A Formula for Vertical Flux of Horizontal Momentum Generated by Topography

by

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Abstract

An approximate formula for the vertical flux of horizontal momentum (momentum flux for brevity) generated by topography is derived. A steady non-rotating hydrostatic 3-dimensional linear problem is considered. The height of topography is a function of both horizontal coordinates, and its environmental flow has vertical shears of both magnitude and direction. Because of the directional shear, critical levels are present continuously in the vertical direction. In the presence of the critical levels, the momentum flux vector is azimuthally filtered continuously in the vertical direction, and becomes a function of the vertical coordinate.

The formula shows that, except for the azimuthal filtering, the momentum flux vector is approximately the same as that for a uniform environmental flow, whose velocity is equal to the original environmental flow velocity at the ground.

1. Introduction

Topography immersed in an environmental flow generates gravity waves. The gravity waves induce a vertical transport of horizontal momentum (hereafter, momentum flux for brevity).

In a steady non-rotating 2-dimensional linear problem, the governing equation of waves projected on the wave number space becomes singular at a critical level, if exists. The critical level is a height at which the environmental flow speed vanishes. The momentum flux is absorbed at the critical level if the Richardson number is greater than $1/4$ (Boo-ker and Bretherton, 1967). As a result, above the critical level, the momentum flux is essentially zero. Below the critical level, the momentum flux is independent of the vertical coordinate.

In a steady non-rotating 3-dimensional linear problem, the governing equation becomes singular where the environmental flow velocity is perpendicular to a given wave vector (Sawyer, 1962). Since, in general, the environmental flow velocity turns with height, the critical levels exist continuously in the vertical direction. At the critical level corre-

sponding to a wave vector, the component of momentum flux vector projected on this wave is absorbed. As a result, the momentum flux vector is azimuthally filtered continuously in the vertical direction and becomes a function of the vertical coordinate (Broad 1995; Pertsev 1989; Shutts 1998).

For a general 3-dimensional topography and a general environmental flow with vertical shears of both magnitude and direction, it is almost impossible to obtain an analytical expression for the momentum flux vector, even in the steady non-rotating linear framework.

Recently, however, Shutts (1995) analytically demonstrated that, except for the azimuthal filtering, the momentum flux vector for an environmental flow with a uniform shear is almost the same as that for a uniform environmental flow whose velocity is equal to the original flow velocity at the ground.

In this note, in the steady non-rotating hydrostatic linear framework, the above-mentioned result of Shutts (1995) is extended. That is, the following formula is analytically derived. Neglecting the $O((\Lambda/N)^2)$ terms, the momentum flux vector for an environmental flow with a constant buoyancy frequency N is the same as that for a uniform environ-

mental flow whose velocity is equal to the original flow velocity at the ground, except for the azimuthal filtering. Here, Λ is the scale of the shear, and Λ/N is assumed to be extremely smaller than unity.

The organization of this note is as follows. In Section 2, an equation of the momentum flux vector expressed in terms of the vertical velocity is derived. In section 3, the case of a uniform shear is reviewed. In sections 4 and 5, a layer model is introduced. In Sections 6 and 7, the formula is derived. In Section 8, concluding remarks are given.

2. Fundamental Equations

We begin with the following hydrostatic non-rotating steady Boussinesq system of equations:

$$u_i \partial_i u_i + w \partial u_i / \partial z + \partial_i \phi = 0, \quad (2-1a)$$

$$\partial \phi / \partial z = b, \quad (2-1b)$$

$$u_i \partial_i b + w \partial b / \partial z = 0, \quad (2-1c)$$

$$\partial_i u_i + \partial w / \partial z = 0. \quad (2-1d)$$

The subscripts i, j , which run from 1 to 2, denote the horizontal directions, with the convention of summation, e.g., $u_i \partial_i u_i = \sum_{i=1}^2 u_i \partial_i u_i$. u_i and ∂_i are the horizontal velocity and horizontal differential operator, respectively. w and z are the vertical velocity and vertical coordinate, respectively. The vertical coordinate z is the pseudo-height of Hoskins and Bretherton (1972). ϕ is the geopotential, and b represents buoyancy, which is proportional to the potential temperature. The following basic state (denoted by an upper case), which is horizontal, non-vanishing and dependent only on z , is assumed.

$$U_i = U_i(z) \neq 0. \quad (2-2)$$

The basic flow encounters a topography prescribed by

$$z = h(x_1, x_2). \quad (2-3)$$

In (2-3), x_i ($i=1,2$, $-\infty < x_i < +\infty$) are the horizontal coordinates. The topography disturbs the basic flow. The linearized equations of disturbances (which are hereafter denoted by a lower case) become

$$U_j \partial_j u_i + \Lambda_i w + \partial_i \phi = 0, \quad \Lambda_i = dU_i / dz, \quad (2-4a)$$

$$\partial \phi / \partial z = b, \quad (2-4b)$$

$$U_j \partial_j b + N^2 w = 0, \quad (2-4c)$$

$$\partial_i u_i + \partial w / \partial z = 0. \quad (2-4d)$$

In (2-4c), N^2 is the square of buoyancy frequency of the basic state. Since the coefficients of Eqs.(2-4),

that is, U_i , Λ_i and N^2 , depend only on z , it is possible to project Eqs.(2-4) on a Fourier base $(1/2\pi) \exp[iK_i x_i]$. Here K_i ($i=1,2$, $-\infty < K_i < +\infty$) is the horizontal wave number vector.

$$i(\mathbf{K} \cdot \mathbf{U})u_i + \Lambda_i w + iK_i \phi = 0, \quad (2-5a)$$

$$d\phi/dz = b, \quad (2-5b)$$

$$i(\mathbf{K} \cdot \mathbf{U})b + N^2 w = 0, \quad (2-5c)$$

$$i(\mathbf{K} \cdot \mathbf{u}) + dw/dz = 0. \quad (2-5d)$$

Here, $u_i = u_i(K_1, K_2, z)$ is the Fourier transform of $u_i(x_1, x_2, z)$ etc., and $\mathbf{A} \cdot \mathbf{B}$ stands for the inner product $A_i B_i$. Eliminating other variables than w from (2-5), the vertical structure equation for w is obtained as

$$d^2 w / dz^2 + L^2 w = 0, \quad (2-6)$$

where $K = (\mathbf{K} \cdot \mathbf{K})^{1/2}$, $U = \{\mathbf{K} \cdot \mathbf{U}(z)\}/K$,

$$d^2 U / dz^2 = \{\mathbf{K} \cdot d^2 \mathbf{U}(z) / dz^2\} / K.$$

Eq.(2-6) becomes singular at the height where the inner product $\mathbf{K} \cdot \mathbf{U}(z)$ vanishes. This height, which is called the critical level and is denoted as z_c , is dependent on the azimuthal angle α of K_i ,

$$\mathbf{K} \cdot \mathbf{U}(z_c) = 0, \quad z_c = z_c(\alpha), \quad \alpha = \tan^{-1}(K_2/K_1). \quad (2-7)$$

From (2-5a) and (2-5d), u_i can be expressed in terms of w as

$$K^2 u_i = i(dw/dz)K_i - iw\{(\mathbf{K} \times \mathbf{\Lambda})/(\mathbf{K} \cdot \mathbf{U})\} \epsilon_{ij} K_j. \quad (2-8)$$

Here, ϵ_{ij} is defined as $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$, and $\mathbf{A} \times \mathbf{B}$ stands for the outer product $\epsilon_{ij} A_i B_j$. Using the expression (2-8), the vertical flux of horizontal momentum (hereafter momentum flux for brevity) is written as

$$M_i = \int dK_1 \int dK_2 \{u_i w^*\} \\ = (i/2) \int dK_1 \int dK_2 (K_i/K^2) W[w^*, w]. \quad (2-9)$$

Here, A^* is the complex conjugate of A , and $W[A, B] = A dB/dz - B dA/dz$ is the Wronskian of A and B . From (2-6), $W[w^*, w]$ is independent of z for $z < z_c(\alpha)$. According to Booker and Bretherton (1967), the momentum flux is negligible above the critical level $z_c(\alpha)$. As a result, the momentum flux (2-9), which becomes a function of z , can be rewritten as

$$M_i(z) = (i/2) \int dK \int d\alpha \Theta(z_c - z) (K_i/K) W[w^*, w], \\ z_c = z_c(\alpha). \quad (2-10)$$

Here Θ is a step function defined as $\Theta(\xi)=1$ for $\xi>0$ and $\Theta(\xi)=0$ for $\xi<0$. The step function represents azimuthal filtering of the momentum flux for wave number K_i above the critical level $z_c(\alpha)$.

3. Uniform-Shear Flow

For a basic flow with both N and $\Lambda_i=dU_i/dz$ constant, Eq.(2-6) is reduced to

$$\begin{aligned} d^2w/dZ^2 + (N/\Lambda)^2 w/Z^2 &= 0, \\ Z &= z + U(0)/\Lambda, \quad \Lambda = (\mathbf{K} \cdot \Lambda)/K. \end{aligned} \quad (3-1)$$

Assuming the existence of the critical level $z_c(\alpha)$ (i. e., $Z_0 = Z(z=0) < 0$), the general solution to (3-1) for $z < z_c(\alpha)$ is given by

$$\begin{aligned} w &= A|Z|^{1/2} \exp[i\mu \log|Z|] \\ &\quad + B|Z|^{1/2} \exp[-i\mu \log|Z|], \\ \mu &= (N^2/\Lambda^2 - 1/4)^{1/2}. \end{aligned} \quad (3-2)$$

Since $|\Lambda/N|$ can be estimated as

$$\begin{aligned} |\Lambda/N| &\cong |\Lambda_i/N| = O(\epsilon), \\ \epsilon &\cong (10/10^4)/10^{-2} = 10^{-1}, \end{aligned} \quad (3-3)$$

the parameter μ in (3-2) is real and positive. In order for the upper radiation condition to be satisfied, Eq.(3-2) must have the following form:

$$\begin{aligned} w &= A|Z|^{1/2} \exp[-i\beta\mu \log|Z|], \\ \beta &= \text{sign}\{U(0)\}. \end{aligned} \quad (3-4)$$

From the lower boundary condition $w=i(\mathbf{K} \cdot \mathbf{U})h$ on $z=0$, the coefficient A is determined and w becomes

$$\begin{aligned} w &= iKU(0)h|Z/Z_0|^{1/2} \exp[-i\beta\mu \log|Z/Z_0|], \\ Z_0 &= U(0)/\Lambda. \end{aligned} \quad (3-5)$$

Substituting (3-5) into (2-10), the momentum flux is given by

$$\begin{aligned} M_i(z) &= -\int dK \int d\alpha \Theta(z_c - z) N |h|^2 K_i KU(0) \{1 \\ &\quad - \Lambda^2/4N^2\}^{1/2}. \end{aligned} \quad (3-6)$$

Using (3-3), Eq.(3-6) can be estimated as

$$\begin{aligned} M_i(z) &= -\int dK \int d\alpha \Theta(z_c - z) N |h|^2 K_i KU(0) \{1 + O(\epsilon^2)\}, \\ z_c &= z_c(\alpha). \end{aligned} \quad (3-7)$$

Neglecting the $O(\epsilon^2)$ terms, the momentum flux for the uniform shear basic flow, $U_i = U_i(0) + \Lambda_i z$, is equal to that of the following formula for a uniform

basic flow, $U_i = U_i(0)$, except for the azimuthal filtering,

$$M_i = -\int dK \int d\alpha N |h|^2 K_i KU. \quad (3-8)$$

4. Layer Model

Since $|N/U| \cong 10^{-3}$ and $(d^2U/dz^2)/U \cong 10^{-8}$ under usual atmospheric conditions, the ratio of $(d^2U/dz^2)/U$ to $|N/U|^2$ is estimated as

$$\{(d^2U/dz^2)/U\}/|N/U|^2 \cong 10^{-2} = O(\epsilon^2). \quad (4-1)$$

Because of (4-1), neglecting the terms of $O(\epsilon^2)$, Eq. (2-6) can be approximated as

$$d^2w/dz^2 + L^2 w = 0, \quad L^2 = N^2/U^2 \quad (4-2)$$

In order to solve (4-2) for general basic flows with N constant, we consider the following M layers model:

$$\begin{aligned} 0 &= z_0 < z_1 < z_2 < \cdots < z_{M-1} < z_M = z_c(\alpha), \\ d^2w_n/dz^2 + L_n^2 w_n &= 0 \end{aligned} \quad (4-3)$$

for $z_{n-1} < z < z_n$, $n=1, \dots, M$,

$$L_n^2 = N^2/\{U_{n-1} + \Lambda_n(z - z_{n-1})\}^2,$$

$$U_n = \mathbf{K} \cdot \mathbf{U}(z_n)/K = \beta |\mathbf{K} \cdot \mathbf{U}(z_n)|/K,$$

$$\beta = \text{sign}\{U(0)\}, \quad \Lambda_n = (U_n - U_{n-1})/(z_n - z_{n-1}).$$

The projection U_n of $U_i(z_n)$ on K_i is of sign definite for $z < z_c(\alpha)$. The sign is denoted by β . Eq.(4-3) is rewritten as

$$\begin{aligned} d^2w_n/dz_n^2 + (N/\Lambda_n)^2 w_n/z_n^2 &= 0, \\ Z_n &= z - z_{n-1} + U_{n-1}/\Lambda_n. \end{aligned} \quad (4-4)$$

In this model, the shear $\Lambda_i = dU_i/dz$ is uniform in each layer. The velocity U_i is continuous at each layer boundary. Because of the continuity of U_i , the kinematical boundary conditions are

$$w_{n-1} = w_n \quad \text{on } z = z_{n-1}, \quad n=2, \dots, M, \quad (4-5)$$

$$w_1 = iKU(0)h \quad \text{on } z = z_0 = 0. \quad (4-6)$$

While because of the presence of Λ_i , the dynamical boundary conditions are

$$\begin{aligned} U_{n-1} dw_{n-1}/dz - \Lambda_{n-1} w_{n-1} &= U_{n-1} dw_n/dz - \Lambda_n w_n \\ \text{on } z &= z_{n-1}, \quad n=2, \dots, M. \end{aligned} \quad (4-7)$$

Under the boundary conditions (4-5) and (4-7), the Wronskian $W[w^*, w]$ is of course continuous at each boundary.

5. Level Setting

In (4-3), the levels z_n , $n=1, \dots, M-1$ are set as

follows.

$$\begin{aligned} |U_n/U_{n-1}| &= \exp[-\pi/2\mu_n] \\ \text{if } |U_n| < |U_{n-1}|, \\ |U_n/U_{n-1}| &= \exp[+\pi/2\mu_n] \\ \text{if } |U_n| > |U_{n-1}|, \quad \mu_n &= (N^2/\Lambda_n^2 - 1/4)^{1/2} \quad (5-1) \end{aligned}$$

Since $\pi/2\mu_n \equiv \pi\Lambda_n/2N \equiv O(\epsilon)$, the conditions (5-1) imply that

$$|U_n/U_{n-1}| \equiv 1 \pm \pi\Lambda_n/2N \equiv 1 + O(\epsilon). \quad (5-2)$$

Under usual atmospheric conditions, the variation of the projected velocity $U(z) = \mathbf{K} \cdot \mathbf{U}(z)/K$ in the range of local wave length is small

$$\begin{aligned} U(z+\lambda) &\equiv U(z) + \lambda\Lambda(z) \\ &\equiv U(z)\{1 + O(|\Lambda/N|)\} \equiv U(z)\{1 + O(\epsilon)\}, \quad (5-3) \end{aligned}$$

where $\lambda = U(z)/N$ is the local wave length (divided by 2π). Here we may fairly assume that the variation of the projected shear $\Lambda(z) = \mathbf{K} \cdot \Lambda(z)/K$ is also so small,

$$\Lambda(z+\lambda) \equiv \Lambda(z)\{1 + O(\epsilon)\}. \quad (5-4)$$

From (5-4), the depth of layer $z_n - z_{n-1}$, becomes $O(\lambda_n)$.

$$\begin{aligned} z_n - z_{n-1} &= (U_n - U_{n-1})/\Lambda_n \\ &\equiv O(\pi U_{n-1}/2N) \equiv O(\lambda_n). \quad (5-5) \end{aligned}$$

From (5-4) and (5-5), and using Taylor's mean value theorem, the difference between the velocity $(U_n + U_{n-1})/2$ and true velocity $U_{n-0.5} = \mathbf{K} \cdot \mathbf{U}((Z_n + Z_{n-1})/2)/K$ at the middle point $z = (Z_n + Z_{n-1})/2$ is estimated as

$$\begin{aligned} (U_n + U_{n-1})/2 - U_{n-0.5} \\ &\equiv \{U_{n-1} + \Lambda_n(z_n - z_{n-1})/2\} \\ &\quad - \{U_{n-1} + (1 + O(\epsilon))\Lambda_n(z_n - z_{n-1})/2\} \\ &\equiv O(\epsilon) \cdot \Lambda_n(z_n - z_{n-1})/2 = O(\epsilon) \cdot (U_n - U_{n-1})/2. \quad (5-6) \end{aligned}$$

Eq.(5-6) together with Eq.(5-2) implies that the error of the layer-modeled velocity is

$$\{(U_n + U_{n-1})/2 - U_{n-0.5}\}/U_{n-0.5} \equiv O(\epsilon^2) \quad (5-7)$$

From (5-7), the error in approximating the continuous model of $U_1(z)$ by the layer model of piece-wise uniform shear with the level setting (5-1) is $O(\epsilon^2)$.

6. Solution

Solutions to (4-4) have the following form.

$$\begin{aligned} w_n &= A_n F_n(z) + B_n G_n(z) \\ \text{for } z_{n-1} < z < z_n, \quad n &= 1, \dots, M, \\ F_n(z) &= |Z_n(z)|^{1/2} \exp[i\mu_n \log |Z_n(z)|], \\ G_n(z) &= |Z_n(z)|^{1/2} \exp[-i\mu_n \log |Z_n(z)|]. \quad (6-1) \end{aligned}$$

The boundary conditions (4-5) and (4-7) become

$$\begin{aligned} A_{n-1} F_{n-1}(z_{n-1}) + B_{n-1} G_{n-1}(z_{n-1}) \\ = A_n F_n(z_{n-1}) + B_n G_n(z_{n-1}), \quad (6-2a) \end{aligned}$$

$$\begin{aligned} (1 + i2\epsilon_{n-1})A_{n-1} F_{n-1}(z_{n-1}) \\ - (1 - i2\epsilon_{n-1})B_{n-1} G_{n-1}(z_{n-1}) \\ = (1 + i2\epsilon_n)A_n F_n(z_{n-1}) \\ - (1 - i2\epsilon_n)B_n G_n(z_{n-1}) + O(\epsilon^2), \quad (6-2b) \end{aligned}$$

where $2\epsilon_n = -\beta(\Lambda_n/N) - |\Lambda_n/2N| \equiv O(\epsilon)$, and here and hereafter the term $O(\epsilon^2)$ is of the order of ϵ^2 compared with the other terms. From the boundary conditions (6-2a) and (6-2b), the coefficients A_n and B_n in (6-1) must satisfy.

$$\begin{aligned} A_{n-1} F_{n-1}(z_{n-1}) &= \{1 + i(\epsilon_n - \epsilon_{n-1})\} A_n F_n(z_{n-1}) \\ &\quad + i(\epsilon_n - \epsilon_{n-1}) B_n G_n(z_{n-1}) + O(\epsilon^2), \\ B_{n-1} G_{n-1}(z_{n-1}) &= \{1 - i(\epsilon_n - \epsilon_{n-1})\} B_n G_n(z_{n-1}) \\ &\quad - i(\epsilon_n - \epsilon_{n-1}) A_n F_n(z_{n-1}) + O(\epsilon^2). \quad (6-3) \end{aligned}$$

Eq.(6-3) is rewritten as

$$\begin{aligned} [A_{n-1} F_{n-1}(z_{n-2}), B_{n-1} G_{n-1}(z_{n-2})] \\ = \Xi(n-1, n-2) \{ \mathbf{I} + i(\epsilon_n - \epsilon_{n-1}) \mathbf{D} \} [A_n F_n(z_{n-1}), \\ B_n G_n(z_{n-1})] + O(\epsilon^2). \quad (6-4) \end{aligned}$$

where $\mathbf{I}$ is the identity matrix, and the other matrices Ξ and $\mathbf{D}$ are defined as

$$\begin{aligned} \Xi(n, n-1)_{11} &= F_n(z_{n-1})/F_n(z_n) \\ &= |U_{n-1}/U_n|^{1/2} \exp[i\mu_n \log |U_{n-1}/U_n|], \\ \Xi(n, n-1)_{22} &= G_n(z_{n-1})/G_n(z_n) \\ &= |U_{n-1}/U_n|^{1/2} \exp[-i\mu_n \log |U_{n-1}/U_n|], \\ \Xi(n, n-1)_{12} &= \Xi(n, n-1)_{21} = 0, \\ \mathbf{D}_{11} &= \mathbf{D}_{12} = -\mathbf{D}_{21} = -\mathbf{D}_{22} = 1. \quad (6-5) \end{aligned}$$

The solution in the highest layer has the following form.

$$w_M = A_M |Z_M(z)|^{1/2} \exp[-i\beta\mu_M \log |Z_M(z)|]. \quad (6-6)$$

In the following, the case of $\beta = \text{sign}\{\mathbf{K} \cdot \mathbf{U}(0)\} = -1$ is considered (in the other case of $\beta = 1$, the same result is obtained).

7. Formula

Substituting (6-6) into (6-4), the coefficients A_1 and B_1 in the lowest layer are related with A_M as

$$\begin{aligned} A_1 F_1(z_0) &= |U_0/U_{M-1}|^{1/2} \exp[i \sum_{n=1}^{M-1} \xi(n, n-1)] \{1 \\ &\quad + i(\epsilon_M - \epsilon_1)\} A_M F_M(z_{M-1}) + O(\epsilon^2), \\ B_1 G_1(z_0) &= -|U_0/U_{M-1}|^{1/2} \sum_{k=2}^M \exp[i \{\sum_{n=k}^{M-1} \xi(n, n-1) \\ &\quad - \sum_{m=1}^{k-1} \xi(m, m-1)\}] i(\epsilon_k - \epsilon_{k-1}) A_M F_M(z_{M-1}) \\ &\quad + O(\epsilon^2), \end{aligned} \quad (7-1)$$

where

$$\begin{aligned} \xi(n, n-1) &= \mu_n \log |Z_n(z_{n-1})/Z_n(z_n)| \\ &= \mu_n \log |U_{n-1}/U_n|. \end{aligned} \quad (7-2)$$

Substituting (7-1) into the lower boundary condition (4-6) gives

$$\begin{aligned} K^2 |U_{M-1} U_0| |h|^2 / |A_M F_M(z_{M-1})|^2 \\ = |1 + i(\epsilon_M - \epsilon_1) \\ - i \sum_{k=2}^M (\epsilon_k - \epsilon_{k-1}) \exp[-2i \sum_{m=1}^{k-1} \xi(m, m-1)]|^2 \\ + O(\epsilon^2). \end{aligned} \quad (7-3)$$

Further substituting (5-1) and (7-2) into (7-3) gives

$$|A_M|^2 / K^2 |U_0| |A_M| |h|^2 = 1 + O(\epsilon^2). \quad (7-4)$$

From (7-4), the Wronskian $W[w^*, w]$ is calculated as

$$\begin{aligned} W[w^*, w] &= -2i \mu_M |A_M|^2 \\ &= -2i n K^2 |U_0| |h|^2 + O(\epsilon^2). \end{aligned} \quad (7-5)$$

Substituting (7-5) into (2-10) yields

$$\begin{aligned} M_1(z) &= - \int dK \int d\alpha \Theta(z_c - z) N |h|^2 K_1 K U(0) \\ &\quad + O(\epsilon^2), \quad z_c = z_c(\alpha). \end{aligned} \quad (7-6)$$

As in the uniform shear case (3-7), neglecting the $O(\epsilon^2)$ terms, the momentum flux for a general basic flow $U_1(z)$ is the same as that for a uniform basic flow $U_1 = U_1(0)$ except for the azimuthal filtering.

8. conclusion

In this note, an approximate formula for the vertical flux of horizontal momentum generated by a topography is derived. A steady non-rotating hydrostatic 3-dimensional linear problem is consid-

ered. The "3-dimensional" means that the height of topography is a function of both the horizontal coordinates, and that the environmental flow velocity has shears of both magnitude and direction. Because of the directional shear, the momentum flux is azimuthally filtered. As a result, the momentum flux vector becomes a function of the vertical coordinate.

For a general environmental flow, it is almost impossible to obtain an analytical expression for the momentum flux. The derived formula, however, shows that, except for the azimuthal filtering, the momentum flux approximately (neglecting $O(\epsilon^2)$ terms) becomes the same as that for a uniform environmental flow, whose flow velocity is equal to the prescribed environmental flow velocity at the ground. Here, ϵ is the scale of ratio of the vertical shear to the buoyancy frequency.

In other words, except for the azimuthal filtering, the momentum flux is approximately obtained from a uniform environmental flow. The momentum flux for a uniform environmental flow is analytically expressible in terms of the uniform flow velocity, environmental buoyancy frequency and topography. Therefore, the formula can approximately analytically express the momentum flux in terms of the environmental flow velocity at the ground, buoyancy frequency and topography, together with the azimuthal filtering.

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地形によって惹起された水平運動量の鉛直輸送の公式

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地形によって惹起された水平運動量の鉛直輸送（以下簡単に運動量輸送）の近似公式を導く。定常で非回転で静水圧近似の、三次元線形問題を考える。地形の高さは二つの水平座標の関数である。一般流は大きさと方向の両方のシアを持つ。方向シアの為に、臨界高度が鉛直方向に連続的に存在する。この為、運動量輸送ベクトルは鉛直方向に連続的に方向濾過される。その結果、鉛直座標の関数と成る。

公式は次の事を示す。方向濾過を別にすれば、運動量輸送ベクトルは近似的に一樣一般流によって惹き起こされたものに等しい。この一樣一般流は、元の一般流の地表での値に等しいものである。