On the dependence of topographically forced flows on the form of dissipation

by

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Abstract

The dependence of topographically forced flows in a quasigeostrophic 2-layer model on the form of dissipation is investigated. In particular, the following 2 cases are examined. (1) The dissipation is proportional to the potential vorticity, and (2) the dissipation is proportional to the Laplacian of the streamfunction, *i.e.*, Ekman dampings on the upper and lower boundaries.

In the case (1), the bifurcation diagram is qualitatively the same as Pedlosky (1981)'s in which the dissipation is a sum of Ekman dampings on the upper and lower boundaries and an interfacial friction between the upper and lower layers. That is, there exist 2 steady solution bifurcation points, between which 2 stable solutions and 1 unstable one coexist for the same parameter values of the basic zonal flow.

On the other hand, in the case (2), a wave-wave interaction occurs. As a result, although the steady solution diagram is not altered qualitatively, Hopf bifurcation points may emerge on the stable steady solution curve, and the steady solutions between them may lose their stability.

So far as the considerations in this note are concerned, the stability of steady solutions is dependent on the form of dissipation although the steady solution diagram itself is not altered qualitatively.

1. Introduction

Since Phillips (1954), the evolution of baroclinic waves has been investigated by means of a quasigeostrophic 2-layer model. In the model, the parameterization of dissipation due to smaller scale waves has some ambiguities.

Pedlosky (1970, 1971, 1972) investigated the nonlinear evolution of linearly unstable baroclinic waves in the quasigeostrophic 2-layer model, and obtained the following results. If the magnitude of dissipation is sufficiently large, then the wave approaches a

steady state. On the contrary, if the magnitude is sufficiently small, then the wave approaches a cyclic state. Moreover, Pedlosky and Frenzen (1980) showed that the wave does not approach a steady or a cyclic state, but behaves chaotically for some intermediate value of dissipation. Thus, the evolution of baroclinic waves has been shown to depend crucially upon the magnitude of dissipation.

In the above-mentioned investigations, the dissipation was assumed to be proportional to the Laplacian of the streamfunction, *i.e.*, Ekman damping. Other forms of dissipation were not examined.

However, using a weakly nonlinear technique, Klein and Pedlosky (1992) revealed

that the wave approaches a steady state for any value of magnitude of dissipation if dissipation is taken to be proportional to the potential vorticity. That is, the evolution of baroclinic waves has been shown to depend crucially upon the form as well as the magnitude of dissipation.

On the other hand, using the weakly nonlinear technique, Pedlosky (1981) obtained a bifurcation diagram of topographically forced flows in a 2-layer model. The bifurcation diagram shows multiple equilibria and a catastrophe between them. One of the two stable equilibria corresponds to a blocked flow, and the other to a nearly zonal one.

In Pedlosky (1991), the dissipation was taken to be a sum of Ekman dampings on the upper and lower boundaries and an interfacial friction between the upper and lower layers. In this note, it is examined whether the bifurcation diagram of Pedlosky (1981) is dependent upon the form of dissipation. In particular, the following 2 cases are examined. (1) The dissipation is proportional to the potential vorticity, and (2) the dissipation is proportional to the Laplacian of the streamfunction, i.e., Ekman dampings on the upper and lower boundaries.

The organization of this note is as follows. In section 2, the governing equation is presented. In sections 3 and 4, the linear baroclinic and topographic problems are reviewed, respectively. In section 5, the weakly nonlinear evolution equations are derived. In section 6, steady solutions are calculated. In section 7, stability properties of the steady solutions are investigated. In section 8, concluding remarks are given.

2. Governing equation

The model is governed by the quasigeostrophic 2-layer potential vorticity equation in a β channel $\{(x,y): 0 \leq y \leq Y, 0 \leq x \leq X \text{ and cyclic in } x\}$,

$$\begin{aligned} &(\partial/\partial t) \sum_j Q_{ij}(\Delta) \psi_j + J(\psi_i, \sum_j Q_{ij}(\Delta) \psi_j + \beta y + h_i) \\ &+ r \sum_j D_{ij}(\Delta) \psi_j = f_i, \end{aligned} \quad (2-1)$$

with boundary conditions for the streamfunc-

tion ψ_i , $\partial \psi_i / \partial x = 0$ on $y=0, Y$, where $i=1$ and 2 denote the upper and lower layers, respectively. $Q_{ij}(\Delta)$ is a symmetric 2×2 matrix given by

$$Q_{ij}(\Delta) = \Delta \delta_{ij} + F_{ij}, \quad \delta_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_{ij} = F \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

where Δ is the horizontal Laplacian, and F is the Froude number. r is the coefficient of dissipation, and the 2×2 matrix $D_{ij}(\Delta)$ determines the form of dissipation.

In the absence of the topography, that is, $h_i(x,y) = 0$, the external force f_i induces a steady zonal flow $\Psi_i(y)$ which satisfies

$$r \sum_j D_{ij} (d^2/dy^2) \Psi_j(y) = f_i. \quad (2-2)$$

The force f_i is assumed to be such that the forced zonal flow $U_i = -d\Psi_i(y)/dy$ is independent of y . In the presence of the topography, we write the solution to Eq. (2-1) as

$$\psi_i = \Psi_i + \psi_i'. \quad (2-3)$$

Then the evolution equation of ψ_i' is derived from Eq. (2-1) as follows:

$$\begin{aligned} &(\partial/\partial t) \sum_j Q_{ij}(\Delta) \psi_j' + J(\Psi_i, \sum_j Q_{ij}(\Delta) \psi_j' + h_i) \\ &+ J(\psi_i', \sum_j Q_{ij}(\Delta) \Psi_j + \beta y) \\ &+ J(\psi_i', \sum_j Q_{ij}(\Delta) \psi_j' + h_i) + r \sum_j D_{ij}(\Delta) \psi_j' = 0. \end{aligned} \quad (2-4)$$

3. Linear baroclinic problem

On the assumption that the topography is small, the linearized equation of Eq. (2-4) is given by

$$\begin{aligned} &(\partial/\partial t) \sum_j Q_{ij}(\Delta) \psi_j' + r \sum_j D_{ij}(\Delta) \psi_j' + U_i (\partial h_i / \partial x) \\ &+ U_i (\partial / \partial x) \sum_j Q_{ij}(\Delta) \psi_j' + \beta (\partial \psi_i' / \partial x) \\ &- \sum_j F_{ij} U_j (\partial \psi_i' / \partial x) = 0. \end{aligned} \quad (3-1)$$

In the absence of the topography, Eq. (3-1) becomes a partial differential equation of ψ_i' with constant coefficients,

$$\begin{aligned} &(\partial/\partial t) \sum_j Q_{ij}(\Delta) \psi_j' + r \sum_j D_{ij}(\Delta) \psi_j' \\ &+ U_i (\partial / \partial x) \sum_j Q_{ij}(\Delta) \psi_j' + \beta (\partial \psi_i' / \partial x) \\ &- \sum_j F_{ij} U_j (\partial \psi_i' / \partial x) = 0. \end{aligned} \quad (3-2)$$

In this case, we can assume that ψ_i' has the following form

$$\psi_i' = A_i \exp(ikx) \sin y + \text{c.c.},$$

$$k = 2m\pi/X, \quad l = n\pi/Y, \quad m, n = 1, 2, 3, \dots \quad (3-3)$$

Substituting Eq. (3-3) into Eq. (3-2), we obtain the evolution equation of A_i as follows:

$$\Sigma_j Q_{ij}(-a^2) dA_j/dt + r \Sigma_j D_{ij}(-a^2) A_j + ikU \Sigma_j P_{ij}(-a^2) A_j = 0, \quad (3-4)$$

where we write $U_i = Uu_i$, $k^2 + l^2 = a^2$ and

$$Q_{ij}(-a^2) = \begin{bmatrix} -a^2 - F & F \\ F & -a^2 - F \end{bmatrix}, \quad (3-5)$$

$$P_{ij}(-a^2) = \begin{bmatrix} -a^2 u_1 - Fu_2 + \beta/U & Fu_1 \\ Fu_2 & -a^2 u_2 - Fu_1 + \beta/U \end{bmatrix}. \quad (3-6)$$

With $r \ll 1$, A_i can grow only when the matrix $ikU \Sigma_j Q_{ij}(-a^2)^{-1} P_{jk}(-a^2)$ has negative real eigenvalues. From Eq. (3-5) and Eq. (3-6), the eigenvalues σ of this matrix are given by the following equation:

$$\sigma = ikU(u_1 + u_2)/2$$

$$- ik\beta \{ (a^2 + F)/a^2(a^2 + 2F) \}$$

$$\pm ik[a^4(a^4 - 4F^2)U^2(u_1 - u_2)^2 + 4F^2\beta^2]^{1/2}$$

$$/ \{ 2a^2(a^2 + 2F) \}. \quad (3-7)$$

Eq. (3-7) implies that the waves grow only when

$$a^2 < 2F$$

$$\text{and } U(u_1 - u_2) > U_c = 2\beta F / \{ a^2(4F^2 - a^4)^{1/2} \}. \quad (3-8)$$

The possible minimal value of U_c is

$$U_{cmin} = \beta/F. \quad (3-9)$$

4. Linear topographic problem

In this note, we consider such a topography that is given by

$$h_i(x, y) = H_i \{ \exp(ikx) + \exp(-ikx) \} \sin y, \quad (4-1)$$

where $k = 2\pi m/X$, $l = \pi n/Y$, $m, n = 1, 2, 3, \dots$ and $H_i = \delta_{i2}H$. If the magnitude of the basic zonal flow is less than the minimal value, i.e., $U(u_1 - u_2) < U_{cmin}$, then there is no growing wave component except for the projection on the bases $\exp(\pm ikx) \sin y$. Hence we consider only such a disturbance that

$$\psi_i' = A_i \exp(ikx) \sin y + \text{c.c.} \quad (4-2)$$

By substituting Eq. (4-2) into Eq. (3-1), the evolution equation of A_i is obtained as

$$dA_i/dt + r \Sigma_j \Sigma_k Q_{ij}(-a^2)^{-1} D_{jk}(-a^2) A_k + ikU \Sigma_j \Sigma_k Q_{ij}(-a^2)^{-1} P_{jk}(-a^2) A_k = -ikUH \Sigma_j Q_{ij}(-a^2)^{-1} \delta_{j2} u_2, \quad (4-3)$$

where $a^2 = k^2 + l^2$. With $r > 0$, the disturbance will approach a steady state. If r is sufficiently small and the magnitude of the basic zonal flow U is such that the matrix $ikU \Sigma_j Q_{ij}(-a^2)^{-1} P_{jk}(-a^2)$ has a zero eigenvalue, then a steady wave which has the same vertical structure as the corresponding right zero-eigenvector will eventually grow up. From Eq. (3-7), the matrix $ikU \Sigma_j Q_{ij}(-a^2)^{-1} P_{jk}(-a^2)$ has a zero eigenvalue when

$$\beta/U = (1/2) (a^2 + F) (u_1 + u_2) + [F^2(u_1 + u_2)^2 + a^2(a^2 - 2F)(u_1 - u_2)^2]^{1/2}/2,$$

$$a^2 \geq 2F, \quad (4-4)$$

with the corresponding right eigenvector

$$r_i \propto \begin{bmatrix} Fu_1 \\ a^2 u_1 + Fu_2 - \beta/U \\ a^2 u_2 + Fu_1 - \beta/U \\ Fu_2 \end{bmatrix} \quad (4-5)$$

From Eq. (3-9), the baroclinic stability condition becomes $U(u_1 - u_2) < \beta/F$. It is easily seen that the linear resonant value U given by Eq. (4-4) satisfies this condition, $\beta/U > F(u_1 - u_2)$.

5. Weakly nonlinear problem

In the following, we consider a case in

which the magnitude of the basic zonal flow is nearly equal to the resonant value of Eq.(4-4), and denote it by

$$(1+\eta)U, \quad |\eta| \ll 1, \quad (5-1)$$

where U is given by Eq.(4-4). Further let it be supposed that

$$\psi_i' = O(\epsilon), \quad \eta = O(\epsilon^2), \quad |\epsilon| \ll 1. \quad (5-2)$$

If we wish to describe the system with one time scale, then from Eq.(4-3),

$$\partial/\partial t = \epsilon^2 \partial/\partial \tau, \quad r = \epsilon^2 r^{(2)}, \quad H = \epsilon^3 H^{(3)}, \quad (5-3)$$

where $\partial/\partial \tau$, $r^{(2)}$ and $H^{(3)}$ are of $O(1)$, and then Eq.(2-4) becomes

$$\begin{aligned} & \epsilon^2 (\partial/\partial \tau) \Sigma_j Q_{ij}(\Delta) \psi_j' + \epsilon^2 r^{(2)} \Sigma_j D_{ij}(\Delta) \psi_j' \\ & + \beta (\partial \psi_i' / \partial x) \\ & + (1 + \epsilon^2 \eta^{(2)}) U (\partial/\partial x) \Sigma_j u_i Q_{ij}(\Delta) \psi_j' \\ & - (1 + \epsilon^2 \eta^{(2)}) U \Sigma_j F_{ij} u_j (\partial \psi_i' / \partial x) \\ & + \epsilon^3 (1 + \epsilon^2 \eta^{(2)}) U u_i (\partial h_i^{(3)} / \partial x) \\ & + J(\psi_i', \Sigma_j Q_{ij}(\Delta) \psi_j' + \epsilon^3 h_i^{(3)}) = 0. \end{aligned} \quad (5-4)$$

We expand the disturbance ψ_i' as

$$\psi_i' = \epsilon \psi_i^{(1)} + \epsilon^2 \psi_i^{(2)} + \epsilon^3 \psi_i^{(3)} + \dots, \quad (5-5)$$

substitute this expansion into Eq.(5-4), and equate the terms of the like order of ϵ . First, the $O(\epsilon)$ equation becomes

$$\Sigma_j L_{ij} (\partial/\partial x, \partial/\partial y) \psi_j^{(1)} = 0, \quad (5-6)$$

where $L_{ij} (\partial/\partial x, \partial/\partial y)$

$$= U (\partial/\partial x) [u_i Q_{ij}(\Delta) + \delta_{ij} \{ (\beta/U) - \Sigma_k F_{ik} u_k \}]. \quad (5-7)$$

Since the magnitude of the basic zonal flow is nearly equal to the linear resonant value, we take as the solution of Eq.(5-6)

$$\psi_i^{(1)} = r_i A(\tau) \exp(ikx) \sin y + \text{c.c.}, \quad (5-8)$$

where the amplitude $r_i A(\tau)$ is proportional to the right eigenvector r_i in Eq.(4-5), and thereby the linear operator in Eq.(5-7) becomes

$$\begin{aligned} L_{ij}(ik, il) &= ikU [u_i Q_{ij}(-a^2) \\ &+ \delta_{ij} \{ (\beta/U) - \Sigma_k F_{ik} u_k \}] = ikU P_{ij}(-a^2). \end{aligned}$$

Next, the $O(\epsilon^2)$ equation becomes

$$\Sigma_j L_{ij} (\partial/\partial x, \partial/\partial y) \psi_j^{(2)} = 0. \quad (5-9)$$

Redefining $\psi_i^{(1)}$, we can take the following solution to Eq.(5-9)

$$\psi_i^{(2)} = \psi_i^{(2)}(y, \tau). \quad (5-10)$$

Next, the $O(\epsilon^3)$ equation becomes

$$\begin{aligned} & \Sigma_j L_{ij} (\partial/\partial x, \partial/\partial y) \psi_j^{(3)} + U u_i (\partial h_i^{(3)} / \partial x) \\ & + (\partial/\partial \tau) \Sigma_j Q_{ij}(\Delta) \psi_j^{(1)} + r^{(2)} \Sigma_j D_{ij}(\Delta) \psi_j^{(1)} \\ & + \Sigma_j \eta^{(2)} U \{ \partial L_{ij} (\partial/\partial x, \partial/\partial y) / \partial U \} \psi_j^{(1)} \\ & + J(\psi_i^{(1)}, \Sigma_j Q_{ij}(\Delta) \psi_j^{(2)}) + J(\psi_i^{(2)}, \Sigma_j Q_{ij}(\Delta) \psi_j^{(1)}) \\ & = 0. \end{aligned} \quad (5-11)$$

Here we introduce the left zero-eigenvector of $P_{ij}(-a^2)$ which can be written as $l_i = r_i / u_i$ and normalized as $\Sigma_i l_i r_i = 1$. According to the procedure to remove the secular terms, multiplying Eq.(5-11) by $l_i \exp(-ikx)$ sinly, and taking the summation with respect to i and integration with respect to (x, y) in $\{0 \leq x \leq X, 0 \leq y \leq Y\}$, we obtain the following evolution equation for A :

$$\begin{aligned} & Q dA/d\tau + r^{(2)} D A - ik\beta \eta^{(2)} A + ikU H^{(3)} \\ & + ik l A \Sigma_i E_i \Phi_i = 0. \end{aligned} \quad (5-12)$$

where $Q = \Sigma_i \Sigma_j l_i Q_{ij}(-a^2) r_j$,

$$D = \Sigma_i \Sigma_j l_i D_{ij}(-a^2) r_j \quad (5-13)$$

and $\Phi_i = (2/Y) \int_0^Y \text{dysin} 2ly \psi_i^{(2)}$,

$$E_i = \Sigma_j \{ r_j Q_{ji}(-a^2) l_i - l_j r_j Q_{ji}(-4l^2) \}. \quad (5-14)$$

Next, the $O(\epsilon^4)$ equation becomes

$$\begin{aligned} & \Sigma_j L_{ij} (\partial/\partial x, \partial/\partial y) \psi_j^{(4)} \\ & + \Sigma_j \eta^{(2)} U \{ \partial L_{ij} (\partial/\partial x, \partial/\partial y) / \partial U \} \psi_j^{(2)} \\ & + (\partial/\partial \tau) \Sigma_j Q_{ij}(\Delta) \psi_j^{(2)} + r^{(2)} \Sigma_j D_{ij}(\Delta) \psi_j^{(2)} \\ & + J(\psi_i^{(1)}, h_i^{(3)}) + J(\psi_i^{(1)}, \Sigma_j Q_{ij}(\Delta) \psi_j^{(3)}) \\ & + J(\psi_i^{(3)}, \Sigma_j Q_{ij}(\Delta) \psi_j^{(1)}) = 0. \end{aligned} \quad (5-15)$$

Integrating Eq.(5-15) with respect to x yields

$$\begin{aligned} & (\partial/\partial \tau) \Sigma_j Q_{ij}(\Delta) \psi_j^{(2)} + r^{(2)} \Sigma_j D_{ij}(\Delta) \psi_j^{(2)} \\ & + ik l r_i H_i^{(3)} (A - \bar{A}) \sin 2ly \\ & - (1/X) \int_0^X dx (\partial/\partial y) [(\psi_i^{(1)} / U u_i) \Sigma_j L_{ij} (\partial/\partial x, \partial/\partial y) \psi_j^{(3)}] \end{aligned}$$

$$=0. \quad (5-16)$$

On the other hand, multiplying Eq.(5-11) by $\psi_i^{(1)}/Uu_i$, integrating with respect to x and differentiating with respect to y yields

$$\begin{aligned} & (1/X) \int_0^X dx (\partial/\partial y) [(\psi_i^{(1)}/Uu_i) \Sigma_j L_{ij} (\partial/\partial x, \partial/\partial y) \psi_j^{(3)}] \\ & + ikl(\bar{A} - A) r_i H_i^{(3)} \sin 2ly \\ & + (1/U) (r_i/u_i) \Sigma_j Q_{ij} (-a^2) r_j \{A (d\bar{A}/d\tau) \\ & + (dA/d\tau) \bar{A}\} \sin 2ly \\ & + (2lr^{(2)}/U) (r_i/u_i) \Sigma_j D_{ij} (-a^2) r_j A \bar{A} \sin 2ly = 0. \end{aligned} \quad (5-17)$$

Adding Eq.(5-16) to Eq.(5-17), multiplying by $\sin 2ly$, integrating with respect to y , and further eliminating $A (d\bar{A}/d\tau) + (dA/d\tau) \bar{A}$ by using Eq.(5-12), we obtain the following evolution equation for Φ

$$\begin{aligned} & \Sigma_j Q_{ij} (-4l^2) (d\Phi/d\tau) + r^{(2)} \Sigma_j D_{ij} (-4l^2) \Phi \\ & + i(klH^{(3)}/Q) (r_i/u_i) \Sigma_j Q_{ij} (-a^2) r_j (A - \bar{A}) \\ & + (2lr^{(2)}/U) (r_i/u_i) \Sigma_j |D_{ij} (-a^2) - (D/Q) Q_{ij} (-a^2)| r_j A \bar{A} \\ & = 0. \end{aligned} \quad (5-18)$$

If we write the real and imaginary parts of A explicitly, then from Eq.(5-12) and Eq.(5-18) the evolution equations become

$$\begin{aligned} & d\text{Re}A/d\tau + r\text{Re}A - \eta\text{Im}A + \text{Im}A \Sigma_i E_i B_i = 0, \\ & d\text{Im}A/d\tau + r\text{Im}A + \eta\text{Re}A - \text{Re}A \Sigma_i E_i B_i = h, \\ & dB_i/d\tau + r \Sigma_j M_{ij} B_j + hN_i \text{Im}A + r\omega_i \{\text{Re}A^2 + \text{Im}A^2\} = 0 \end{aligned} \quad (5-19)$$

$$\text{where } r = r^{(2)}(D/Q), \quad \eta = -\eta^{(2)}(k\beta/Q),$$

$$h = -H^{(3)}kU/Q, \quad B_i = -\Phi_i(kl/Q) \quad (5-20)$$

$$\begin{aligned} & \text{and } M_{ij} = (Q/D) \Sigma_k Q_{ik} (-4l^2)^{-1} D_{kj} (-4l^2), \\ & N_i = -2(kl^2/UQ) \Sigma_j \Sigma_k Q_{ij} (-4l^2)^{-1} l_j Q_{jk} (-a^2) r_k, \\ & \omega_i = -(2kl^2/DU) \Sigma_j \Sigma_k Q_{ij} (-4l^2)^{-1} l_j |D_{jk} (-a^2) \\ & - (D/Q) Q_{jk} (-a^2)| r_k. \end{aligned} \quad (5-21)$$

A and B_i represent the amplitude of the disturbance and the correction of the basic zonal flow, respectively.

6. Steady solution

From Eq.(5-19), any steady solution must satisfy the following equations

$$\begin{aligned} & r\text{Re}\bar{A} - \eta\text{Im}\bar{A} + \text{Im}\bar{A} \Sigma_i E_i B_i = 0, \\ & r\text{Im}\bar{A} + \eta\text{Re}\bar{A} - \text{Re}\bar{A} \Sigma_i E_i B_i = h, \\ & r \Sigma_j M_{ij} B_j + hN_i \text{Im}\bar{A} + r\omega_i \{\text{Re}\bar{A}^2 + \text{Im}\bar{A}^2\} = 0. \end{aligned} \quad (6-1)$$

Writing $\Sigma_i E_i B_i = B$, from Eq.(6-1) the steady solutions are given by the following equation:

$$G(\eta, B) = B(\eta - B)^2 + r^2 B + bh^2 = 0, \quad (6-2)$$

$$\begin{aligned} & \text{Re}\bar{A} = -(1/bh)B(\eta - B), \quad \text{Im}\bar{A} = -(r/bh)B, \\ & \bar{B}_i = -2(kl^2/bQU)BS_i \end{aligned} \quad (6-3)$$

$$\text{where } b = -(2kl^2/QU) \Sigma_i E_i S_i,$$

$$S_i = \Sigma_j \Sigma_k D_{ij} (-4l^2)^{-1} l_j D_{jk} (-a^2) r_k. \quad (6-4)$$

From Eq.(6-2), we can see that any bifurcation point lies on the following hyperbola in (η, B) -plane,

$$\partial G(\eta, B)/\partial B = (\eta - B)(\eta - 3B) + r^2 = 0. \quad (6-5)$$

From Eq.(6-2) and Eq.(6-5), the steady solution bifurcation diagram is shown to be the same as Pedlosky (1981)'s except for the stability irrespective of the dissipation form $D_{ij}(\Delta)$ (see Fig.1).

Fig. 1 Steady solution diagram (a case of $b < 0$). (1) $\eta = B$ and (2) $\eta = 3B$. S_1 and S_2 indicate steady solution bifurcation points.

7. Stability properties of the steady solutions

(7-4)

7-1. Eigenvalues determining the stability properties

As shown in section 6, any steady solution $\{\text{Re}A, \text{Im}A, B_i\}$ belongs to a 3-dimensional sub-space spanned by $\text{Re}A, \text{Im}A$ and S_i . The vector S_i is independent of the control parameter η . Hereafter we denote this sub-space as $[A, S_i]$. We consider a projection of Eq.(5-19) on $[A, S_i]$. If a steady solution is unstable in $[A, S_i]$, then this steady solution is also unstable in the full space. Replacing B_i by its projection B_i' on $[A, S_i]$, and writing $B_i' = (B/\sum_j E_j S_j) S_i$, the projection of Eq.(5-19) becomes

$$\begin{aligned} d\text{Re}A/d\tau + r\text{Re}A - \eta\text{Im}A + B\text{Im}A &= 0, \\ d\text{Im}A/d\tau + r\text{Im}A + \eta\text{Re}A - B\text{Re}A &= h, \\ dB/d\tau + rMB + hN\text{Im}A + r\omega\{\text{Re}A^2 + \text{Im}A^2\} &= 0, \end{aligned} \quad (7-1)$$

$$\text{where } M = \sum_i \sum_j S_i M_{ij} S_j / \sum_k S_k S_k,$$

$$N = \sum_i S_i N_i \sum_j E_j S_j / \sum_k S_k S_k$$

$$\text{and } \omega = \sum_i S_i \omega_i \sum_j E_j S_j / \sum_k S_k S_k \quad (7-2)$$

From Eq.(5-21), Eq.(6-4) and Eq.(7-2), we see that

$$\begin{aligned} (N + \omega)/b &= M \\ &= (Q/D) \sum_i \sum_j \sum_k S_i Q_{ij} (-4I^2)^{-1} I_j D_{jk} (-a^2) r_k \\ &\quad / \sum_i S_i S_i, \\ N/b &= \sum_i \sum_j \sum_k S_i Q_{ij} (-4I^2)^{-1} I_j Q_{jk} (-a^2) r_k \\ &\quad / \sum_i S_i S_i. \end{aligned} \quad (7-3)$$

Using Eq.(7-3), steady solutions of Eq.(7-1) are given by Eq.(6-2) and Eq.(6-3) as expected.

stability properties of steady solutions is determined by eigenvalues of the following matrix:

$$\begin{bmatrix} r + \sigma & -\eta + B & \text{Im}A \\ \eta - B & r + \sigma & -\text{Re}A \\ 2r\omega\text{Re}A & hN + 2r\omega\text{Im}A & rM + \sigma \end{bmatrix}$$

The eigenvalues σ are given by

$$\begin{aligned} E(\sigma) &= \sigma^3 + r(2+M)\sigma^2 + [(2M+1)r^2 \\ &\quad + (\eta-B)^2 - (N/b)B(\eta-B)]\sigma \\ &\quad + rM\partial G(\eta, B)/\partial B = 0. \end{aligned} \quad (7-5)$$

As expected, from Eq.(7-5), at least one eigenvalue denoted by σ_1 vanishes if and only if on the bifurcation hyperbola $\partial G(\eta, B)/\partial B = 0$,

$$\sigma_1 = 0 \longleftrightarrow \partial G(\eta, B)/\partial B = 0. \quad (7-6)$$

7-2. Region for σ to be complex in (η, B) -plane

In order to find the region for σ to be complex, we differentiate Eq.(7-5) with respect to σ ,

$$\begin{aligned} \partial E(\sigma)/\partial \sigma &= 3\sigma^2 + 2r(2+M)\sigma + (2M+1)r^2 \\ &\quad + (\eta-B)^2 - (N/b)B(\eta-B). \end{aligned} \quad (7-7)$$

From Eq.(7-7), $\sigma = \sigma_{\pm}$ at which $E(\sigma)$ has extrema are given by

$$\begin{aligned} \sigma_{\pm} &= (1/3)[-r(2+M) \pm \{r^2(M-1)^2 \\ &\quad - 3(\eta-B)^2 + 3(N/b)B(\eta-B)\}^{1/2}]. \end{aligned} \quad (7-8)$$

Substituting Eq.(7-8) into Eq.(7-5), the extrema $E(\sigma_{\pm})$ are calculated as

$$\begin{aligned} E(\sigma_{\pm}) &= \pm (2/9) \{- (r^2/3)(M-1)^2 + (\eta-B)^2 \\ &\quad - (N/b)B(\eta-B)\} \cdot \{r^2(M-1)^2 - 3(\eta-B)^2 \\ &\quad + 3(N/b)B(\eta-B)\}^{1/2} \\ &\quad + (2r^3/27)(M-1)^3 + (2r/3)(M-1)(\eta-B)^2 \\ &\quad + r\{(N/3b)(2+M) - 2M\}B(\eta-B). \end{aligned} \quad (7-9)$$

The function $D(\eta, B)$ defined as

$$D(\eta, B) = E(\sigma_+)E(\sigma_-) \quad (7-10)$$

determines the region for σ to be complex. From Eq.(7-9), the curve $D(\eta, B) = 0$ is a union of a line $\eta = B$ and a curve with asymptotes $\eta = B$ and $\eta = \{1 + (N/b)\}B$. In the region $D(\eta, B) > 0$, there exist two complex eigenvalues denoted by $\sigma_{2,3}$,

$$\sigma_{2,3} \text{ are complex for } D(\eta, B) > 0. \quad (7-11)$$

For sufficiently small r ,

the region $D(\eta, B) > 0$

$\simeq$ the region $(\eta - B)[\eta - \{1 + (N/b)\}B] > 0$.

7-3. Curve for $\text{Re}\{\partial_{2,3}\}$ to vanish in (η, B) -plane

If we explicitly write $\sigma = \sigma_r + i\sigma_i$ for complex eigenvalues, then Eq. (7-5) becomes

$$\begin{aligned} & \sigma_r[\sigma_r^2 + r(2+M)\sigma_r + (2M+1)r^2 + (\eta-B)^2 \\ & - (N/b)B(\eta-B)] + rM\partial G(\eta, B)/\partial B \\ & = \{3\sigma_r + r(2+M)\}\sigma_i^2, \end{aligned} \quad (7-12)$$

$$\begin{aligned} & 3\sigma_r^2 + 2r(2+M)\sigma_r + (2M+1)r^2 + (\eta-B)^2 \\ & - (N/b)B(\eta-B) = \sigma_i^2. \end{aligned} \quad (7-13)$$

Substituting Eq. (7-13) into Eq. (7-12), we obtain the equation for σ_r ,

$$\begin{aligned} & \sigma_r[4\sigma_r^2 + 4r(2+M)\sigma_r + (2M+1)r^2 \\ & + r^2(2+M)^2 + (\eta-B)^2 - (N/b)B(\eta-B)] \\ & + rP(\eta, B) = 0, \end{aligned} \quad (7-14)$$

$$\begin{aligned} & \text{where } P(\eta, B) = (M+1)^2 r^2 \\ & + (\eta-B)[\eta - \{1-M+(2+M)(N/2b)\}B]. \end{aligned} \quad (7-15)$$

Let it be supposed that σ_r is negative (positive) on and inside (outside) of the hyperbola $P(\eta, B) = 0$,

$$\begin{aligned} & \sigma_r < 0 \text{ for } P(\eta, B) \leq 0 \\ & \text{or } \sigma_r > 0 \text{ for } P(\eta, B) \geq 0. \end{aligned} \quad (7-16)$$

Then Eq. (7-16) implies that

$$\begin{aligned} & 4\sigma_r^2 + 4r(2+M)\sigma_r + (2M+1)r^2 + r^2(2+M)^2 \\ & + (\eta-B)^2 - (N/b)B(\eta-B) \leq 0. \end{aligned} \quad (7-17)$$

Substituting Eq. (7-17) into Eq. (7-13), we obtain

$$\{\sigma_r + r(2+M)\}^2 + \sigma_i^2 \leq 0. \quad (7-18)$$

Eq. (7-18) contradicts the fact that σ is complex. Therefore we can conclude that in the complex region

$$\begin{aligned} & \sigma_r > 0 \text{ for } P(\eta, B) < 0, \\ & \sigma_r < 0 \text{ for } P(\eta, B) > 0. \end{aligned} \quad (7-19)$$

8. Dependence on the dissipation form $D_{ij}(\Delta)$

8-1. Case of dissipation proportional to potential vorticity

If the dissipation is proportional to the potential vorticity, i.e., $D_{ij}(\Delta) = Q_{ij}(\Delta)$, then the wave-wave interaction term vanishes, that is, $\omega_i = 0$ and Eq. (5-19) can be reduced to a 3-dimensional system. In this case, the 3-dimensional projected equations obtained above are identical to those of the full 4-dimensional system with $\omega = 0$ and $M = N/b = 1$. Then for sufficiently small r , Eqs. (7-11) and (7-19) become

$$\sigma_{2,3} \text{ are complex for } (\eta - B)(\eta - 2B) > 0,$$

$$\text{and } \sigma_r > 0 \text{ for } P(\eta, B) < 0,$$

$$\sigma_r < 0 \text{ for } P(\eta, B) > 0,$$

where $P(\eta, B) = 4r^2 + (\eta - B)(\eta - 3B/2)$. Because the hyperbola $P(\eta, B) = 0$ lies in the real region $(\eta - B)(\eta - 2B) < 0$, there can not exist any Hopf bifurcation, and the bifurcation diagram is qualitatively the same as Pedlosky (1981)'s including the stability (see Fig. 2).

Fig. 2 Bifurcation diagram in the case of $D_{ij}(\Delta) = Q_{ij}(\Delta)$ (a case of $b < 0$). (1) $\eta = B$ and (2) $\eta = 3B$. The solid and dashed lines represent stable and unstable solutions, respectively. S_1 and S_2 indicate steady solution bifurcation points.

8-2. Case of dissipation proportional to Laplacian of the streamfunction

In the remainder of this section, we consider a case of $D_{ij}(\Delta) = \Delta \delta_{ij}$. From Eqs. (6-4), (7-2) and (7-3), S_i , N/b and M become

$$\begin{aligned} S_i &= (a^2/4l^2) l_i r_i, \\ N/b &= (4l^2/a^2) \sum_i \sum_j \sum_k (r_i^2/U_i) Q_{ij} (-4l^2)^{-1} l_j Q_{jk} (-a^2) r_k \\ &\quad / \sum_i (r_i^2/u_i)^2, \\ M &= Q(4l^2/a^2) \sum_i \sum_j (r_i^2/U_i) Q_{ij} (-4l^2)^{-1} (r_j^2/u_j) \\ &\quad / \sum_i (r_i^2/u_i)^2. \end{aligned} \quad (8-1)$$

Further we restrict our attention to a case of $a^2 = 2F$. Then we can see from Eqs. (4-4) and (4-5) that the eigenvectors r_i and l_i become

$$r_i \propto [u_1 - u_2] \quad \text{and} \quad l_i \propto [1 \ -1], \quad (8-2)$$

and Eqs. (8-1) become

$$\begin{aligned} M &= -8l^2 \sum_i \sum_j u_i Q_{ij} (-4l^2)^{-1} u_j / \sum_i u_i^2, \\ N/b &= M - \delta. \end{aligned} \quad (8-3)$$

Here we write

$$\{(2l^2)/(4l^2 + 2F)\} \{(u_1 - u_2)^2 / \sum_i u_i^2\} = \delta > 0.$$

As is easily seen from Eq. (8-3), M is bounded as

$$2 > M > 8l^2 / (4l^2 + 2F). \quad (8-4)$$

When M is minimum, (δ) has a maximum value

$$\delta = \delta_{\max} = 4l^2 / (4l^2 + 2F) < 1. \quad (8-5)$$

For these values of M and δ ,

$$1 - M + (2 + M)(N/2b) = \delta^2 - \delta + 1 < 1. \quad (8-6)$$

Moreover, for these values of M and δ ,

$$N/b = M - \delta = 4l^2 / (4l^2 + 2F) > 0. \quad (8-7)$$

For sufficiently small r , Eqs. (7-11) and (8-7) imply that the steady solution curve corresponding to blocked flows belongs to the complex eigenvalue region. Consequently, Eqs. (7-19) and Eq. (8-6) imply that these complex eigenvalues become purely imaginary at 2 points on this steady solution curve, i.e., 2 Hopf bifurcation points emerge and the steady solutions between them lose their stability (see Fig. 3).

Fig. 3 Hopf bifurcations in the case of $D_{ij}(\Delta) = \delta_{ij}\Delta$ (a case of $b < 0$). (1) $\eta = \{1 - M + (2 + M)(N/2b)\}$, (2) $\eta = B$, (3) $\eta = \{1 + (N/b)B\}$ and (4) $\eta = 3B$. The solid and dashed lines represent stable and unstable solutions, respectively. S_1 and S_2 indicate steady solution bifurcation points. H_1 and H_2 indicate Hopf bifurcation points.

9. Conclusion

In the above sections, we examined the dependence of the topographically forced flows in a 2-layer model on the form of dissipation. As Pedlosky (1981) showed, the bifurcation diagram is dependent on the magnitude of the dissipation, i.e., the steady solution bifurcations occur only when the coefficient of dissipation is less than a critical value $r = r_c$. In this note the bifurcation diagram was shown to be dependent also on the form of dissipation.

If the dissipation is taken to be proportional to the potential vorticity, then the wave-wave interaction terms vanish. As a result, there exists no Hopf bifurcation, and the diagram becomes qualitatively the same as Pedlosky (1981)'s in which the dissipation is a sum of Ekman dampings on the upper and lower boundaries and an interfacial friction between the upper and lower layers; whereas, if the dissipation is taken to be proportional to the Laplacian of the streamfunction, i.e., Ekman dampings on the upper and lower bound-

aries, then the wave-wave interaction terms survive, in general. As a result, 2 Hopf bifurcation points may emerge in a sub-space to which the steady solutions belong, and the steady solutions between them lose their stability. Thus, how to parameterize the dissipation due to smaller scale waves is a cru-

cial problem.

In the case of Ekman dampings, Hopf bifurcations occur for appropriately strong wave-wave interaction and sufficiently small dissipation coefficient. The wave-wave interaction must be appropriately strong so that the curve on which the eigenvalues, if com-

Fig. 4a The magnitude of the wave-wave interaction is not appropriate. Since the hyperbola $P(\eta, B) = 0$ on which the eigenvalues, if complex, become purely imaginary lies in the real region $D(\eta, B) < 0$, there can not exist any Hopf bifurcation. $G(\eta, B) = 0$ is the steady solution curve.

Fig. 5a The dissipation coefficient is large enough. Since the hyperbola $P(\eta, B) = 0$ on which the eigenvalues become purely imaginary does not intersect the steady solution curve $G(\eta, B) = 0$, there does not exist any Hopf bifurcation.

Fig. 4b The magnitude of the wave-wave interaction is appropriate. Since the hyperbola $P(\eta, B) = 0$ lies in the complex region $D(\eta, B) > 0$, there may exist 2 Hopf bifurcation points denoted by H_1 and H_2 .

Fig. 5b The dissipation coefficient is small enough. Since the hyperbola $P(\eta, B) = 0$ intersects the curve $G(\eta, B) = 0$, there exist 2 Hopf bifurcation points denoted by H_1 and H_2 .

plex, become purely imaginary can shift into the complex-eigenvalue region (see Fig.4). As shown in section 8-2, this indeed does occur.

The dissipation coefficient must be so small that the above curve does indeed intersect the steady solution curve (see Fig.5).

The magnitude of dissipation due to smaller scale waves has some ambiguity, and is usually assumed to be sufficiently small in the large scale dynamics. So, in this note, we also assume so small a value of dissipation that the above intersection occurs. From the equations in sections 7 and 8, we can show that this intersection indeed occurs for a dissipation coefficient one order less than r_c , whose value is meteorologically meaningful (see Pedlosky 1981).

As was shown in the above sections, the dissipation proportional to the potential vorticity has qualitatively the same effect as Ekman dampings plus an interfacial friction. This is compatible with Klein and Pedlosky (1992)'s study for the nonlinear evolution of baroclinic waves. Moreover, the result that the dynamics with the dissipation proportional to the potential vorticity becomes simpler than

with Ekman dampings is also shared with Klein and Pedlosky (1992).

It was not yet been clarified whether these are merely mathematical problems or have deep physical meanings. This belongs to our future study.

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地形によって励起された流れの散逸形式への依存性に就いて

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準地衡風近似の二層模型で、地形によって励起された流れの散逸形式への依存性を解析的に考察した。特に(1)散逸が渦位に比例する場合と(2)散逸が流線関数のラプラシアンに比例する場合即ちエクマン散逸の場合を調べた。(1)の場合には分岐図は、エクマン散逸と上下層の界面摩擦を考慮した Pedlosky (1981) と定性的に同じものと成る。即ちブロッキング流および帯状流に対応する二つの安定定常解が同じ基本帯状流の値に対して同時に存在する。(2)の場合には波波相互作用項が出現し、その結果分岐図に二つの周期解分岐点が生まれ、此の分岐点の間の安定定常解が不安定化する可能性が出て来る。