A Note on Elliptical Modons

by

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Abstract

For any stationary solution of the potential vorticity equation, the streamfunction Ψ and the potential vorticity Q are functionally related, *i.e.*, $Q=F(\Psi)$. For analytical modons which are considered as one of the models of atmospheric blocking, the functional form of $Q=F(\Psi)$ is assumed to be linear and therefore the modon boundary is assumed to be circular. Modons with boundaries other than a circle have been numerically investigated by several authors. Boyd and Ma (1990) calculated numerically the functional form of $Q=F(\Psi)$ for elliptical modons elongated in the basic flow direction on an f plane. In this note, the functional form of $Q=F(\Psi)$ is calculated analytically for modons with an elliptical boundary slightly different from a circle. For modons elongated in the basic flow direction, the result agrees qualitatively with the above-mentioned numerical one. On the other hand, for modons elongated in the direction perpendicular to the basic flow, the functional form of $Q=F(\Psi)$ is shown to be qualitatively different from that for modons elongated in the basic flow.

1. Introduction

It is well known that modons are considered as one of the models of atmospheric blocking (e.g., McWilliams, 1980; Haines and Marshall, 1987; Butchart *et al.*, 1989). Modons are stationary (or propagating) solutions of the potential vorticity equation. The stationarity implies that the streamfunction Ψ and the potential vorticity Q are functionally related. Although atmospheric blockings in reality have several shapes, modons are assumed to have a circular boundary and therefore $Q=F(\Psi)$ is assumed to be a linear function (e.g., Larichev and Reznik, 1976; Tribbia, 1984; Verkley, 1987). The reason why circular modons are considered is that modons with boundaries other than a circle are not well understood

analytically. Taking account of the variability of the form of atmospheric blocking, it is desirable to consider several modon boundaries other than a circle in order to grasp atmospheric blocking in terms of modons. In fact, modons with boundaries other than a circle have been numerically investigated by several researchers (e.g., Tanveer, 1986; Eydeland and Turkington, 1988).

Recently Boyd and Ma (1990) considered elliptical modons on an f plane. In their paper, modons are assumed to have an elliptical boundary elongated in the basic flow direction and the functional form of $Q=F(\Psi)$ within the modon boundary is calculated numerically (the functional form of $Q=F(\Psi)$ outside the modon boundary is determined to be linear by the boundary condition at infinity. In particular, on an f plane, $Q=\text{constant}$ outside the modon boundary). In their calculation, $Q=$

Fig. 1 Ψ of an (almost) circular modon (a), of an elliptical modon of small eccentricity (b) and of an elliptical modon of large eccentricity (c). (after Boyd and Ma, 1990)

$F(\Psi)$ deviates from a linear function as the ellipticity increases. To be more precise, the functional form of $Q=F(\Psi)$ for an elliptical modon is nearly linear with a steeper slope than a circular modon for a sufficiently small absolute value of Ψ , but the absolute value of Q damps exponentially as the absolute value of Ψ increases. Although the distribution of Ψ is not qualitatively changed (see Fig.1), the distribution of Q inside the modon boundary is qualitatively changed as a result of this deviation from linearity of $Q=F(\Psi)$ (see Fig. 2).

In this note, the functional form of $Q=F(\Psi)$ is considered analytically for modons with an elliptical boundary slightly different from a circle. For modons elongated in the basic flow direction, a qualitative agreement with the above mentioned numerical calculation is obtained. Concerning modons elongated in the direction perpendicular to the basic flow, it is shown that $Q=F(\Psi)$ also deviates from a linear function but this deviation is qualitatively different from that of modons elongated in the basic flow direction. In the case of equivalent barotropic modons on a β plane, almost the same result as on an f plane is obtained.

The construction of this note is as follows. In section 2 the derivation of analytical modons with a circular boundary is briefly reviewed. In section 3 the functional form of $Q=F(\Psi)$ is analytically calculated for modons with an

elliptical boundary slightly different from a circle. Section 4 gives the conclusion.

2. Analytical modons on an f plane

In this section, the analytical expression of modons with a circular boundary on an f plane is presented. The fundamental equation is the two-dimensional conservation equation of potential vorticity on an f plane,

$$\partial Q / \partial t + J(\Phi, Q) = 0, \quad Q = \Delta \Phi + f, \quad (1)$$

where

$$J(A, B) = (\partial A / \partial x)(\partial B / \partial y) - (\partial B / \partial x)(\partial A / \partial y), \\ \Delta = (\partial / \partial x)^2 + (\partial / \partial y)^2,$$

x and y are the eastward and northward coordinate respectively, t is time, Φ is the stream function and Q is the potential vorticity. In Eq.(1) we can assume $f=0$ without loss of generality. Under the following coordinate transformation:

$$x \rightarrow x' = x - Ut, \quad y \rightarrow y' = y, \quad t \rightarrow t' = t, \quad (2)$$

the above potential vorticity equation is transformed into

$$\partial Q / \partial t' + J(\Phi + Uy', Q) = 0, \quad (3)$$

where x and y in $J(,)$ are replaced by x' and y' . An eastward propagating solitary wave solution with a constant velocity U in the old

Fig. 2 Q of an (almost) circular modon (a), of an elliptical modon of small eccentricity (b) and of an elliptical modon of large eccentricity (c). (after Boyd and Ma, 1990)

coordinate system becomes a stationary solitary wave solution in the new one. For such a solution, Eq. (3) turns out to be

$$J(\Psi, Q) = 0, \quad \Psi = \Phi + Uy'. \quad (4)$$

Eq. (4) implies that Q is a function of Ψ . Hereafter primes of the new coordinates x' , y' and t' are dropped. For analytical modons, the whole domain is partitioned into an inner region $\{(x, y) : r < a\}$ and an outer region $\{(x, y) : r > a\}$ (hereafter denoted by subscript _{in} and _{ex} respectively), and the following functional form is assumed,

$$\begin{aligned} Q_{\text{in}} &= -k^2 \Psi_{\text{in}} & \text{for } r < a, \\ Q_{\text{ex}} &= 0 & \text{for } r > a, \end{aligned} \quad (5)$$

where k^2 is a positive constant, $r = (x^2 + y^2)^{1/2}$ and a is a positive constant called modon radius. These are second order partial differential equations for Ψ with boundary conditions that $\Psi = 0$ on $r = a$, $\Delta \Psi$ is continuous at $r = a$ and $\Psi \rightarrow Uy$ as $r \rightarrow \infty$. The solution is given by the following equations:

$$\Psi(r, \theta) = R(r) \sin \theta,$$

$$\begin{aligned} \text{where } R_{\text{in}}(r) &= [2U / \{kJ'_1(ka)\}] J_1(kr) & r < a, \\ R_{\text{ex}}(r) &= U \{r - (a^2/r)\} & r > a, \\ \text{and } k &\text{ is the smallest root of } J_1(ka) = 0. \end{aligned} \quad (6)$$

Here $\theta = \tan^{-1}(y/x)$ and $J_1(z)$ is Bessel function of the first kind of order 1.

3. Elliptical modons

In this section we calculate analytically the functional form of $Q = F(\Psi)$ for modons with an elliptical boundary. The modon boundary is assumed to be a function of a parameter ϵ given by the following equation:

$$x = a(1 + \epsilon) \cos \theta, \quad y = a \sin \theta. \quad (7)$$

The modon boundary is circular for $\epsilon = 0$, and the boundary deviates from a circle as ϵ increases or decreases. Eq. (7) implies

$$\begin{aligned} r &= a + \epsilon a \cos^2 \theta + (\text{term of } \epsilon^2) \\ &\quad + (\text{term of } \epsilon^3) + \dots = r_B(\theta). \end{aligned} \quad (8)$$

First, the functional form of $Q_{\text{ex}} = F_{\text{ex}}(\Psi_{\text{ex}})$ in the outer region is considered. Since modons are assumed to be isolated,

$$\lim_{r \rightarrow \infty} \Psi = \lim_{r \rightarrow \infty} (\Phi + Uy) = Uy.$$

This equation implies

$$\lim_{r \rightarrow \infty} Q = \lim_{r \rightarrow \infty} (\Delta \Psi) = \Delta(Uy) = 0.$$

Therefore, in the outer region

$$Q_{\text{ex}} = F_{\text{ex}}(\Psi_{\text{ex}}) = 0, \quad r > a, \quad (9)$$

regardless of the shape of the modon boundary.

Next, the functional form of $Q_{\text{in}} = F_{\text{in}}(\Psi_{\text{in}})$ in the inner region is considered. Taking account of the dependence on ϵ , we write this function as

$$Q_{\text{in}}(\varepsilon) = F_{\text{in}}(\Psi_{\text{in}}(\varepsilon), \varepsilon). \quad (10)$$

Expanding with respect to ε , Eq. (10) can be rewritten as follows

$$\begin{aligned} & F_{\text{in}}(\Psi_{\text{in}}(0), 0) - \Delta \Psi_{\text{in}}(0) \\ & - \varepsilon \Delta \partial \Psi_{\text{in}}(0) / \partial \varepsilon + \varepsilon \partial F_{\text{in}}(\Psi_{\text{in}}(0), 0) / \partial \varepsilon \\ & + \varepsilon \{ \partial \Psi_{\text{in}}(0) / \partial \varepsilon \} \partial F_{\text{in}}(\Psi_{\text{in}}(0), 0) / \partial \Psi_{\text{in}} \\ & + (\text{term of } \varepsilon^2) + (\text{term of } \varepsilon^3) + \dots = 0. \end{aligned} \quad (11)$$

Since Eq. (11) holds for any value of ε , each term of ε^n ($n=0, 1, 2, \dots$) must vanish because of the independency of ε^n . Using the fact that

$$\Delta \Psi_{\text{in}}(0) = F_{\text{in}}(\Psi_{\text{in}}(0), 0) = -k^2 \Psi_{\text{in}}(0),$$

the linear terms of ε in Eq. (11) become

$$\varepsilon \Delta \Psi_{\text{in}}^{(1)} = \varepsilon \partial F_{\text{in}}(\Psi_{\text{in}}^{(0)}, 0) / \partial \varepsilon - \varepsilon k^2 \Psi_{\text{in}}^{(1)}, \quad (12)$$

where $\Psi_{\text{in}}^{(1)} = \partial \Psi_{\text{in}}(0) / \partial \varepsilon$, and $\Psi_{\text{in}}^{(0)} = \Psi_{\text{in}}(0)$ is the analytical modon solution obtained in section 2. Eq. (12) implies that $\Psi_{\text{in}}^{(1)}$ must satisfy

$$\Delta \Psi_{\text{in}}^{(1)} + k^2 \Psi_{\text{in}}^{(1)} = \partial F_{\text{in}}(\Psi_{\text{in}}^{(0)}, 0) / \partial \varepsilon. \quad (13)$$

Since $\partial F_{\text{in}}(\Psi_{\text{in}}^{(0)}, 0) / \partial \varepsilon$ is a function of $\Psi_{\text{in}}^{(0)}$, this can be expanded as

$$\partial F_{\text{in}}(\Psi_{\text{in}}^{(0)}, 0) / \partial \varepsilon = \sum_{n=0}^{\infty} \alpha_n \Psi_{\text{in}}^{(0)n}. \quad (14)$$

In Eq. (14), α_0 must vanish because of the conditions $Q=0$ and $\Psi=0$ on the modon boundary. Then Eq. (13) is rewritten as follows

$$\Delta \Psi_{\text{in}}^{(1)} + k^2 \Psi_{\text{in}}^{(1)} = \sum_{n=1}^N \alpha_n \Psi_{\text{in}}^{(0)n}, \quad (15)$$

where ∞ is replaced by an arbitrary large finite number N .

From Eq. (9) and Eq. (15), $\Psi^{(1)}$ must satisfy the following equations

$$\begin{aligned} \Delta \Psi_{\text{in}}^{(1)} + k^2 \Psi_{\text{in}}^{(1)} &= \sum_{n=1}^N \alpha_n \Psi_{\text{in}}^{(0)n}, \\ \Delta \Psi_{\text{ex}}^{(1)} &= 0. \end{aligned} \quad (16)$$

Noticing that $\Psi^{(0)} = R(r) \sin \theta$ and

$$\begin{aligned} \sin^{2n} \theta &= \sum_{m=0}^n C_{2m} \cos 2m\theta, \\ \sin^{2n+1} \theta &= \sum_{m=0}^n D_{2m+1} \sin(2m+1)\theta, \end{aligned}$$

where coefficients C_{2m} and D_{2m+1} are some constants, the partial differential equation for $\Psi_{\text{in}}^{(1)}$ is rewritten as (if N is even)

$$\begin{aligned} \Delta \Psi_{\text{in}}^{(1)} + k^2 \Psi_{\text{in}}^{(1)} &= \sum_{m=0}^{N/2} \gamma_{2m} R_{2m}(r) \cos 2m\theta \\ &+ \sum_{m=0}^{N/2-1} \sigma_{2m+1} R_{2m+1}(r) \sin(2m+1)\theta, \end{aligned} \quad (17)$$

where the coefficients γ_{2m} and σ_{2m+1} are linear combinations of α_n , and $R_{2m}(r)$ and $R_{2m+1}(r)$ are linear combinations of $R(r)^n$. Hereafter only the case of even N is considered. For the case of odd N , the following argument is essentially the same. With the boundary condition $\Psi^{(1)} < \infty$ as $r \rightarrow 0$, the solution of Eq. (17) is given by the following equation:

$$\begin{aligned} \Psi_{\text{in}}^{(1)} &= \sum_{m=0}^{N/2} \gamma_{2m} f_{2m}(r) \cos 2m\theta \\ &+ \sum_{m=0}^{N/2-1} \sigma_{2m+1} f_{2m+1}(r) \sin(2m+1)\theta \\ &+ \sum_{n=0}^{\infty} (A_{\text{in}}^n \cos n\theta + B_{\text{in}}^n \sin n\theta) J_n(kr), \end{aligned} \quad (18)$$

where $f_{2m}(r)$ and $f_{2m+1}(r)$ are solutions of

$$\begin{aligned} (\Delta + k^2) \{ \gamma_{2m} f_{2m}(r) \cos 2m\theta \} \\ = \gamma_{2m} R_{2m}(r) \cos 2m\theta \end{aligned}$$

and

$$\begin{aligned} (\Delta + k^2) \{ \sigma_{2m+1} f_{2m+1}(r) \sin(2m+1)\theta \} \\ = \sigma_{2m+1} R_{2m+1}(r) \sin(2m+1)\theta \end{aligned} \quad (19)$$

respectively, and $J_n(z)$ are Bessel functions of the first kind of order n . As for the outer region, with the boundary condition $\Psi^{(1)} \rightarrow 0$ as $r \rightarrow \infty$, the solution of Eq. (16) is given by

$$\Psi_{\text{ex}}^{(1)} = \sum_{n=0}^{\infty} (A_{\text{ex}}^n \cos n\theta + B_{\text{ex}}^n \sin n\theta) r^{-n}. \quad (20)$$

Therefore, with the boundary conditions at $r \rightarrow 0$ and $r \rightarrow \infty$, $\Psi^{(1)}$ is given by the following equations:

$$\begin{aligned} \Psi_{\text{in}}^{(1)} &= \sum_{m=0}^{N/2} \gamma_{2m} f_{2m}(r) \cos 2m\theta \\ &+ \sum_{m=0}^{N/2-1} \sigma_{2m+1} f_{2m+1}(r) \sin(2m+1)\theta \\ &+ \sum_{n=0}^{\infty} (A_{\text{in}}^n \cos n\theta + B_{\text{in}}^n \sin n\theta) J_n(kr), \end{aligned}$$

$$\Psi_{\text{ex}}^{(1)} = \sum_{n=0}^{\infty} (A_{\text{ex}}^n \cos n\theta + B_{\text{ex}}^n \sin n\theta) r^{-n}. \quad (21)$$

The boundary conditions at the modon boundary for Ψ are that

$$\Psi_{\text{in}} = \Psi_{\text{ex}} = 0, \quad \nabla \Psi_{\text{in}} = \nabla \Psi_{\text{ex}}$$

at $r = r_B(\theta) = a + \varepsilon a \cos^2 \theta + (\text{term of } \varepsilon^2) + \dots$. (22)

Expanding Eq. (22) with respect to ε , it can be rewritten as

$$\begin{aligned} \Psi_{\text{in}}^{(0)}(r=a) + \varepsilon a \cos^2 \theta \{ \partial \Psi_{\text{in}}^{(0)}(r=a) / \partial r \} \\ + \varepsilon \Psi_{\text{in}}^{(1)}(r=a) + (\text{term of } \varepsilon^2) + \dots = 0, \\ \Psi_{\text{ex}}^{(0)}(r=a) + \varepsilon a \cos^2 \theta \{ \partial \Psi_{\text{ex}}^{(0)}(r=a) / \partial r \} \\ + \varepsilon \Psi_{\text{ex}}^{(1)}(r=a) + (\text{term of } \varepsilon^2) + \dots = 0, \\ \nabla \Psi_{\text{in}}^{(0)}(r=a) + \varepsilon a \cos^2 \theta \{ \partial \nabla \Psi_{\text{in}}^{(0)}(r=a) / \partial r \} \\ + \varepsilon \nabla \Psi_{\text{in}}^{(1)}(r=a) + (\text{term of } \varepsilon^2) + \dots \\ = \nabla \Psi_{\text{ex}}^{(0)}(r=a) + \varepsilon a \cos^2 \theta \{ \partial \nabla \Psi_{\text{ex}}^{(0)}(r=a) / \partial r \} \\ + \varepsilon \nabla \Psi_{\text{ex}}^{(1)}(r=a) + (\text{term of } \varepsilon^2) + \dots. \end{aligned} \quad (23)$$

Taking the linear terms of ε in Eq. (23), and noticing that $\Psi^{(0)}$ is continuous up to the second derivative at $r=a$, the boundary conditions for $\Psi^{(1)}$ are given as

$$\begin{aligned} a \cos^2 \theta \{ \partial \Psi^{(0)} / \partial r \} + \Psi_{\text{in}}^{(1)} \\ = a \cos^2 \theta \{ \partial \Psi^{(0)} / \partial r \} + \Psi_{\text{ex}}^{(1)} = 0, \\ \nabla \Psi_{\text{in}}^{(1)} = \nabla \Psi_{\text{ex}}^{(1)} \quad \text{at } r=a. \end{aligned} \quad (24)$$

Noticing that $\Psi^{(0)} = R(r) \sin \theta$ and $\cos^2 \theta \sin \theta = (1/4)(\sin \theta + \sin 3\theta)$, Eq. (24) are rewritten as follows:

$$\begin{aligned} (a/4) R'(a) \sin \theta + (a/4) R'(a) \sin 3\theta \\ + \Psi_{\text{in}}^{(1)}(r=a) = 0, \\ (a/4) R'(a) \sin \theta + (a/4) R'(a) \sin 3\theta \\ + \Psi_{\text{ex}}^{(1)}(r=a) = 0, \\ \nabla \Psi_{\text{in}}^{(1)}(r=a) = \nabla \Psi_{\text{ex}}^{(1)}(r=a). \end{aligned} \quad (25)$$

$\Psi^{(1)}$ in Eq. (21) must satisfy these boundary conditions (25). First, by the first and second equation of Eq. (25),

$$\begin{aligned} A_n^{\text{in}} &= 0 \quad \text{for } n > N, \\ B_n^{\text{in}} &= 0 \quad \text{for } n \geq N, \\ B_{\text{ex}}^n &= 0 \quad \text{except for } n=1, 3 \end{aligned}$$

$$\text{and } A_{\text{ex}}^n = 0 \quad \text{for any } n. \quad (26)$$

Next, from the above boundary conditions (25) for terms proportional to $\cos N\theta$ in Eq.

(21),

$$\begin{aligned} \gamma_N f_N(a) + A_{\text{in}}^N J_N(ka) &= 0, \\ \gamma_N f_N'(a) + A_{\text{in}}^N k J_N'(ka) &= 0. \end{aligned} \quad (27)$$

Eqs. (27) imply that $\gamma_N=0$ and $A_{\text{in}}^N=0$. On the other hand, from Eq. (16) and Eq. (17) we can see that $\alpha_N R(r)^N = \gamma_N R_N(r)$. Therefore α_N must vanish. By the same reasoning for terms proportional to $\sin(N-1)\theta$, $\cos(N-2)\theta$, $\sin(N-3)\theta$, ... in Eq. (21), we obtain

$$\alpha_N=0, \alpha_{N-1}=0, \alpha_{N-2}=0, \dots, \alpha_4=0,$$

$$\begin{aligned} A_{\text{in}}^n &= 0 \quad \text{except for } n=0, 2 \\ \text{and } B_{\text{in}}^n &= 0 \quad \text{except for } n=1, 3. \end{aligned} \quad (28)$$

Finally, for terms proportional to $\cos 2\theta$ and independent of θ in Eq. (21),

$$\begin{aligned} \gamma_2 f_2(a) + A_{\text{in}}^2 J_2(ka) &= 0, \\ \gamma_2 f_2'(a) + A_{\text{in}}^2 k J_2'(ka) &= 0, \\ \gamma_0 f_0(a) + A_{\text{in}}^0 J_0(ka) &= 0, \\ \gamma_0 f_0'(a) + A_{\text{in}}^0 k J_0'(ka) &= 0. \end{aligned} \quad (29)$$

Eqs. (29) imply that $\gamma_0=\gamma_2=0$ and $A_{\text{in}}^0=A_{\text{in}}^2=0$. On the other hand, from Eq. (16) and Eq. (17) we can see that γ_0 and γ_2 are proportional to α_2 . So, α_2 must vanish. Then, from Eq. (10) and Eq. (11) the functional form of Q_{in} are calculated as follows:

$$\begin{aligned} Q_{\text{in}} &= F_{\text{in}}(\Psi_{\text{in}}(0), 0) \\ &+ \varepsilon \partial F_{\text{in}}(\Psi_{\text{in}}(0), 0) / \partial \varepsilon \\ &+ \varepsilon \{ \partial \Psi_{\text{in}}(0) / \partial \varepsilon \} \partial F_{\text{in}}(\Psi_{\text{in}}(0), 0) / \partial \Psi_{\text{in}} \\ &+ (\text{term of } \varepsilon^2) + \dots \\ &= -k^2 \Psi_{\text{in}}^{(0)} + \varepsilon \sum_{n=1,3} \alpha_n \Psi_{\text{in}}^{(0)n} \\ &\quad - \varepsilon k^2 \Psi_{\text{in}}^{(1)} + (\text{term of } \varepsilon^2) + \dots \\ &= - (k^2 - \varepsilon \alpha_1) \{ \Psi_{\text{in}}^{(0)} + \varepsilon \Psi_{\text{in}}^{(1)} \} \\ &\quad + \varepsilon \alpha_3 \{ \Psi_{\text{in}}^{(0)} + \varepsilon \Psi_{\text{in}}^{(1)} \}^3 + (\text{term of } \varepsilon^2) + \dots \\ &= - (k^2 - \varepsilon \alpha_1) \Psi_{\text{in}} + \varepsilon \alpha_3 \Psi_{\text{in}}^3 \\ &\quad + (\text{term of } \varepsilon^2) + \dots \end{aligned}$$

This equation holds for any value of ε . In the remainder of this section, we assume that $|\varepsilon|$

$\ll 1$. Then, the above calculations imply that to the $O(\varepsilon^1)$ approximation, the functional form of $Q_{\text{in}} = F_{\text{in}}(\Psi_{\text{in}})$ retains only cubic non-linear term, and $\Psi^{(1)}$ retains only terms proportional to $\sin\theta$ and $\sin 3\theta$,

$$\begin{aligned} Q_{\text{in}} &= F_{\text{in}}(\Psi_{\text{in}}) = -(k^2 - \varepsilon\alpha_1)\Psi_{\text{in}} + \varepsilon\alpha_3\Psi_{\text{in}}^3, \\ \Psi_{\text{in}}^{(1)} &= \{\sigma_1 f_1(r) + B_{\text{in}}^1 J_1(kr)\}\sin\theta \\ &\quad + \{\sigma_3 f_3(r) + B_{\text{in}}^3 J_3(kr)\}\sin 3\theta, \\ \Psi_{\text{ex}}^{(1)} &= B_{\text{ex}}^1 r^{-1} \sin\theta + B_{\text{ex}}^3 r^{-3} \sin 3\theta. \end{aligned} \quad (30)$$

By the boundary conditions (25) for the terms proportional to $\sin 3\theta$ in Eq. (30), σ_3 and B_{in}^3 and B_{ex}^3 must satisfy the following equations:

$$\begin{aligned} \sigma_3 f_3(a) + B_{\text{in}}^3 J_3(ka) + (a/4)R'(a) &= 0, \\ B_{\text{ex}}^3 a^{-3} + (a/4)R'(a) &= 0, \\ \sigma_3 f'_3(a) + B_{\text{in}}^3 kJ'_3(ka) &= -3B_{\text{ex}}^3 a^{-4}. \end{aligned} \quad (31)$$

From Eq. (31), we obtain

$$\begin{aligned} \sigma_3 \{f'_3(a)J_3(ka) - f_3(a)kJ'_3(ka)\} \\ = (1/4)R'(a)\{3J_3(ka) + kaJ'_3(ka)\} \\ = (1/2)UkaJ_2(ka). \end{aligned} \quad (32)$$

Here we use the fact $R'(a) = 2U$ and the recurrence relation of Bessel functions

$$J'_n(z) = J_{n-1}(z) - (n/z)J_n(z). \quad (33)$$

From Eq. (19) $f_3(r)$ satisfies

$$\begin{aligned} \{\Delta + k^2\}\{\sigma_3 f_3(r)\sin 3\theta\} &= \sigma_3 R_3(r)\sin 3\theta \\ &= -(1/4)\alpha_3 R(r)^3 \sin 3\theta. \end{aligned} \quad (34)$$

Multiplying Eq. (34) by $J_3(kr)\sin 3\theta$ and integrating in the inner region, we obtain

$$\begin{aligned} -(1/4)\alpha_3 \iint_{r < a} dx dy J_3(kr) R(r)^3 \sin^2 3\theta \\ = \sigma_3 \pi a \{f'_3(a)J_3(ka) - f_3(a)kJ'_3(ka)\}. \end{aligned} \quad (35)$$

From Eq. (32) and Eq. (35) α_3 is determined as

$$\alpha_3 = -2U\pi ka^2 J_2(ka)$$

$$\begin{aligned} & / \iint_{r < a} dx dy J_3(kr) R(r)^3 \sin^2 3\theta \\ & = (k^4 a^2 / 4U^2) J_2(ka)^4 / \int_{r < a} dr r J_3(kr) J_1(kr)^3. \end{aligned} \quad (36)$$

Here we use Eq. (6) and the recurrence relation of Bessel functions

$$J'_n(z) = (n/z)J_n(z) - J_{n+1}(z). \quad (37)$$

Since from Eq. (6) k is the smallest root of $J_1(ka) = 0$, $J_1(kr)$, $J_2(kr)$ and $J_3(kr)$ are positive definite for $r < a$, and therefore Eq. (36) implies that α_3 is positive. On the other hand by the boundary conditions (25) for the terms proportional to $\sin\theta$ in Eq. (30), σ_1 and B_{in}^1 and B_{ex}^1 must satisfy the following equations:

$$\begin{aligned} \sigma_1 f_1(a) + B_{\text{in}}^1 J_1(ka) + (a/4)R'(a) &= 0, \\ B_{\text{ex}}^1 a^{-1} + (a/4)R'(a) &= 0, \\ \sigma_1 f'_1(a) + B_{\text{in}}^1 kJ'_1(ka) &= -B_{\text{ex}}^1 a^{-2}. \end{aligned} \quad (38)$$

From Eq. (38) we obtain

$$\begin{aligned} \sigma_1 \{f'_1(a)J_1(ka) - f_1(a)kJ'_1(ka)\} \\ = (1/4)R'(a)\{J_1(ka) + kaJ'_1(ka)\} \\ = (1/2)UkaJ_0(ka). \end{aligned} \quad (39)$$

Here we use the fact $R'(a) = 2U$ and Eq. (33). From Eq. (19) $f_1(r)$ satisfies

$$\begin{aligned} \{\Delta + k^2\}\{\sigma_1 f_1(r)\sin\theta\} &= \sigma_1 R_1(r)\sin\theta \\ &= \{\sigma_1 R(r) + (3/4)\alpha_3 R(r)^3\}\sin\theta. \end{aligned} \quad (40)$$

Multiplying Eq. (40) by $J_1(kr)\sin\theta$ and integrating in the inner region, we obtain

$$\begin{aligned} \iint_{r < a} dx dy J_1(kr) \{\alpha_1 R(r) + (3/4)\alpha_3 R(r)^3\} \sin^2\theta \\ = \sigma_1 \pi a \{f'_1(a)J_1(ka) - f_1(a)kJ'_1(ka)\}. \end{aligned} \quad (41)$$

From Eq. (39) and Eq. (41) α_1 is determined as

$$\begin{aligned} \alpha_1 &= -\{(3/4)\alpha_3 \int_{r < a} dr r J_1(kr) R(r)^3 \\ &\quad - (U/2)ka^2 J_0(ka)\} \\ &\quad / \int_{r < a} dr r J_1(kr) R(r). \end{aligned} \quad (42)$$

Substituting Eq. (6) and Eq. (36) into Eq. (42), α_1 is finally determined as follows:

Fig. 3 $Q=F(\Psi)$ of an (almost) circular modon (a), of an elliptical modon of small eccentricity (b) and of an elliptical modon of large eccentricity (c). (after Boyd and Ma, 1990)

$$\alpha_1 = -k^2 a^2 J_2(ka)^2 \int_{r < a} dr r J_1(kr)^3 \{ 3 J_1(kr) - J_3(kr) \} / 4 \int_{r < a} dr r J_1(kr)^2 \int_{r < a} dr r J_3(kr) J_1(kr)^3. \quad (43)$$

Here we use Eq. (33) and Eq. (37). Eq. (43) implies that α_1 is negative.

Summarizing the above calculation, for small $|\varepsilon|$ the functional form of $Q=F(\Psi)$ is determined as follows:

$$\begin{aligned} Q_{\text{in}} &= F_{\text{in}}(\Psi_{\text{in}}) \\ &= -(k^2 - \varepsilon \alpha_1) \Psi_{\text{in}} + \varepsilon \alpha_3 \Psi_{\text{in}}^3 + O(\varepsilon^2) \\ &= -(k^2 - \varepsilon \alpha_1) \Psi_{\text{in}} [1 - \varepsilon \{ \alpha_3 / (k^2 - \varepsilon \alpha_1) \} \Psi_{\text{in}}^2] \\ &\quad + O(\varepsilon^2) \\ &= -(k^2 - \varepsilon \alpha_1) \Psi_{\text{in}} \text{Exp}[-\varepsilon \{ \alpha_3 / (k^2 - \varepsilon \alpha_1) \} \Psi_{\text{in}}^2] + O(\varepsilon^2) \end{aligned}$$

$$Q_{\text{ex}} = F_{\text{ex}}(\Psi_{\text{ex}}) = 0,$$

where

$$\begin{aligned} \alpha_3 &= (k^4 a^2 / 4 U^2) J_2(ka)^4 \\ &\quad / \int_{r < a} dr r J_3(kr) J_1(kr)^3 > 0, \\ \alpha_1 &= -k^2 a^2 J_2(ka)^2 \int_{r < a} dr r J_1(kr)^3 \{ 3 J_1(kr) - J_3(kr) \} \\ &\quad / 4 \int_{r < a} dr r J_1(kr)^2 \int_{r < a} dr r J_3(kr) J_1(kr)^3 \\ &< 0. \end{aligned} \quad (44)$$

For small positive ε , to the $O(\varepsilon^1)$ approximation the function $Q_{\text{in}} = F_{\text{in}}(\Psi_{\text{in}})$ behaves linearly with a steeper slope $k^2 + \varepsilon |\alpha_1|$ than k^2 for small absolute value of Ψ_{in} , and Q_{in} behaves exponentially with a damping sense as Ψ_{in} increases (or decreases). As a result the distribution of Q_{in} may be somewhat homogenized in the vicinity of the vortex centre compared with circular modons. This agrees qualitatively with the result of Boyd and Ma (1990) (see Fig. 3).

The calculations in this section can be carried out also on a β plane and almost the same results are obtained for β plane modons as for f plane ones.

4. Conclusion

For elliptical modons on an f plane with sufficiently small eccentricity, the functional form of $Q=F(\Psi)$ is calculated analytically. And a qualitative agreement with the numerical result of Boyd and Ma (1990) is obtained. Although Boyd and Ma (1990) considered only elliptical boundaries elongated in the x direction, the reasoning in section 3 is also valid for $\varepsilon < 0$, i.e., for an elliptical boundary elongated in the y direction. Therefore the functional form of $Q_{\text{in}} = F_{\text{in}}(\Psi_{\text{in}})$ for modons with an elliptical boundary elongated in the y direction is the same as for modons elongated in the x direction except for the sign of ε , i.e., given by Eq. (44) with $\varepsilon < 0$. That is, $Q_{\text{in}} = F_{\text{in}}(\Psi_{\text{in}})$

for y -elongated modons behaves linearly with a less steep slope $k^2 - |\varepsilon| |\alpha_1|$ than k^2 for a small absolute value of Ψ_{in} , and Q_{in} behaves exponentially with a growing sense as Ψ_{in} increases (or decreases). As a result the homogenization of Q_{in} in the vicinity of the vortex centre which may occur for x -elongated modons does not occur for y -elongated ones.

Boyd and Ma (1990) investigated also the case of large eccentricity and concluded that the vorticity concentrates in thin ridges around the edges of each vortex as the eccentricity increases. Although the equation $\partial F_{in}(\Psi_{in}(0), 0)/\partial \varepsilon = \sum_{n=1,3} \alpha_n \Psi_{in}^{(0)n}$ holds for any value of ε , the functional form of $Q_{in} = F_{in}(\Psi_{in})$ obtained in this note is valid only for $|\varepsilon| \ll 1$, and therefore we are not able to say anything about the interesting result of Boyd and Ma (1990) for large eccentricity.

References

- Boyd, J. P. and H. Ma, 1990: Numerical study of elliptical modons using a spectral method. *J. Fluid Mech.*, **221**, 597-611.
- Butchart, N., K. Haines and J. C. Marshall, 1989: A theoretical and diagnostic study of solitary waves and atmospheric blocking. *J. Atmos. Sci.*, **46**, 2063-2078.
- Eydeland, A. and B. Turkington, 1988: A computational method of solving free boundary problems in vortex dynamics. *J. Comp. Phys.*, **78**, 194-214.
- Haines, K. and J. C. Marshall, 1987: Eddy forced coherent structures as a prototype of atmospheric blocking. *Quart. J. Roy. Meteor. Soc.*, **113**, 681-704.
- Larichev, V. D. and G. M. Reznik, 1976: Two dimensional Rossby soliton: an exact solution. *Rep. U.S.S.R. Acad. Sci.*, **231**(5), 1077-1079.
- McWilliams, J. C., 1980: An application of equivalent modons to atmospheric blocking. *Dyn. Atmos. Oceans*, **5**, 43-66.
- Tanveer, S., 1986: A steadily translating pair of equal and opposite vortices with vortex sheets on their boundaries. *Stud. Appl. Maths.*, **74**, 139-154.
- Tribbia, J. J., 1984: Modons in spherical geometry. *Geophys. Astrophys. Fluid Dyn.*, **30**, 131-168.
- Verkley, W. T. M., 1987: Stationary barotropic modons in westerly background flows. *J. Atmos. Sci.*, **44**, 2383-2398.

楕円形モドンに就いての一考察

金久博忠

大気ブロッキング現象のモデルの一つと考えられているモドンは、渦位保存式の定常 (或は並進) 孤立波動解であり、渦位 Q は流線関数 Ψ の関数となるが、解析的には数学的取り扱い易さの為にモドン境界が円形であり Q が Ψ の線形関数である場合が考えられている。円形以外の境界を持つモドンに就いては、いくつかの数値的研究が為されているが、BoydとMa (1990) は、 f 平面上で基本流方向に長軸を持つ楕円境界のモドンに対して $Q = F(\Psi)$ の関数形を数値的に計算した。此处では、円形と僅かに異なる楕円境界を持つモドンに対して $Q = F(\Psi)$ の関数形を解析的に計算して、長軸が基本流方向の場合には上に述べた数値計算と定性的に一致する結果を得た。基本流に対して垂直方向に長軸を持つ楕円形モドンに対する $Q = F(\Psi)$ の関数形は、基本流方向に長軸を持つ楕円形モドンに対するものとは定性的に異なる事が分かった。