Dynamics of the Eye of Storm

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Abstract

We find the mathematically correct solutions of the equations of motion for circular isobaric system in the case without friction and verify that an area into which no wind blows is formed around the center corresponding to the eye of storm. And from this solutions it is shown that under certain conditions the vertical air current may exist within the eye.

1. Introduction

The first problem about the eye of storm is why the wind speed suddenly drops within the eye. A. Oberbeck ⁽¹⁾, D. Kitao ⁽²⁾, F. H. Bigelow ⁽³⁾, B. Haurwitz ⁽⁴⁾, S. Shono ⁽⁵⁾, and others published valuable memoirs, in which they gave mathematical theories about the vortical motion in the atmosphere. But in none of their theories is given any satisfactory reason why the wind falls abruptly in force within the central part. T. Okada ⁽⁶⁾ said that the existence of a central calm within a well-developed typhoon might easily be understood from the analogy of a well-known hydrodynamical problem of a discontinuous motion in two dimensions of an ideal fluid. In the theories mentioned above about the vortical motion, frictional resistance was taken into consideration, but recently K. Wadati ⁽⁷⁾ and his collaborator, neglecting it, integrated approximately the differential equations of motion in the case of moving circular isobaric system and showed that an area into which no wind blows, is formed around the center. In this paper, we shall find the mathematically correct solutions of the equations of motion for staying circular isobaric system in the case without frictional resistance and verify that an area into which no wind blows is formed around the center corresponding to the eye of storm, divided from the outer zone by a discontinuity of wind.

The second problem about the eye of storm is the temperature and relative humidity within the eye. S. M. Ballou ⁽⁸⁾ and W. Trabert ⁽⁹⁾ and others assumed the existence of a descending air current within the eye, to explain the excessive high temperature and dryness within it. The example of excessive high temperature and dryness is, as often mentioned in textbooks, that found during the wellknown Manila typhoon of October 20th 1882. During the passage of this typhoon center over Manila the temperature rose by 6.5° C., that is, from 25° C. to 31.5° C. and the relative humidity fell from 95% to 43%. But E. Deppermann ⁽¹⁰⁾ and T. Okada ⁽⁶⁾, investigating many typhoons, verified that the excessive high temperature and dryness are not essential characteristics of the eye of typhoon and therefore there is no reason to assume a descending air current within the eye. In this report,

from our solutions of equations of motion, it will be seen that under certain conditions the descending air current may exist within the eye.

2. The solutions of equations corresponding to the eye of storm

Suppose a cylindrical coordinate (r, θ, z) , the z axis of which is directed to zenith. We denote the r -, θ -, and z -component of the wind by v_r , v_θ and v_z , and pressure, density and the Coriolis' factor by P , ρ and λ . And we shall consider the special case of the circular isobaric system, accompanying motion symmetrical with respect to the center, in which v_r , v_θ , v_z , P and ρ may be considered as functions of r and z only. In this case, if we assume that $\frac{dw}{dt}$ and the z component of the Coriolis' factor may be neglected, the equations of steady motion of atmosphere become as follows:

$$(1) \quad v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} - \lambda v_\theta = -\frac{1}{\rho} \frac{\partial P}{\partial r}$$

$$(2) \quad v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} + \lambda v_r = 0$$

$$(3) \quad g = -\frac{1}{\rho} \frac{\partial P}{\partial z}$$

And the equation of continuity is

$$(4) \quad \frac{1}{r} \frac{\partial}{\partial r} (\rho r v_r) + \frac{\partial}{\partial z} (\rho v_z) = 0$$

Now we shall consider the case where $\frac{1}{\rho} \frac{\partial P}{\partial r}$ is a function of r only. For instance if we consider the polytropic atmosphere of order n , this will be the case. For, in this case, temperature T , pressure P and density ρ are expressed as

$$T = T_0 \left(1 - \frac{z}{H_n}\right), \quad P = P_0 \left(1 - \frac{z}{H_n}\right)^{n+1}, \quad \rho = \rho_0 \left(1 - \frac{z}{H_n}\right)^n$$

in which T_0 , P_0 and ρ_0 are values of T , P and ρ at $z=0$ respectively and $H_n = \frac{R T_0 (n+1)}{g}$.

From these relations, we obtain

$$(5) \quad \frac{1}{\rho} \frac{\partial P}{\partial r} = \frac{1}{\rho_0} \frac{\partial P_0}{\partial r} = R(n+1) \frac{\partial T_0}{\partial r}$$

If $\frac{1}{\rho} \frac{\partial P}{\partial r}$ is a function of r only, v_r and v_θ may also be regarded as functions of r only, and the equations (1) and (2) will become

$$(6) \quad v_r \frac{dv_r}{dr} - \frac{v_\theta^2}{r} - \lambda v_\theta = -\frac{d\pi}{dr}$$

$$(7) \quad v_r \left(\frac{dv_\theta}{dr} + \frac{v_\theta}{r} + \lambda \right) = 0$$

where $\pi = \int \frac{dP}{\rho}$. If we put $v_r=0$ from (7), we can get the equation of the gradi-

ent wind from (6). On the other hand, we can obtain from (7)

$$(8) \quad \frac{dv_\theta}{dr} + \frac{v_\theta}{r} + \lambda = 0$$

If the equation (6) is added to the equation (8), multiplied by v_θ and this summation is integrated, we will get the integral of energy as

$$(9) \quad \frac{1}{2}(v_r^2 + v_\theta^2) + \pi = h$$

where h is an integrating constant. Integrating (8), we obtain the integral of absolute angular momentum as

$$(10) \quad rv_\theta + \frac{\lambda r^2}{2} = c$$

where c is an integrating constant.

Now h and c are constants on each stream line, but if we consider the motion symmetrical with respect to the center, we may regard them as constants in space. From (9) and (10), we get

$$(11) \quad v_r = -\sqrt{2(h - \pi) - \left(\frac{c}{r} - \frac{\lambda r}{2}\right)^2}$$

$$(12) \quad v_\theta = \frac{c}{r} - \frac{\lambda r}{2}$$

which give the horizontal components of wind velocity. Such solutions that fulfil the conditions $v_r = 0$, $v_\theta = 0$ and $\pi = \pi_0$ (const.) along the circle $r = r_0$ become from (11) and (12) as follows,

$$(13) \quad v_r = -\sqrt{2(\pi_0 - \pi) - \left(\frac{\lambda r_0^2}{2r} - \frac{\lambda r}{2}\right)^2}$$

$$(14) \quad v_\theta = \frac{\lambda r_0^2}{2r} - \frac{\lambda r}{2}$$

Here we consider the following two curves on a plane of r - η Cartesian coordinate

$$\begin{cases} \eta_1 = 2(\pi_0 - \pi) \\ \eta_2 = \left(\frac{\lambda r_0^2}{2r} - \frac{\lambda r}{2}\right)^2 \end{cases}$$

In the case where the pressure distribution p is cyclonic as in typhoon and $\pi(r_0) = \pi_0$ ($r \geq r_0$), $\pi'(r_0) = 0$ and $\frac{d\pi}{dr} > 0$ ($0 < r < r_0$), η_1 represents a curve as shown by (I) in Fig.1, which contacts with the r axis at $r = r_0$. And η_2 represents a curve as shown by (II) in the same Fig., which also contacts with the r axis at $r = r_0$. If we denote the abscissae of the intersecting points of both curves (I) and (II) by a and r_0 as in Fig.1, v_r will be a real value only when $a \leq r \leq r_0$, and the motion will take place within the interval of $a \leq r \leq r_0$. And if we denote the terms under the root sign of (13)

Fig. 1

lines starting from (16), we classify this problem into the following two cases and treat them one by one.

i). When $\varphi(r)$ has a zero-point of the first order at $r=a$ and so $\varphi(a)=0$ and $\varphi'(a) \neq 0$.

In this case, if we put

$$F(r) = (r-a)f(r)$$

then $f(r) > 0$ for the interval $a \leq r \leq r_0$ and the equation for the stream line becomes

$$\left(\frac{dr}{d\theta}\right)^2 = (r-a)f(r)$$

Here, we introduce a subsidiary variable ψ and put

$$\left(\frac{dr}{d\psi}\right)^2 = (r-a), \quad \left(\frac{d\psi}{d\theta}\right)^2 = f(r)$$

As $f(r) > 0$ within the interval of $a \leq r \leq r_0$, ψ will be a monotonously increasing function of θ , if we choose $\frac{d\psi}{d\theta} = +\sqrt{f(r)}$. So, the essential features of the stream lines will be found from

$$\left(\frac{dr}{d\psi}\right)^2 = r-a$$

Integrating this equation

$$r = a + \frac{(\psi - \psi_0)^2}{4}$$

in which ψ_0 is an integrating constant. If we represent one of the curves represented by this equation in a polar coordinate, taking r and ψ as its radius and argument respectively, we obtain a figure as shown by (I) in Fig. 2. As will be seen in this, the air particle which has reached the circle $r=a$ blows outward again.

ii). When $\varphi(r)$ has a zero-point of the second order at $r=a$ and so $\varphi(a)=\varphi'(a)=0$

$$(15) \quad \varphi(r) = 2(\pi_0 - \pi) - \left(\frac{\lambda r_0^2}{2r} - \frac{\lambda r}{2}\right)^2$$

then $\varphi(a)=\varphi(r_0)=\varphi'(r_0)=0$ and $\varphi(r)$ will have a zero-point of second order at $r=r_0$. On the other hand, the differential equation for the stream line become

$$(16) \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{r^2\varphi(r)}{\left(\frac{\lambda r_0^2}{2r} - \frac{\lambda r}{2}\right)^2}$$

If we put the function in the right-hand side of (16) as $F(r)$, we shall see that $F(r) > 0$ for the interval $a < r \leq r_0$ while $F(a) = 0$. To trace the actual pattern of stream

=0.

In this case, the curves (I) and (II) in Fig.1 contact each other. If we put

$$F(r) = (r-a)^2 f(r)$$

then $f(r) > 0$ for the interval $a \leq r \leq r_0$ and the equation of stream line will become

$$\left(\frac{dr}{d\theta}\right)^2 = (r-a)^2 f(r)$$

After the same procedure as in i), we introduce a subsidiary variable ψ and put

$$\left(\frac{dr}{d\psi}\right)^2 = (r-a)^2, \quad \left(\frac{d\psi}{d\theta}\right)^2 = f(r)$$

Then, characteristics of the stream lines will be known from

$$\left(\frac{dr}{d\psi}\right)^2 = (r-a)^2$$

Integrating this

$$r = a + e^{-(\psi - \psi_0)}$$

in which ψ_0 is an integrating constant. If we trace one of the curves represented by this in a polar coordinate system taking r and ψ as the radius and argument, we obtain a figure as shown by (II) in Fig.2. As will be seen in this, the stream line approaches asymptotically the circle $r=a$.

Fig. 2

On the other hand, from (6), (8), (14) and (15)

$$\varphi(r) = 2(\pi_0 - \pi) - v_\theta^2$$

$$\varphi'(r) = -2 \frac{\partial \pi}{\partial r} - 2 v_\theta \frac{\partial v_\theta}{\partial r}$$

$$= -\frac{2}{\rho} \frac{\partial P}{\partial r} + 2 v_\theta \left(\frac{v_\theta}{r} + \lambda \right)$$

$$= 2 \left(\frac{v_\theta^2}{r} + \lambda v_\theta - \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right)$$

And so $\varphi'(a)=0$ shows that at $r=a$, v_θ is equal to the velocity of the gradient wind. And as from $v_r = -\sqrt{\varphi(r)}$

$$\frac{\partial v_r}{\partial r} = -\frac{1}{2} \frac{\varphi'(r)}{\sqrt{\varphi(r)}}$$

if $\varphi(a)=0$ and $\varphi'(a) \neq 0$, it follows $\left(\frac{\partial v_r}{\partial r}\right)_{r=a} = \infty$. Next, from (3) and (4)

$$\frac{\partial}{\partial z} (\rho v_z) = \frac{\partial}{\partial z} \left\{ \frac{1}{gr} \frac{\partial}{\partial r} (r p v_r) \right\}$$

Integrating this equation, we obtain

$$(17) \quad v_z = \frac{1}{\rho g} \left\{ \left(\frac{v_r}{r} + \frac{\partial v_r}{\partial r} \right) (P - P_0) + v_r \left(\frac{\partial P}{\partial r} - \left(\frac{\partial P}{\partial r} \right)_0 \right) \right\}$$

where P_0 and $\left(\frac{\partial P}{\partial r}\right)_0$ are the values of P and $\frac{\partial P}{\partial r}$ at $z=0$. From (17), if $\frac{\partial v_r}{\partial r} = \infty$, it follows $v_z = \infty$.

In conclusion, the air particle which has reached the circle $r=a$ blows outward again and $v_z = \infty$ at $r=a$, when $\varphi(a)=0$ and $\varphi'(a) \neq 0$. This is impossible in the symmetrical steady motion. On the other hand, the stream line approaches the circle $r=a$ asymptotically and v_z at $r=a$ is finite, when $\varphi(a)=\varphi'(a)=0$. Therefore, (13), (14) and (17) are our solutions of the equations for symmetrical steady motion, only when $\varphi(a)=\varphi'(a)=0$. In this case, the motion does not take place within the circle $r=a$. To verify that this area may be regarded as the eye of storm, we shall investigate the distribution of the meteorological elements to be obtained from the above solutions.

3. Distribution of the wind and pressure

When the density is uniform, we shall calculate the distribution of the wind and pressure. When the density is uniform, (13), (14), (17) and (15) become as follows,

$$(18) \quad v_r = -\sqrt{\frac{2}{\rho}(P(r_0) - P(r)) - \left(\frac{\lambda r_0^2}{2r} - \frac{\lambda r}{2}\right)^2}$$

$$(19) \quad v_\theta = \frac{\lambda r_0^2}{2r} - \frac{\lambda r}{2}$$

$$(20) \quad v_z = -\left(\frac{v_r}{r} + \frac{\partial v_r}{\partial r}\right)z$$

$$(21) \quad \varphi(r) = \frac{2}{\rho}(P(r_0) - P(r)) - \left(\frac{\lambda r_0^2}{2r} - \frac{\lambda r}{2}\right)^2$$

in which $P'(r) > 0$ ($a \leq r \leq r_0$), $P(r) = P(r_0)$ ($r > r_0$) and $P'(r_0) = 0$. As the motion takes place within the interval $a \leq r \leq r_0$ and $\varphi(r_0) = \varphi'(r_0) = \varphi(a) = \varphi'(a) = 0$, $\varphi(r)$ must be expressed in the form

$$(22) \quad \varphi(r) = (r_0 - r)^2(r - a)^2 g(r)$$

where $g(r)$ is an arbitrary positive function. From (21) and (22), we get

$$(23) \quad P(r) = P(r_0) - \frac{\rho}{2}\left(\frac{\lambda r_0^2}{2r} - \frac{\lambda r}{2}\right)^2 - \frac{\rho}{2}(r_0 - r)^2(r - a)^2 g(r)$$

Therefore, the pressure can not be given as an arbitrary function of r , but it must be expressed in the form of (23). Now, for the sake of simplicity, we consider the case of $g(r) = A^2(\text{const.})$. In this case (18), (19), (20) and (23) become

$$(24) \quad v_r = -A(r_0 - r)(r - a)$$

$$(25) \quad v_\theta = \frac{\lambda r_0^2}{2r} - \frac{\lambda r}{2}$$

$$(26) \quad v_z = A \left\{ \frac{(r_0 - r)(r - a)}{r} + a + r_0 - 2r \right\} z$$

$$(27) \quad P(r) = P(r_0) - \frac{\rho}{2}(v_r^2 + v_\theta^2)$$

We assume $a=70$ km, $r_0=500$ km and $A=10^{-12}$ (c.g.s. unit). As $\frac{dv_r}{dr} = A(2r-a-r_0)$, v_r becomes maximum at $r = \frac{a+r_0}{2} = 285$ km. From (24), we obtain

r	70 km	100	200	285	300	400	500
v_r	0 m/s	-1.2	-3.7	-4.62	-4.6	-3.3	0

As $\frac{dv_\theta}{dr} = -\frac{\lambda r_0^2}{2r^2} - \frac{\lambda}{2} < 0$, the smaller r becomes, the larger v_θ becomes. From (25), we obtain

r	70 km	100	200	300	400	500
v_θ	m/s 127.6	87.5	38.3	19.4	8.2	0

In this table, $v_\theta=127.6$ m/s. is too large to be compared with the observed values. And, in this case, the differential equation for the stream line becomes

$$\frac{dr}{d\theta} = -\frac{2Ar^2(r-a)}{\lambda(r_0+r)}$$

Integrating this

$$\frac{r-a}{r} e^{\frac{ar_0}{r_0+a}} \frac{1}{r} = e^{-\frac{2Aa^2}{\lambda(r_0+a)}(\theta-\theta_0)}$$

where θ_0 is an integrating constant.

Next, we shall calculate the distribution of the vertical velocity v_z . Deforming (26)

$$v_z = -\frac{3A}{r} \left(r - \frac{a+r_0+\sqrt{a^2-ar_0+r_0^2}}{3} \right) \left(r - \frac{a+r_0-\sqrt{a^2-ar_0+r_0^2}}{3} \right) z$$

$$\approx -\frac{3A}{r} (r-34)(r-346)z \quad (\text{unit km})$$

From this equation we know that $v_z=0$ at $r=346$ km. And then

$$\frac{\partial v_z}{\partial r} = -\frac{3A}{r^2} \left(r - \sqrt{\frac{ar_0}{3}} \right) \left(r + \sqrt{\frac{ar_0}{3}} \right)$$

$$\approx -\frac{3A}{r^2} (r-108)(r+108) \quad (\text{unit km})$$

From this equation we see that v_z becomes maximum at $r=108$ km. From the above equation, we get the vertical velocity at $z=1$ km as follows.

r	70 km	100	108	200	300	346	400	500
v_z	4.3 cm/s	4.9	4.92	3.6	1.2	0	-1.5	-4.3

We know therefore that there is an ascending current within the interval $a \leq r < 346$ km and a descending current within the interval $346 \text{ km} < r \leq r_0$.

Lastly we shall consider the distribution of the pressure. Putting

$P(r_0)=1013$ mb and $\rho=0.0012$ g/cm³, if we substitute the values calculated above of v_r and v_θ for those of (27), we get

r	70 km	100	200	300	400	500
P	923.5mb	970.9	1004.9	1010.9	1012.6	1013

From this table, we obtain a figure as shown in Fig.3.

Fig. 3

the eye of typhoon. But a question arises as to the value of a , which may have an arbitrary positive value in our theory, but the observed eye of typhoon is relatively small.

The gradient wind may be considered when vertical current does not exist, but the above solutions may be considered when there may be vertical current. And it is remarkable that the above distribution of the wind and pressure is a special case in which an arbitrary function $g(r)$ is equal to constant A^2 .

4. Descending current within the eye of typhoon

Substituting $r=a$ in the formula (23) we get

$$P(r_0) - P(a) = \frac{\rho}{2} \left(\frac{\lambda r_0^2}{2a} - \frac{\lambda a}{2} \right)^2$$

From this equation, the positive value of a becomes

$$(28) \quad a = \sqrt{r_0^2 + x} - \sqrt{x}$$

in which

$$(29) \quad x = \frac{2(P(r_0) - P(a))}{\lambda^2 \rho}$$

From (28),

$$\frac{\partial a}{\partial x} = \frac{1}{2} \left(\frac{1}{\sqrt{r_0^2 + x}} - \frac{1}{\sqrt{x}} \right) < 0$$

Therefore the larger x becomes, the smaller a becomes. That is, from (29), the larger $P(r_0) - P(a)$ becomes, the smaller the radius of the eye becomes and it becomes larger, as the storm goes to the north. In reality, the eye of storm can distinctly be observed when $P(r_0) - P(a)$ is large, and it becomes indistinct as

The above distribution of the wind and pressure does not necessarily accord with the observed values, but they are not so irrational values. Thus, in the case of circular isobaric system the motion takes place within the interval $a \leq r \leq r_0$ and no wind blows within the circle $r=a$. This area in which no wind blows, may be regarded as a model of

the storm goes to the north. That is, the eye of storm can distinctly be observed only when it is relatively small. Our theory is on this point not perfect.

Next, it may be seen from our theory that horizontal divergence and consequently a descending current may arise within the eye of storm, as the radius $r=a$ becomes larger, and horizontal convergence and consequently an ascending current may arise within the eye, as the radius $r=a$ becomes smaller.

In reality, in a great many storm, $P(r_0)-P(a)$ may become smaller, as they go to the north, and therefore there may be descending currents within the eyes.

5. Conclusion

When the storm moves over the surface of the sea where frictional resistance is smaller than on the land surface, the eye of storm may be distinctly observed. From this point of view, in our theory the differential equations of motion without frictional resistance were considered and we saw that in the case of circular isobaric system, there might be a central area in which no wind blows.

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