor. The protector sheath is made of 18·8 stainless steel (Ni: 18%, Cr: 8%), and the filling material is composed of silicon grease for heat sink (air is also used as the filling material in another thermometer). Table 1 shows the physical properties of air, and silicon grease, and 18·8 stainless steel, respectively.

To investigate the effects of the properties of each cylinder on the response characteristics of the model thermometer, we also set various thermometers (II)-(XIII). The thermometers (II)-(VII) in Table 2 have silicon grease of various thermal properties but the same thickness in filling materials. The set of (σ, μ) or (τ, ν) in the thermometers (II)-(VII) is multiplied or divided by $\sqrt{2}$ of that in the thermometer (I). The effect of the thickness of the filling material is studied, by changing the parameter a/c , in the thermometers (VIII)-(XIII) as shown in Table 3, where the thermometers (VIII) and (IX) have silicon grease fillings and the ones (X)-(XIII) have air fillings.

4.2 Characteristics of the prototype thermometer (I)

Fig. 2 shows the graph of $Bi (= K_3/hc)$ vs $(\mu c \alpha_1)^{-2}$ for some model thermometers and a solid cylinder, in which $\omega = \sigma = \nu = \mu = 1$. As $\lambda_1 = c_2 \kappa_3^{-1} (\mu c \alpha_1)^{-2}$, λ_1 can be obtained by multiplying the scale of coordinate in Fig. 2 by $c^2 \kappa_3^{-1}$. (Multiplication of

5.6 yields λ_1 in unit of second for the 18·8 stainless steel.) Comparisons between the experiment and the model will be made in terms of C_1 and λ_1 , because these two quantities can describe the experimental results with sufficient accuracy (see Characteristic-2).

We can derive important features of λ_1 by examining the dependence of λ_1 on Bi in Fig. 2. The entire Bi region can be approximately separated into three regions: When $Bi < 0.1$, $(\mu c \alpha_1)^{-2} \propto Bi$, so that $\lambda_1^{-1} \propto c/h$; similarly, $\lambda_1^{-1} \propto (hK_3)^{-1/2} c^{3/2}$ when $0.1 < Bi < 1.0$; and $\lambda_1^{-1} \propto c^2 K_3^{-1}$ when $1.0 < Bi$. Hence, λ_1^{-1} does not depend upon the thermal conductivity K_3 of the outer cylinder in the region $Bi < 0.1$, and does not upon the thermal emissivity h in the region $Bi > 1.0$, while λ_1^{-1} depends upon both quantities, K_3 and h , in the intermediate region of $0.1 < Bi < 1.0$. These features are qualitatively common in the models and a solid cylinder, because the graphs of Bi vs $(\mu c \alpha_1)^{-2}$ are conformable to one another as seen in Fig. 2. Differences among them are the values of the three regions of Bi .

Next we consider the coefficient $C_1(r)$, particularly the one in the first cylinder. Our concern consists in the areally averaged value in the resistance element from $r=0$ to $r=a$, $2C_1(0)J_1(\nu \alpha_1 a)/(\nu \alpha_1 a)$, but we will use the center value $C_1(0)$. This is because $2J_1(\nu \alpha_1 a)/(\nu \alpha_1 a)$ is nearly equal to 1 when Bi is not so large. Fig. 3 shows the curves of $C_1(0)$ for the various model thermometers and a solid cylinder. There is a distinct feature in $C_1(=C_1(0))$ of the model thermometer; it shows a maximum value for the model thermometers, whereas it monotonically increases with Bi for a solid cylinder. Thus the behavior beyond the maximum value exhibits the particular characteristics of the composite cylinder which contains a thermal insulating cylinder. We will refer to the Bi at which C_1 takes the maximum value as the critical Biot number ($Bicr$).

Fig. 4 shows dimensionless temperature ratios at $r=0$ for the thermometer(I) and a solid cylinder. The time of the one- and two-

Table 1 Physical properties of air, silicon grease, and 18·8 stainless steel. Density, thermal conductivity, and diffusivity of silicon grease are the arithmetic means of those of three silicon greases for heat sink, KS609, KS612, and G746, made by the Sinetu Chemistry Co. Ltd., and specific heat is calculated from those of their base oils. σ and μ are the values when the third cylinder is made of 18·8 stainless steel.

Material	Physical Properties					
	Density	Specific Heat	Thermal Conductivity	Diffusivity	σ	μ
	ρ (kg/m ³)	S (kcal/kgK)	K (kcal/mhrK)	κ (m ² /hr)		
Air	1.17	0.24	0.022	0.079	0.71×10^{-4}	0.22
Silicon Grease	2500	0.36	0.54	0.6×10^{-5}	0.2	0.19
Stainless Steel	7800	0.11	14.0	0.016		

Table 2 Parameters for thermometers (I)~(VII). The second cylinders in all these thermometers are of silicon grease. The last two rows show the ratios of thermal conductivity to that of (I) when K_2 or K_3 is fixed.

Parameter	Type						
	I	II	III	IV	V	VI	VII
a/c	.70	.70	.70	.70	.70	.70	.70
b/c	.80	.80	.80	.80	.80	.80	.80
σ	.20	.28	.20	.28	.14	.20	.14
μ	.19	.27	.19	.27	.13	.19	.13
τ	.20	.20	.28	.28	.20	.14	.14
ν	.19	.19	.27	.27	.19	.13	.13
$K_1 K_3 (K_2:\text{fix})$		1 1/2	1/2 1	1/2 1/2	1 2	2 1	2 2
$K_1 K_2 (K_3:\text{fix})$		2 2	1/2 1	1 2	1/2 1/2	2 1	1 1/2

Table 3 Parameters for thermometers (I), (VIII)~(XIII). The second cylinder is silicon grease in (I), (VIII), and (IX), while it is air in (X)~(XIII). The thermal properties of the first and the third are equivalent in all these thermometers.

Parameter	Type						
	I	VIII	IX	X	XI	XII	XIII
a/c	.70	.75	.60	.78	.75	.70	.60
b/c	.80	.80	.80	.80	.80	.80	.80
σ	.20	.20	.20	.7E-3	.7E-3	.7E-3	.7E-3
μ	.19	.19	.19	.2E+1	.2E+1	.2E+1	.2E+1

Fig. 2 $(\mu c \alpha)^{-2}$ versus Bi for various model thermometers.

term approximations, t_1 and t_2 , the errors of which are within 0.5%, are shown by arrows and triangles, respectively, in Fig. 4 (Multiplication by the e-folding time τ yields the Fourier number $(\kappa_3(\mu c)^2 t)$. As the coefficient C_1 is the intercept of the line of one-term approximation $\ln \Theta(t) = -(t - t_1)/\lambda_1 + \ln \Theta(t_1)$, $\ln C_1 = t_1/\lambda_1 + \ln \Theta(t_1)$. Hence, C_1 becomes large when the inclination λ_1^{-1} , t_1 or $\Theta(t_1)$ is large.

In a solid cylinder t_1 increases very slowly ($t_1 = 0.03, 0.05, 0.05$ for $Bi = 0.1, 1.0, 10$ in Fig. 4), while $\Theta(t_1)$ decreases rapidly ($\Theta(t_1) = 0.99, 0.80, 0.45$ for $Bi = 0.1, 1.0, 10$). However, the rapid increase of λ_1^{-1} (see Fig. 2) outweighs the decrease of $\Theta(t_1)$ when $0.1 < Bi < 10$, so that C_1 becomes an increasing function of Bi and shows a sharp rise in the region of $0.1 < Bi < 10$. When a thermally insulating cylinder is sandwiched between thermal conducting ones as in the thermom-

eter (I), t_1 and $\Theta(t_1)$ behave differently, though λ_1^{-1} shows a similar increase (see Fig. 2); t_1 rapidly decreases with Bi ($t_1 = 0.19, 0.12, 0.04$ for $Bi = 0.1, 1.0, 10$), while $\Theta(t_1)$ increases after decreasing ($\Theta(t_1) = 0.89, 0.72, 0.92$ for $Bi = 0.1, 1.0, 10$). Thus in the small Bi region the effect of λ_1^{-1} is greater than those of t_1 and Θ , but the effect of t_1 comes to surpass the others as Bi increases, and thereby C_1 begins to decrease at a certain Bi . This characteristic of C_1 comes to approach that of a solid cylinder as the degree of the thermal insulation of the second cylinder weakens. That is, the critical Biot number (Bi_{cr}) increases, and the decrease in C_1 after the maximum value becomes small.

4.3 Effects of the thermal conductivity.

If we assume that the thermometers (I)-(VII) are different only in thermal conductivity, then three cases can be thought

Fig. 3 $C_1(0)$ versus Bi for various model thermometers.

according to which is the fixed thermal conductivity. We treat here only two cases: one is fixed K_3 and the other is fixed K_2 . We concentrate on λ_1 though the difference of C_1 among the thermometers (I)-(VII) is not small. This is because the behavior of C_1 is too complicated to treat as seen in Fig. 3.

(a) When K_3 is fixed.

In this case the thermal conductivities of the first and second cylinders, K_1 and K_2 , become twice or half that of the thermometer (I) in every thermometer. The relation of thermometer types to the thermal conductivity ratios are shown in the last row in Table 2. Since Bi is common to all the thermometers owing to fixed K_3 , the order of magnitudes of λ_1 can be obtained easily, (II) < (IV) < (VI) < (I) < (III) < (VII) < (V), where the ratios of λ_1 to that of the thermometer (I) fall within the range 0.998-1.02 at $Bi = 0.02$; 0.942-1.13 at $Bi = 0.1$; and 0.677-1.68 at $Bi = 1.0$.

λ_1 of the thermometer (VI) is greater than that of the thermometer (IV), where the ratios of K_1 and K_2 between them are 1/2 and 2, respectively. The same relation holds between the thermometers (VII) and (III). This indicates that the effect of K_2 on λ_1 is greater than that of K_1 . In a large Bi region λ_1 is determined mainly according to the magnitude of K_2 as (II) ~ (IV) << (VI) ~ (I) ~ (III) << (VII) ~ (V). On the other hand, when $Bi < 0.05$, there is substantially no difference among the thermometers (I)-(VII).

(b) When K_2 is fixed.

The thermal conductivities of the first and third cylinders, K_1 and K_3 , become twice or half that of the thermometer (I) as shown in the second to the last row in Table 2. The order of λ_1 becomes (VII) < (V) < (VI) < (I) < (III) < (II) < (IV). Note that Bi is not common for this case. When we take $Bi = 0.02, 0.1, 1.0$ for the thermometer (I), the ratios of λ_1 to

Fig. 4 Dimensionless temperature ratio at $r=0$ for the thermometer (I) (C, right ordinate) and a solid cylinder (S, left ordinate). Numbers near the curves represent Bi . $\uparrow$ (or $\downarrow$) and Δ (or ∇) are the times of one-term and two-term approximations (0.5% error), t_1 and t_2 respectively. Time in the abscissa is divided by τ at $Bi=0.1$.

that of the thermometer (I) fall within the ranges, 0.998-1.00, 0.990-1.02, and 0.995-1.09, respectively. The range of the ratios of λ_1 is very small, nearly 1, for any Bi , in contrast with when K_3 is fixed. Also in this case λ_1 is more sensitive to K_3 than to K_1 , but the difference between the effects of K_3 and K_1 is much smaller than that between the effects of K_2 and K_1 under fixed K_3 .

4.4 Effects of the filling material

We consider here the effect of filling material in comparison with that of thermal conductivity. Consider the thermometers (VII) and (IV). Compared with the thermometer (I), the former is half in thickness in the filling

material (silicon grease), and the latter is twice in value in K_2 (see Table 2 and 3). Though λ_1 of the thermometer (VII) is larger than that of the thermometer (VI), the difference in λ_1 is very small; it is less than 1% for $Bi < 0.1$, 4% at $Bi = 1.0$, and 6% at $Bi = 100$. A similar feature can be seen for the thermometers (IX) and (V). Compared with the thermometer (I), the former has double thickness in the filling material (silicon grease), and the latter is half in value in K_2 . The difference is 3.6% at $Bi = 0.1$, 12% at $Bi = 1.0$, and 14% at $Bi = 100$. These features indicate that the increase in thickness of the second cylinder and the decrease in thermal conductivity of the second cylinder exert an

approximately equivalent effect on λ_1 .

When air is used as the filling material instead of silicon grease, the response characteristics change greatly as shown in Fig. 2. So they do, even when the thickness becomes 1/5 of that of the thermometer (I) as in the thermometer (X). This is because air insulates much more than silicon grease for heat sink. The dependence of λ_1 on the thickness of air is similar to the case of silicon grease.

5. Comparison between the model and the experiment

It needs Bi or the mean thermal emissivity h in the experiment to compare the model with the experiment. There are, however, two difficulties in evaluating h : one is the effect of turbulence, which can raise h as much as about 50% (McAdams, 1954), and the other is that of the film temperature. Since h depends upon fluid viscosity, which changes with time in the experiment of indicial response, h can not be maintained constant. (The changes of h in air and water inverse to each other because viscosity increases with temperature in air but decreases in water.) For these reasons, we roughly estimate h and adopt the value at 20°C with no turbulence.

When a wind of 6 m/s blows perpendicularly to the cylinder of 1 cm diameter, the Reynolds number becomes 3800 and the corresponding value of h is 66 kcal/m²hrK (Eckert and Drake, 1951) or 75 kcal/m²hrK (McAdams, 1954). We adopt these values for the reasons mentioned above, though a wind of 6 m/s blow parallel to the thermometer in the experiment by Tokuue *et al.* (1984). Then Bi takes the value of 0.024-0.027 for the prototype thermometer(I). On the other hand, in the experiment in water the relative of water was referred to. So, we guess it about 10-30 cm/s. Then the Reynolds number becomes 1000-3000 and $h = 160-280$ kcal/m²hrK (McAdams, 1954), which makes Bi 0.057-0.10 for the prototype thermometer(I).

Since the time of one-term approximation is shorter than the e-folding time, $t_1 < \tau$, in the experiment as seen in Fig. 1, λ_1 can be

calculated from Eq.(21) as 30 sec at $Bi \sim 0.024-0.027$ (in water), and 126 sec at $Bi \sim 0.057-0.10$ (in air), respectively. The thermometers whose second cylinder consists of air cannot have less λ_1 than 31 sec even if the thickness is as thin as 0.1 mm. Hence we here do not treat these ones.

On the other hand, the thermometers, (I)-(IX), the filling materials of which are silicon grease, satisfy both relations between Bi and λ as seen in Fig. 2, the differences of λ_1 among them being very small in the region of $Bi < 0.1$, for instance less than 1% at $Bi = 0.05$. In addition, their range of λ falls within the range caused by the estimation error of Bi , and hence we cannot identify the type of thermometer most closely similar to that used in the experiment from a comparison of λ_1 .

Nevertheless, these facts reveal to us important characteristics, that is, response speed of the model electric resistance thermometer cannot be improved much even if the parameters are doubled or halved, so long as $Bi < 0.1$. For example, if another kind of stainless steel such as AISI 420 or 430 is used instead of the 18·8 stainless steel (AISI 304), the thermal conductivity increases by about 50%, and the density and specific heat scarcely change (Lewis, 1977). The improvement of the e-folding time, however, exhibits no more than 1% at $Bi = 0.02$.

In comparing the model and the experiment in regard to C_1 we use the center value $C_1(0)$ instead of the areally averaged value, because there is virtually no difference between the two values in the small Bi region as has been stated. For simplicity we will write $C_1(0)$ as C_1 . If the standard errors in Characteristic-2 are brought about by the dispersion of C_1 , then the values of C_1 can be estimated as 1.12 ± 0.04 in water and 1.11 ± 0.05 in air. Of all the models there is no thermometer with C_1 agreeing with the experimental value in air, while there are some with C_1 's agreeing with the experimental value in water. Even if Bi in air is doubled, the increase of C_1 is too little. This disagreement may be ascribed to the fact that the

model thermometers may be too simplified, compared with the actual thermometer, though there is good agreement for λ_1 . We assume, however, that there are some models which intrinsically represent the actual thermometer.

The e-folding time in air is greater by 5 times than that in water, so that the values of C_1 cannot be similar to each other, for, in the range of $Bi < 0.1$, C_1 is an increasing function of Bi for the silicon grease thermometers as seen in Fig. 4, and the difference in Bi by 3 times is expected to make the difference in C_1 about several percent. If we estimate C_1 in air by a slightly less value as 1.05, 94% of C_1 in water, then three thermometers (V), (VII), and (IX) agree with the experiments both in air and in water.

In the thermometers (V) and (VII) the thermal conductivities of the filling material are half as great as that of the silicon grease used in the model (I). Since the thermal conductivities of the silicon grease for heat sink are not so wide range, neither of the thermometers are good as the model of the electric resistance thermometer, the filling material of which is made of silicon grease. On the other hand, the thermometer (IX) has the same thermal conductivity in the filling material as the silicon grease for heat sink, so that the thermometer (IX) is the most closely similar model.

6. Discussion

As shown above, an insulating second

cylinder makes C_1 a decreasing function of Bi over $Bicr$, which phenomenon is an intrinsic feature of a composite cylinder, because this is not seen in a solid cylinder. In every model thermometer λ_1 is nearly half that of a solid cylinder at $Bicr$ (see Figs. 2 and 3). The temperature profile in the second cylinder is expected to drastically change when Bi exceeds $Bicr$. Consider the thermometer (I), whose $Bicr$ is about 0.7. The temperature difference between the inner and outer surfaces of the second cylinder, i.e., $r=a$ and $r=b$, is very small at $Bi=0.01$, and still smaller at $Bi=0.1$, but it shows a very large value at $Bi=1.0$ as shown in Fig. 5. At $Bi=1.0$ the temperature change at the outer surface ($r=b$) already decreases below 37% when the temperature change at the inner surface ($r=a$) reaches 90%. In a solid cylinder such a large temperature difference does not occur even if Bi is very large (see the example at $Bi=10$ in Fig. 5). The large temperature difference in the second cylinder implies that the insulating material truly works as an insulator for $Bi > Bicr$. Thus $Bicr$ can be used as an index to the insulating efficiency, since smaller $Bicr$ means better insulation.

We can induce the criterion of Bi for the approximation to a first order system by using the theoretical results obtained above. When $Bi \rightarrow 0$, then $C_1 \rightarrow 1$, $t_1 \rightarrow 0$, and Θ can be represented by the first term of Eq.(20). Expanding the eigenvalue equation (9), we obtain the first eigenvalue α and the e-folding time τ :

$$\alpha = \sqrt{2Bi \left(\frac{\omega}{\mu} \right) \{ \sigma \nu a^2 + \sigma \omega (b^2 - a^2) + \mu \omega (c^2 - b^2) \}} , \quad (22)$$

$$\tau = \frac{\rho_1 S_1 a^2 + \rho_2 S_2 (b^2 - a^2) + \rho_3 S_3 (c^2 - b^2)}{2hc} \quad (23)$$

where ρ_j and S_j represent the density and the specific heat of j -th cylinder, respectively. Eq.(23) is identical with the e-folding time derived from the first order system, in which the governing equation is represented by

$$D \left(\frac{d\Theta}{dt} \right) + 2\pi hc \Theta = 0, \quad (24)$$

Where D is the total heat capacity of the thermometer. Thus the approximation to the

Fig. 5 Dimensionless temperature ratios at the inner(a) and outer(b) surfaces of the second cylinder for the thermometer (I) (solid curves) and a solid cylinder(dotted curves). Numbers near the curves represent Bi . Symbols and abscissa are the same as in Fig. 4. $\tau (= \tau)$ is the value at $r=0$ and represented by the Fourier number.

first order system can be applied only to the cases where the cooling (or heating) speed (i.e., Bi) is so small that the eigenvalue α approximately satisfies Eq.(22). We do not quantitatively derive the criterion of Bi for the first order system, because the criterion depends upon the degree of accuracy in the approximation.

The insulating material of the second cylinder does not always work as an insulator, but shows conducting properties for small Bi . Generally speaking, non-metallic materials have smaller heat capacities than metals. Thus, when Bi is small enough and Eq.(23) is valid, the e-folding time of the

composite cylinder which includes nonmetallic materials can be smaller than that of a metal composite cylinder. In fact, the thermometers (X)-(XIII), in which the second cylinder are filled with air, have smaller e-folding times than that of solid cylinder, and the ratios of their e-folding times to that of a solid cylinder at $Bi = 0.001$ are 0.98, 0.94, 0.88, and 0.76, respectively. On the contrary, when the second cylinder is filled with silicon grease as in the thermometers (I)-(IX), λ_1 does not cross that of a solid cylinder as $Bi \rightarrow 0$, but asymptotically approaches it.

7. Summary

To account for the experimental characteristic of the Platinum electric resistance thermometer that it does not follow a first order system in usual meteorological conditions, we used a simplified model thermometer and obtained its indicial response by analytically solving its thermal diffusion equation. The model of the electric resistance thermometer is composed of three concentric cylinders of infinite length, *i.e.*, protector sheath, filling materials, and electric resistance element. The first and third cylinders are made of conducting materials, while the second one is made of insulating material. Various thermometers, which differ in the physical properties of the filling material, are used in order to investigate the effects of the protector sheath, filling material and resistance element on the response speed of the thermometer.

The major results are as follows. (1) After a certain time (over a certain Fourier number) the solution can be approximated with one term of the exponential function, $C \exp(-t/\lambda)$, which needs two parameters, coefficient $C(>1)$ and exponent λ . (2) When Bi is small, the response speed hardly depends upon the inner structure. It becomes, however, sensitive to the thickness and thermal properties of the filling material as Bi increases. (3) The thermal properties of the protector sheath and electric resistance element have little effect on the response speed, so long as the filling material is thermally insulating.

Appendix I

We will solve the diffusion equation of three composite cylinders by the same method as used in the case of two composite cylinders (Kataoka, 1933). The definitions of the terms and functions are the same as those in the text. Let $\Phi_j = T_j - T_{i(j=1,2,3)}$ and $\Phi_f = T_f - T_i$, then Eqs.(1)-(5) become

Fig. A-1 Integral path p in α -plane.

$$\frac{\partial \Phi_j}{\partial t} = \kappa_j \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \Phi_j, \quad j=1, 2, 3. \quad (\text{A-1a,b,c})$$

$$\Phi_j = 0, \quad \text{when } t=0, \quad j=1, 2, 3, \quad (\text{A-2a,b,c})$$

$$\Phi_j = \Phi_{j+1}, \quad \text{at } r=r_j, \quad j=1, 2, \quad (\text{A-3a,b})$$

$$K_j \frac{\partial \Phi_j}{\partial r} = K_{j+1} \frac{\partial \Phi_{j+1}}{\partial r}, \quad \text{at } r=r_j, \quad j=1, 2, \quad (\text{A-4a,b})$$

$$K_3 \frac{\partial \Phi_3}{\partial r} = h(\Phi_f - \Phi_3), \quad \text{at } r=r_3. \quad (\text{A-5})$$

First, we consider the system in which the boundary condition at $r=r_3$ is not a radiative type as Eq.(A-5) but a fixed type as

$$\Phi_3 = \Phi_f. \quad (\text{A-6})$$

Then, the solutions are

$$\Phi_1 = A_1 J_0(\nu \alpha r) \exp(-\kappa_2 \alpha^2 t), \quad (\text{A-7a})$$

$$\Phi_2 = \{A_2 J_0(\alpha r) + B_2 Y_0(\alpha r)\} \exp(-\kappa_2 \alpha^2 t), \quad (\text{A-7b})$$

$$\Phi_3 = \{A_3 J_0(\mu \alpha r) + B_3 Y_0(\mu \alpha r)\} \exp(-\kappa_2 \alpha^2 t), \quad (\text{A-7c})$$

The boundary conditions of Eqs.(A-3a, b) and (A-4a, b) give a relation among the coefficients A_j and B_j ,

$$D^t (A_2, B_2, A_3, B_3) = A_1^t (J_0(\nu \alpha a), J_1(\nu \alpha a), 0, 0), \quad (\text{A-8})$$

where superscript t denotes a transpose operator and D is a 4×4 matrix,

$$D = \begin{pmatrix} J_0(\alpha a) & Y_0(\alpha a) & 0 & 0 \\ \omega J_1(\alpha a) & \omega Y_1(\alpha a) & 0 & 0 \\ J_0(\alpha b) & Y_0(\alpha b) & -J_0(\mu \alpha b) & -Y_0(\mu \alpha b) \\ \sigma J_1(\alpha b) & \sigma Y_1(\alpha b) & -J_1(\mu \alpha b) & -Y_1(\mu \alpha b) \end{pmatrix}. \quad (\text{A-9})$$

If we introduce A_1 as

$$A_1 = \frac{\text{Det}(D)}{U(r, \alpha)}, \quad (\text{A-10})$$

where

$$U(r, \alpha) = V(\alpha) P(r, b, \mu \alpha) - \sigma W(\alpha) O(r, b, \mu \alpha), \quad (\text{A-11})$$

then we can obtain A_2 , B_2 , A_3 and B_3 from Eq.(A-9). Inserting them into Eq.(A-7) and reforming Θ_1 , Θ_2 and Θ_3 so as to satisfy the

initial and boundary conditions, *i.e.*, Eqs. (A-2a, b, c) and Eq.(A-6), we have the solutions

$$\Phi_1 = \frac{\Phi_f}{i\pi} \int_p \frac{\omega P(a, a, \alpha) P(b, b, \mu \alpha)}{\alpha U(c, \alpha)} J_0(\nu \alpha r) \exp(-\kappa_2 \alpha^2 t) d\alpha, \quad (\text{A-12a})$$

$$\Phi_2 = \frac{\Phi_f}{i\pi} \int_p \frac{P(b, b, \mu \alpha)}{\alpha U(c, \alpha)} V(r, a, \alpha) \exp(-\kappa_2 \alpha^2 t) d\alpha, \quad (\text{A-12b})$$

$$\Phi_3 = \frac{\Phi_f}{i\pi} \int_p \frac{U(r, \alpha)}{\alpha U(c, \alpha)} \exp(-\kappa_2 \alpha^2 t) d\alpha, \quad (\text{A-12c})$$

where the integral path p is shown in Fig. A-1. These solutions can be easily proved to satisfy the initial conditions of Eqs.(A-2a, b, c) and the boundary conditions of Eqs.(A-3a, b), (A-4a, b) and (A-6).

Next we return to the case whose boundary condition at $r=r_3$ is the radiative type given by Eq.(A-2). It follows from Eqs.(A-12a, b, c) that the solutions are

$$\Phi_1 = \frac{\Phi_f}{i\pi} \int_p \frac{\omega P(a, a, \alpha) P(b, b, \mu \alpha)}{\alpha \left[\frac{K_3}{h} \frac{\partial U(r, \alpha)}{\partial r} + U(r, \alpha) \right]_{r=c}} J_0(\nu \alpha r) \exp(-\kappa_2 \alpha^2 t) d\alpha, \quad (\text{A-13a})$$

$$\Phi_2 = \frac{\Phi_f}{i\pi} \int_p \frac{P(b, b, \mu \alpha)}{\alpha \left[\frac{K_3}{h} \frac{\partial U(r, \alpha)}{\partial r} + U(r, \alpha) \right]_{r=c}} V(r, a, \alpha) \exp(-\kappa_2 \alpha^2 t) d\alpha, \quad (\text{A-13b})$$

$$\Phi_3 = \frac{\Phi_f}{i\pi} \int_p \frac{U(i, \alpha)}{\alpha \left[\frac{K_3}{h} \frac{\partial U(r, \alpha)}{\partial r} + U(r, \alpha) \right]_{r=c}} \exp(-\kappa_2 \alpha^2 t) d\alpha, \quad (\text{A-13c})$$

where the integral path p is the same as that in Eqs.(A-11a, b, c). These solutions can also be proved to meet the initial conditions of Eqs.(A-2a, b, c) and the boundary conditions

of Eqs.(A-3a, b), (A-4a, b). Since at $r=c$ the integral path p may be replaced by the one around the origin, q , it follows that

$$K_3 \frac{\partial \Phi_3}{\partial r} + h \Phi_3 = \frac{h \Phi_f}{i2\pi} \oint_q \frac{\exp(-\kappa_2 \alpha^2 t)}{\alpha} d\alpha = h \Phi_f.$$

Thus the boundary condition of Eq.(A-5) is also satisfied.

The function in Eqs.(A-13) are even functions of α and have poles only on the

real axis (see appendix II). Hence, evaluating the integral by residues and rewriting the solutions from Φ to Θ , we have

$$\Theta_1 = -2 \sum_n \frac{\omega P(a, a, \alpha) P(b, b, \mu \alpha_n)}{\alpha_n D N(\alpha_n)} J_0(\nu \alpha_n r) \exp(-\kappa_2 \alpha_n^2 t), \quad (\text{A-14a})$$

$$\Theta_2 = -2 \sum_n \frac{P(b, b, \mu \alpha_n) V(r, a, \alpha_n)}{\alpha_n D N(\alpha_n)} \exp(-\kappa_2 \alpha_n^2 t), \quad (\text{A-14b})$$

$$\Theta_3 = -2 \sum_n \frac{U(r, \alpha_n)}{\alpha_n D N(\alpha_n)} \exp(-\kappa_2 \alpha_n^2 t), \quad (\text{A-14c})$$

with

$$DN(\alpha) = \left(\frac{\partial}{\partial \alpha} \right) \left[\frac{K_3}{h} \frac{\partial U(r, \alpha)}{\partial r} + U(r, \alpha) \right]_{r=c} \quad (\text{A-15})$$

where α_n is the n -th root of the eigenvalue equation

$$\left[\frac{K_3}{h} \frac{\partial U(r, \alpha)}{\partial r} + U(r, \alpha) \right]_{r=c} = 0. \quad (\text{A-16})$$

Appendix II

II-A The eigenvalue equation (A-16) has no complex root $\xi + i\eta$ ($\xi \neq 0$).

If we define U_1 , U_2 and U_3 in the ranges $[0, a]$, $[a, b]$ and $[b, c]$, respectively, as

$$U_1(r, \alpha) = \omega P(b, b, \mu \alpha) P(a, b, \alpha) J_0(\nu \alpha r), \quad (\text{A-17a})$$

$$U_2(r, \alpha) = P(b, b, \mu \alpha) V(r, a, \alpha), \quad (\text{A-17b})$$

$$U_3(r, \alpha) = V(b, a, \alpha) P(r, b, \mu \alpha) - \sigma W(b, a, \alpha) O(r, b, \mu \alpha), \quad (\text{A-17c})$$

real axis (see appendix II). Hence, evaluating the integral by residues and rewriting the solutions from Φ to Θ , we have

$$\frac{\partial}{r \partial r} \left(r \frac{\partial U_j}{\partial r} \right) + \epsilon_j^2 \alpha^2 U_j = 0, \quad j=1, 2, 3, \quad (\text{A-18a,b,c})$$

where $(\epsilon_1, \epsilon_2, \epsilon_3) = (\nu, 1, \mu)$. Let $V_j = U_j(r, \beta)$, $j=1, 2, 3$, then Eqs.(A-18a, b, c) and the corresponding equations for V_j yield

$$\left(\frac{\nu}{\omega} \right) (\beta^2 - \alpha^2) \int_0^a r U_1 V_1 dr = \left(\frac{1}{\nu \omega} \right) \left[r V_1 \frac{\partial U_1}{\partial r} - r U_1 \frac{\partial V_1}{\partial r} \right]_0^a, \quad (\text{A-19a})$$

$$(\beta^2 - \alpha^2) \int_a^b r U_2 V_2 dr = \left[r V_2 \frac{\partial U_2}{\partial r} - r U_2 \frac{\partial V_2}{\partial r} \right]_a^b, \quad (\text{A-19b})$$

$$\left(\frac{\mu}{\sigma}\right)(\beta^2 - \alpha^2) \int_b^c r U_3 V_3 dr = \left(\frac{1}{\mu\sigma}\right) \left[r V_3 \frac{\partial U_3}{\partial r} - r U_3 \frac{\partial V_3}{\partial r} \right]_b^c. \quad (\text{A-19c})$$

Since Eqs (A-17a, b, c) lead to the following relations:

$$U_1 = U_2, \quad \left(\frac{1}{\nu\omega}\right) \frac{\partial U_1}{\partial r} = \frac{\partial U_2}{\partial r}, \quad \text{at } r=a, \quad (\text{A-20a})$$

$$U_2 = U_3, \quad \left(\frac{1}{\mu\sigma}\right) \frac{\partial U_2}{\partial r} = \frac{\partial U_3}{\partial r}, \quad \text{at } r=b, \quad (\text{A-20b})$$

and similar equations hold for V_j , the summation of Eqs.(A-19a, b, c) yields

$$\begin{aligned} & (\beta^2 - \alpha^2) \left\{ \left(\frac{\nu}{\omega}\right) \int_0^a r U_1 V_1 dr + \int_a^b r U_2 V_2 dr + \left(\frac{\mu}{\sigma}\right) \int_b^c r U_3 V_3 dr \right\} \\ &= \left(\frac{c}{\mu\sigma}\right) \left[r V_3 \frac{\partial U_3}{\partial r} - r U_3 \frac{\partial V_3}{\partial r} \right]_{r=c}. \end{aligned} \quad (\text{A-21})$$

Let α and β be roots of Eq.(A-16), than

$$\frac{\partial U_3}{\partial r} = -\left(\frac{h}{K_3}\right) U_3, \quad \text{at } r=c, \quad (\text{A-22})$$

and the same equation holds for V_3 . Substituting Eqs.(A-22) and the same equation for

V_3 into Eq.(A-21) we have.

$$(\beta^2 - \alpha^2) \left\{ \left(\frac{\nu}{\omega}\right) \int_0^a r U_1 V_1 dr + \int_a^b r U_2 V_2 dr + \left(\frac{\mu}{\sigma}\right) \int_b^c r U_3 V_3 dr \right\} = 0. \quad (\text{A-23})$$

Eq.(A-23) implies that α and β cannot be complex conjugate to each other. The reason for this is as follows. If α and β are complex conjugate to each other, so are U_i and V_i . Consequently, all the integrals in Eq. (A-23) become positive, and the lefthand side of Eq.(A-23) also becomes non-zero, which result contradicts with Eq.(A-23).

II-B The eigenvalue equation (A-16) has no pure imaginary root.

(i) Let define a function

$$y_n(x) = \frac{Y_n(ix)}{J_n(ix)} - i, \quad n=0, 1, 2, 3 \dots \quad (\text{A-24})$$

then Y_n is a real function when x is real (see Carslaw (1921) in the case of $n=0$). Its derivative is given by

$$\left(\frac{d}{dx}\right) y_n = i \left[J_n(t) \frac{dY_n(t)}{dt} - Y_n(t) \frac{dJ_n(t)}{dt} \right]_{t=ix} / J_n^2(ix) = \frac{2}{\pi x J_n^2(ix)}. \quad (\text{A-25})$$

We consider hereafter that x is positive real. Then we get from Eq.(A-25)

$$y_n(x) = \begin{cases} \text{monotonically increasing,} & n: \text{even,} \\ \text{monotonically decreasing,} & n: \text{odd,} \end{cases} \quad (\text{A-26})$$

because $J_n^2(ix) > 0 (< 0)$ for even (odd) n . Since from Eq.(A-24) $Y_n \rightarrow 0$ as $x \rightarrow \infty$, Eq.(A-26) yields

$$y_n(x) = \begin{cases} < 0, & n: \text{even,} \\ > 0, & n: \text{odd.} \end{cases} \quad (\text{A-27})$$

(ii) Let $a, b(>a), z, \xi$ be positive reals, and a positive definite function be noted by PD ,

$$O(b, a, iz) = J_0(ibz) J_0(iaz) \{y_0(az) - y_0(bz)\} = -PD, \quad (A-28)$$

$$P(b, a, iz) = J_0(ibz) J_1(iaz) \{y_1(az) - y_0(bz)\} = -PD, \quad (A-29)$$

because $J_0(i\xi) = PD$, $J_1(i\xi) = iPD$, $y_0(az) < y_0(bz)$ from Eq.(A-26), $y_1(\xi) > 0$ and $y_0(\xi) < 0$ from Eq.(A-27). Similarly, we can prove

$$V(iz) = iPD, \quad (A-32)$$

$$W(iz) = -PD. \quad (A-33)$$

$$P(a, b, iz) = iPD, \quad (A-30)$$

(iii) Let α be positive real, then it follows from Eqs.(A-28)-(A-33) that

$$Q(b, a, iz) = -PD, \quad (A-31)$$

$$\begin{aligned} & \left(\frac{K_3}{h} \right) \left[\frac{\partial U(r, i\alpha)}{\partial r} + U(r, i\alpha) \right]_{r=c} \\ &= - \left(\frac{K_3}{h} \right) i\mu\alpha \{ V(i\alpha) Q(c, b, i\mu\alpha) + \sigma W(i\alpha) P(b, c, i\mu\alpha) \} \\ & \quad + V(i\alpha) P(c, b, i\mu\alpha) - \sigma W(i\alpha) O(c, b, i\mu\alpha) \\ &= -iPD \{ (iPD) \cdot (-PD) + (-PD) \cdot (iPD) \} + (iPD) \cdot (iPD) - (-PD) \cdot (-PD) \\ &= -PD. \end{aligned} \quad (A-34)$$

Eq.(A-34) means that the eigenvalue equation (A-16) has no positive pure imaginary root. As a result, the eigenvalue equation (A-16) has no negative pure imaginary root either, because Eq.(A-16) is a polynomial function of real coefficients.

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3層の円筒から成る抵抗温度計の過渡応答

柴田清孝

白金抵抗温度計は一階の微分方程式の系で記述されないという実験結果を説明するため抵抗温度計のモデルをつくり、その熱伝導方程式を解析的に解いて過渡応答を求めた。モデルは保護管・充填物質・抵抗線層の3層から成る無限長円筒と単純化されている。この結果を使って充填物質の物理特性の異なる種々のモデルについて過渡応答を評価した。主な結果は次の通りである。(1) 過渡応答はあるフーリエ数以降で $C \exp(-t/\lambda)$ と指数関数の単項式で近似できるが、係数 $C (> 1)$ と指数 λ の2つのパラメータが必要である。(2) 応答速度はビオー数の小さいうちは内部構造の影響をほとんど受けないが、ビオー数が大きくなると充填物質の厚さ、熱的性質に敏感になってくる。(3) 充填物質が熱絶縁的である限り、保護管および抵抗線層の熱的性質は応答速度にほとんど影響を与えない。