Some Characters of the Barotropic Atmosphere with Respect to the Scales of Disturbances

by

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Abstract

The simplified system of dynamic equation is solved by use of the first and the third kind of elliptic functions, to study the non-linear feedback mechanism in the barotropic atmosphere.

These solutions show that there is no shorter period than 5 days in the barotropic variations, unless due to linear effect, that the influences of small-scale disturbances upon the large-scale ones are not so large with respect to the period of variation, that the magnitude of exchange of kinetic energy among disturbances of close scales is larger than the other couple, and that accordingly the contributions of smaller-scale disturbances to the time change of the zonal flow decreases rapidly with the increasing wave number.

It may further be concluded that the period of variation becomes longer the nearer the representative scale approaches the middle scale, and is in inverse proportion to the total kinetic energy. It has been suggested from the above that the domain of wave number less than $m=12$, $n-m=15$ is enough for long-range prediction and time intervals $\Delta t=12$ hours used for integration of equation may be permitted.

1. Introduction

The non-linearity of the equations of motion makes it very difficult to study the behaviour of disturbances in the atmosphere. E. N. Lorenz has recently developed his method to investigate the atmosphere by use of the elementary system of the dynamic equations and studied the non-linear features and stabilities in the barotropic atmosphere.

Now the author extended this method to the case in which the coefficients of these orthogonal functions are the surface harmonic functions, and studied the barotropic characters in the atmosphere for the purpose of minimizing the machine time required for integrating the dynamic equation in the barotropic atmosphere and of throwing some light on the problem of spectral truncation errors.

2. Elementary system of equations

In our case, the dynamic equations governing the atmosphere can be expressed by the vorticity equation
where \( t \) is time, \( \phi \) is a stream function for two-dimensional horizontal flow, \( \mathbf{V} \) is a horizontal differential operator, and \( \mathbf{K} \) is a unit vertical vector.

Equation (1) may be transformed into the following spectral form as stated in the previous report [2]:

\[
\frac{\partial \Psi_{m_1}^{m_2}}{\partial t} = \frac{1}{2} i \Omega \sum_{n_2=m_2+1}^{n_2'} \sum_{n_3=-m_3+1}^{m_3'} \sum_{n_3=-m_3+1}^{m_3'} \Psi_{m_2}^{m_3} \Psi_{m_3}^{m_3} H_{m_1}^{m_2 m_3}
\]

where
\[
\phi = \alpha^2 \Omega \sum_{n=m+1}^{n'} \sum_{m=-m'} \Psi_{m}^{n} Y_{n}^{m},
\]
and \( \alpha = 6,370 \text{ km} \) is the radius of the earth, \( \Omega (= 7.29 \times 10^{-5} \text{ sec}^{-1}) \) is the angular velocity of the earth’s rotation and

\[
Y_{n}^{m} = P_{n}^{m} e^{i m \lambda},
\]
is a surface spherical harmonic of order \( m \) and degree \( n \), with the associated Legendre function of the first kind \( P_{n}^{m} \) and longitude \( \lambda \)

\[
H_{m_1}^{m_2 m_3} = \frac{n_2 (n_2+1) - n_3 (n_3+1)}{n_1 (n_1+1)} L_{m_2 m_3}^{m_1 m_3},
\]

which is called an interaction coefficient. (The readers are referred to the previous paper [2] for further details.)

The term on the right hand side of equation (2) indicates the non-linear effect and can be resolved into the following elementary system of equations:

\[
\begin{align*}
\hat{\phi}_1 &= i \phi_i \phi_i \frac{D_23}{D_1} L, \\
\hat{\phi}_2 &= i \phi_i \phi_i \frac{D_{13}}{D_2} L, \\
\hat{\phi}_3 &= i \phi_i \phi_i \frac{D_{21}}{D_3} L,
\end{align*}
\]

where \( \phi_1, \phi_2, \phi_3 \) denote \( \Psi_{m_1}^{m_2}, \Psi_{m_2}^{m_3}, \Psi_{m_3}^{m_3} \) respectively, (\( \circ \)) denotes the conjugate function of (\( \bullet \)), \( D_1, D_2, D_3 \) are used in place of \( n_1 (n_1+1), n_2 (n_2+1), n_3 (n_3+1) \) respectively, and \( D_{23} = D_2 - D_1, D_{13} = D_1 - D_3, D_{31} = D_3 - D_1 \) and

\[
L = \Omega L_{m_1}^{m_2 m_3}.
\]

In the following treatment one might assume that \( D_1 \geq D_2 \geq D_3 \) and \( m_1 \geq 0, m_2 \geq 0, m_3 \geq 0 \) without loss of generality.

These relations still retain the feature of non-linear effect, i.e., the feed-back
among six or three (with respect to wave number) components of disturbances. Then the mean kinetic energies

$$\sum_j D_j |\phi_j|^2 = E^2,$$

and the mean square of the vorticity,

$$\sum_j D_j |\psi_j|^2 = V^2,$$

are readily seen to be conserved under equations (5), and these are found to express the same relations as those utilized in the study of the motion of the top. Accordingly they can be solved by the use of relations

$$\sum_j |\phi_j|^2 = |\phi|^2,$$

and

$$|\phi_1|^2 = \frac{D_3 D_3}{D_{13} D_{31}} (p_1 - |\phi|^2),$$

$$|\phi_2|^2 = \frac{D_3 D_1}{D_{23} D_{12}} (p_2 - |\phi|^2),$$

$$|\phi_3|^2 = \frac{D_1 D_2}{D_{31} D_{23}} (p_3 - |\phi|^2),$$

with

$$p_1 = \left(\frac{1}{D_2} + \frac{1}{D_3}\right) E^2 - \frac{V^2}{D_2 D_3},$$

$$p_2 = \left(\frac{1}{D_3} + \frac{1}{D_1}\right) E^2 - \frac{V^2}{D_3 D_1},$$

$$p_3 = \left(\frac{1}{D_1} + \frac{1}{D_2}\right) E^2 - \frac{V^2}{D_1 D_2}.$$
as the third equation of integration of the original equations (5), where \( X \) is used in substitution for \( |\phi^2| \) and \( C \) is the integral constant and to be determined from the initial values of stream functions \( \phi_j \)'s.

If one substitute the so-called representative scale \( \lambda (= V^2/E^2) \) into equations (7) and (8),

\[
\begin{align*}
D_2 D_{12} |\phi_1|^2 + D_3 D_{13} |\phi_1|^2 &= E^2 (D_1 - \lambda), \\
D_3 D_{23} |\phi_2|^2 - D_1 D_{23} |\phi_2|^2 &= E^2 (D_2 - \lambda), \\
D_1 D_{13} |\phi_1|^2 + D_2 D_{23} |\phi_2|^2 &= E^2 (\lambda - D_3),
\end{align*}
\]

are given and the left hand sides of the first and third relations of equation (15) are always positive because the order of magnitude satisfies \( D_1 > D_2 > D_3 \), so

\[
D_1 \gg \lambda \gg D_3.
\]

The motion of the special case for \( \lambda = D_1 \), or \( D_3 \) evidently is of permanent character, namely \( |\phi_3|^2 = |\phi_2|^2 = \frac{\partial}{\partial t} |\phi_1|^2 = 0 \) or \( |\phi_1|^2 = |\phi_2|^2 = \frac{\partial}{\partial t} |\phi_3|^2 = 0 \).

Now, let us solve equation (14) by rewriting it into the form

\[
\frac{1}{4} L^2 (X) \cdot X = \varphi (X) \equiv (X - \alpha_1) (X - \alpha_2) (X - \alpha_3).
\]

Fig. 1 indicates the behaviour of the cubic expression \( \varphi (X) \) against \( X \), and \( \alpha_1, \alpha_2, \alpha_3 \) have always real values, because equations (10) require

\[
\begin{align*}
p_1 &< X, \quad p_2 \gg X, \quad p_3 \ll X, \\
(\alpha_2 - X) (X - \alpha_3) &> 0,
\end{align*}
\]

and the value of \( X \) is restricted to the limit from \( p_2 \) to \( p_3 \) or \( p_1 \).

Taking these conditions into consideration, the solutions for equation (14) can be divided into two cases, \( \text{i.e.}, \) one is

a) \( \alpha_2 \gg X \gg \alpha_3 > \alpha_1, \quad X \neq 0 \),

\[
p_{12} \ll 0, \quad p_{23} \gg 0, \quad \text{from} \quad D_1 \gg \lambda \gg D_3.
\]
The text on the page continues as follows:

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(19) \[ t-t_0 = \frac{1}{L \sqrt{a_{21}}} \int \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}, \quad k^2 = \frac{\alpha_{22}}{\alpha_{21}}, \]

or \[ X = \alpha_2 - \alpha_{23} \sin^2[L \sqrt{\alpha_{21}} (t-t_0)], \]

where \( sn, cn, dn \) denote elliptic functions of \( \text{Jacobi} \) with modulus \( k^2 \) and period \( 4K/L \sqrt{\alpha_{21}} \), \( K \) being given by the complete elliptic integral

(20) \[ K = \int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}, \]

and \( X \) adopts the maximum or minimum value at \( t=t_0 \).

From these,

(21) \[
|\phi_1|^2 = \frac{D_2 D_3}{D_{12} D_{21}} \alpha_{21} \left( \frac{p_1}{\alpha_{21}} - d n^2[L \sqrt{\alpha_{21}} (t-t_0)] \right),
\]

\[
|\phi_2|^2 = \frac{D_2 D_1}{D_{23} D_{12}} \alpha_{23} \left( \frac{p_2}{\alpha_{23}} + s n^2[L \sqrt{\alpha_{21}} (t-t_0)] \right),
\]

\[
|\phi_3|^2 = \frac{D_4 D_2}{D_{21} D_{23}} \alpha_{23} \left( \frac{p_3}{\alpha_{23}} - c n^2[L \sqrt{\alpha_{21}} (t-t_0)] \right),
\]

with \( k^2 = \frac{\alpha_{23}}{\alpha_{21}} \), and \( \lambda > D_3 \), because \( 0 < p_{31} = \frac{D_{12} E_2}{D_1 D_2 D_3} (\lambda - D_3) \leq \alpha_{31} \).

And if \( \alpha_1 = p_1, \alpha_2 = p_2, \alpha_3 = p_3 \), equations (21) become

(22) \[
|\phi_1|^2 = \frac{E_2}{D_1 D_{13}} \left( \lambda - D_3 \right) d n^2[L \sqrt{\alpha_{21}} (t-t_0)],
\]

\[
|\phi_2|^2 = \frac{E_2}{D_2 D_{12}} \left( D_1 - \lambda \right) s n^2[L \sqrt{\alpha_{21}} (t-t_0)],
\]

\[
|\phi_3|^2 = \frac{E_2}{D_4 D_{13}} \left( D_1 - \lambda \right) c n^2[L \sqrt{\alpha_{21}} (t-t_0)],
\]

with \( k^2 = \frac{D_{23}}{D_{12}} \frac{D_1 - \lambda}{\lambda - D_3} \),

And, for the next case,

b) \( \alpha_3 > X > \alpha_1 > \alpha_3, \ X_3 \neq 0 \),

\[ t-t_0 = \frac{1}{L \sqrt{a_{23}}} \int \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}, \]

with \( k^2 = \frac{\alpha_{21}}{\alpha_{23}} \),

or \[ X = \alpha_2 - \alpha_{21} \sin^2[L \sqrt{\alpha_{23}} (t-t_0)], \]
And the cases in which the conditions $\alpha_1=\beta_1$, $\alpha_2=\beta_2$, $\alpha_3=\beta_3$ are satisfied, will be the following four cases:

\begin{align}
(25) & \\
\text{i)} \quad & \psi_1 = R, \quad \psi_2 = R, \quad \psi_3 = I, \\
\text{ii)} \quad & \psi_1 = R, \quad \psi_2 = I, \quad \psi_3 = R, \\
\text{iii)} \quad & \psi_1 = I, \quad \psi_2 = R, \quad \psi_3 = R, \\
\text{iv)} \quad & \psi_1 = I, \quad \psi_2 = I, \quad \psi_3 = I,
\end{align}

where $R$ and $I$ mean “real” and “pure imaginary”. These correspond to the minimum cases of LORENZ.

Until now we have derived only the solutions for the time variation of amplitude. Therefore we must derive the formula of time change of the phase angles of disturbances in order to follow the whole process of variations.

If one rewrites the stream function $\psi$ into the formula

\begin{equation}
\psi_j = \frac{1}{2} C_j e^{-i\theta_j}
\end{equation}

with

\begin{align}
\frac{1}{4} C_j^2 &= \bar{\psi}_j \psi_j \equiv |\psi_j|^2, \\
\theta_j &= \frac{1}{2i} \text{Im} (\bar{\psi}_j \psi_j),
\end{align}

where $C_j$ is the amplitude and $\theta_j$ the phase angle of each component, then the time change of phase angle can be expressed as follows:
which can be shown by putting equations (5) into the relation,

\[ \frac{d}{dt} \left( C_j^2 \frac{d\theta_j}{dt} \right) = \frac{1}{2 \chi} \left( \psi_j \bar{\psi}_j - \bar{\psi}_j \psi_j \right), \]

If one takes \( S_1, S_2, S_3 \) as their integration constants, equations (5) can be integrated as follows:

\[ D_j C_j^2 \dot{\theta}_j = D_j C_j^2 \dot{\theta}_2 + D_j C_j^2 \dot{\theta}_3, \]

and

\[ D_j C_j^2 \dot{\theta}_1 = D_j C_j^2 \dot{\theta}_2 + D_j C_j^2 \dot{\theta}_3, \]

or

\[ D_j S_1 = D_j S_2 + D_j S_3, \]

\[ D_j^2 S_1 = D_j^2 S_2 + D_j^2 S_3, \]

namely,

\[ S_2 = \frac{D_1 D_{13} S_1}{D_2 D_{13}}, \]

\[ S_3 = \frac{D_1 D_{12} S_1}{D_2 D_{12}}. \]

Equations (27) can readily be solved if one uses the third kind elliptic function

\[ \pi (u, a) = k^2 \text{sn} \text{cn} \text{dn} \int_0^u \frac{sn^2 \text{udv}}{1 - k^2 sn^2 \text{asnv}}, \]

\[ \dot{\theta}_1 = M_1 \{ u + \pi (u, a_1) \}, \]

\[ \dot{\theta}_2 = M_2 \{ u + \pi (u, a_2) \}, \]

\[ \dot{\theta}_3 = M_3 \{ u + \pi (u, a_3) \}, \]

with

\[ M_1 = - \frac{S_0}{p_1 - a_2}, \]

\[ M_2 = \frac{S_0}{p_2 - a_2}, \]

\[ M_3 = \frac{S_0}{p_3 - a_2}, \]

\[ S_0 = \frac{D_{12} D_{13} S_1}{D_2 D_3}, \]

and \( u = L \sqrt{a_{21}} t, \)

\[ sn^2 a_1 = - \frac{a_{21}}{p_1 - a_2}, \]

\[ sn^2 a_2 = - \frac{a_{21}}{p_2 - a_2}, \]

\[ sn^2 a_3 = - \frac{a_{21}}{p_3 - a_2}, \]
Equations (34) to (36) correspond to the case (a) in the solution for time variation of magnitude $C_j$, and the same results will also be given for the case (b), if

$$\begin{align*}
\ddot{R}^2 &= \alpha_{221}/\alpha_{233}, \\
u &= L \sqrt{\alpha_{22}} t, \\
\sin^2 \alpha_1 &= -\frac{\alpha_{23}}{\rho_1 - \alpha_2}, \\
\sin^2 \alpha_2 &= -\frac{\alpha_{23}}{\rho_2 - \alpha_2}, \\
\sin^2 \alpha_3 &= -\frac{\alpha_{23}}{\rho_1 - \alpha_2},
\end{align*}$$

are taken.

These relations may be used for many purposes, e.g., of evaluating truncation errors due to time extrapolation of the non-linear equation of motion, and non-linear barotropic behaviour of the atmosphere and so on, but we have no detailed numerical table for the third kind of elliptic function, so that it is more convenient to calculate by numerical integration if merely individual computations are required.

3. Various characters of the barotropic disturbances

The non-linear characters of the barotropic disturbances can be observed to be kept still in these simple systems of equations derived hitherto. Therefore, let us hereafter investigate some features in the barotropic atmosphere by use of equations (5) to (36).

a) Stability of the stationary motion

The motion expressed in the system of equations (5) can be stationary only in the case that two components disappear, namely,

$$|\varphi_1|^2 = 0, \quad |\varphi_2|^2 = 0, \quad \text{and} \quad |\varphi_3|^2 = \text{constant} = \alpha,$$

at $t = 0$

Here let us assume that these components of disturbances suffer differential variation $\varepsilon$, and then this stationary motion could be expressed by the equation

$$\begin{align*}
\ddot{\varphi}_1 &= -\frac{D_{23} D_{13}}{D_1 D_2} L^2 \alpha \varphi_1, \\
\ddot{\varphi}_3 &= 0,
\end{align*}$$

if the terms containing $\varepsilon^2$ are neglected. This shows that the stationary motions in our system are always stable, because the coefficient $\frac{D_{23} D_{13}}{D_1 D_2} L^2 \alpha$ is positive, and oscillate around the initial value.

Next, let us consider the semi-stationary system of motion corresponding to the case of $D_1 = D_2$. In this case, the system becomes

$$\dot{\varphi}_1 = i \varphi_2 \frac{D_{23}}{D_1} L,$$
\[
\dot{\psi}_2 = i\psi_1 \tilde{\psi}_3 \frac{D_{12}^3}{D_2} L,
\]
\[
\dot{\psi}_3 = 0,
\]
therefore
\[
\dot{\psi}_1 = -\frac{D_{12}^3 D_{21}^3}{D_1^2 D_2} |\psi_3|^2 L^2 \psi_1,
\]
and this is also a stable oscillatory system.

In fact, these systems cannot become unstable in the conventional meaning of the linear theory, because the total energy is sure to be conserved among the disturbances and merely the existence of a relatively rapid development compared with other systems of disturbances is possible.

\textbf{b) Distributions of kinetic energies}

The right hand sides of equations (21) and (23) can be divided into two parts, one being stationary and the other being dependent upon the time, namely

\begin{align*}
|\psi_1|^2 &= X_1 + Y_1, \\
|\psi_2|^2 &= X_2 + Y_2, \\
|\psi_3|^2 &= X_3 + Y_3,
\end{align*}

with

\begin{align*}
X_1 &= \frac{D_2 D_3}{D_{12} D_{13}} (\alpha_2 - p_1), \\
X_2 &= \frac{D_4 D_1}{D_{23} D_{12}} (p_2 - \alpha_2), \\
X_3 &= \frac{D_1 D_4}{D_{13} D_{23}} (\alpha_3 - p_3),
\end{align*}

and

\begin{align*}
Y_1 &= -\frac{D_3 D_1}{D_{12} D_{13}} \alpha_{23} s n^2 [L \sqrt{\alpha_{21}} (t - t_0)], \\
Y_2 &= \frac{D_2 D_1}{D_{23} D_{12}} \alpha_{23} s n^2 [L \sqrt{\alpha_{21}} (t - t_0)], \\
Y_3 &= -\frac{D_1 D_4}{D_{13} D_{23}} \alpha_{23} s n^2 [L \sqrt{\alpha_{21}} (t - t_0)].
\end{align*}

These relations are used for the case (a), and in the case of (b) the same \(X_1, X_2, X_3\) and

\begin{align*}
Y_1 &= -\frac{D_2 D_3}{D_{12} D_{13}} \alpha_{21} s n^2 [L \sqrt{\alpha_{23}} (t - t_0)], \\
Y_2 &= \frac{D_3}{D_{23} D_{12}} \alpha_{21} s n^2 [L \sqrt{\alpha_{23}} (t - t_0)], \\
Y_3 &= -\frac{D_1 D_4}{D_{13} D_{23}} \alpha_{21} s n^2 [L \sqrt{\alpha_{23}} (t - t_0)],
\end{align*}

should be used instead.
It can be noticed from equations (37) and (38) that 1) the largest and smallest scales of disturbances together form a counterpart to the middle scale disturbance, 2) the ratio of redistribution of kinetic energies is characteristic of that system and independent of $a_{23}$ or $a_{21}$ which is to be determined from the initial values of kinetic energies, and 3) the energy exchange of one component with a disturbance of middle scale takes place in such a manner that the magnitude of exchange is larger for the couple of disturbances of which the difference of scale is smaller as evidently shown by the relation

$$\frac{D_1|\phi_1|^2}{D_2|\phi_2|^2} = \frac{D_{23}}{D_{12}}.$$  

Therefore one can say that energy exchange between disturbances of large-scale and small-scale motions is small in magnitude, and the influence of a couple of two disturbances of small scale upon large-scale ones is small.

Next let us study the magnitude of energy transfer from a certain component of a disturbance of scale $D$ to two definite components of disturbances of scale $D_a$ and $D_b$ ($D_b > D_a$), by use of coefficients of $Y_1$, $Y_2$, $Y_3$ of equations (37) and (38). Here the values $a_{23}$'s and $a_{21}$'s depend upon the initial values of $10_1^1$, $10_2$, $10_3$, and therefore we take $p_{23}$ and $p_{21}$ as the limiting (largest) values of $a_{23}$ and $a_{21}$. Then the maximum values of $Y_1$, $Y_2$, $Y_3$ become

$$D_1Y_1 = -\frac{D_{23}E^2(D_1-\lambda)}{D_{12}D_{13}},$$

$$D_2Y_2 = \frac{E^2(D_1-\lambda)}{D_{12}},$$

$$D_3Y_3 = -\frac{E^2(D_1-\lambda)}{D_{13}},$$

or

$$D_1Y_1 = -\frac{D_{13}D_{23}+E_aD_2D_{12}}{D_{12}D_{13}}D_{23}|\phi_3|^2,$$

$$D_2Y_2 = -\frac{1}{D_{12}}(D_3D_{13}+E_aD_2D_{12})|\phi_2|^2,$$

$$D_3Y_3 = -\frac{1}{D_{13}}\left(\frac{1}{E_a}D_3D_{13}+D_2D_{12}\right)|\phi_3|^2,$$

where

$$E_a = |\phi_2|^2/|\phi_3|^2,$$

for $\lambda > D_2$,

and

$$D_1Y_1 = -\frac{E^2}{D_{13}}(\lambda-D_a),$$

$$D_2Y_2 = \frac{E^2}{D_{23}}(\lambda-D_b),$$

$$D_3Y_3 = -\frac{D_{12}}{D_{13}D_{23}}E^2(\lambda-D_b),$$

or
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Therefore when one considers the influence of a certain small-scale disturbance upon some spectral domain of disturbances, if the wave-number of this disturbance is near to this domain, then such disturbances, \( D_{12} \) being not so large, belong to the groups of \( \lambda < D_2 \) and \( D_{12} Y_1 \sim -2 \frac{D_2}{D_1} \frac{|\phi_1|^2}{|\phi_2|^2} \), and conversely if \( D_{12} \) is large they will belong to the groups of \( \lambda > D_2 \) and \( D_{12} Y_1 \sim 0 \(<1/D_1) \).

Therefore it may be concluded qualitatively from these relations that the spectrum truncation error is confined to the proximity of the end of the spectrum and further that error is slow to be propagated into the domain of larger scale.

As the above conclusion is of importance, we have demonstrated it by two numerical experiments, one being the prediction of the barotropic atmosphere from the standpoint of excluding errors which are concentrate upon the proximity of the end of the spectrum, and the other being performed with the use of a simplified system of barotropic vorticity equation. Namely, Fig. 2 corresponds to the former calculation and shows the correlation coefficients (ordinate) against the number of interaction coefficients used in integration of the barotropic vorticity equation, in which the selection of coefficients was carried out by the magnitude of

\[
D_{12} Y_1 = -\frac{1}{D_{13}} \left( D_2 D_{23} + \frac{1}{E_5} D_1 D_{13} \right) |\phi_2|^2,
\]

\[
D_{2} Y_2 = \frac{1}{D_{23}} \left( E_5 D_2 D_{23} + D_1 D_{13} \right) |\phi_1|^2,
\]

\[
D_{3} Y_3 = -\frac{E_5 D_2 D_{23} + D_1 D_{13}}{D_{12} D_{23}} D_{12} |\phi_1|^2,
\]

where

\[
E_5 = \frac{|\phi_2|^2}{|\phi_1|^2},
\]

and \( \lambda \geq D_2 \) according to

\[
\frac{D_2 D_{23}}{D_1 D_{12}} \leq \frac{|\phi_1|^2}{|\phi_2|^2}.
\]

Therefore when one considers the influence of a certain small-scale disturbance upon some spectral domain of disturbances, if the wave-number of this disturbance is near to this domain, then such disturbances, \( D_{12} \) being not so large, belong to the groups of \( \lambda < D_2 \) and \( D_{12} Y_1 \sim -2 \frac{D_2}{D_1} \frac{|\phi_1|^2}{|\phi_2|^2} \), and conversely if \( D_{12} \) is large they will belong to the groups of \( \lambda > D_2 \) and \( D_{12} Y_1 \sim 0 \(<1/D_1) \).

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\[
\frac{D_{23}}{D_2 D_3 D_1} L,
\]

which is to be given by applying the statistical relation of spectrum distribution to.
equation (5), retaining its property of conservation of mean kinetic energy.

From this figure it will be noticed that the two disturbances of small scale may be neglected and the results are the best for the computation in which about 20,000 interaction coefficients are employed.

One reason for this is supposed to be the various errors accompanying the treatment of small-scale disturbances. However, the real reason is not clear to us.

c) Period of variation of disturbances

It is beyond doubt that as the period of our system is dependent greatly upon the initial values of motion, a perfectly general treatment of this problem is impossible. So, we shall pick up some special cases as limiting cases for their study. It is, then, readily seen from Fig. 1 that the range of variation of modulus $k$ is

$$0 \leq k^2 \leq p_{21} = \frac{D_{23}}{D_{12}} - \frac{\lambda}{D_1} - \frac{\lambda}{D_3},$$  \hspace{1cm} \text{for } \lambda > D_2,$$

and

$$0 \leq k^2 \leq p_{23} = \frac{D_{13}}{D_{23}} - \frac{\lambda}{D_1} - \frac{\lambda}{D_3},$$  \hspace{1cm} \text{for } \lambda < D_2,$$

and in both cases the largest values of the upper limit become unit for $\lambda = D_2$. This indicates that the shortest period of our system can be defined by $\frac{4K(k^2=0)}{L\sqrt{p_{21}}} = \frac{6.2832}{L\sqrt{p_{21}}}$, because $a_{21} < p_{21}$ as can be readily deduced from the characters of cubic expression $\varphi(X)$. Therefore the shortest period of a certain system can be defined by

$$\frac{6.2832}{LE} \sqrt{\frac{D_1 D_2 D_3}{D_{12} D_{23}}},$$

and this does not exceed a certain value $T_0$, which can be evaluated to be about 5.5 days in the actual atmosphere from the following rough estimations:

\begin{align*}
L & = \frac{1}{2} L \Omega \times 2.23 \times 10^{-5} \times 0.864 \times 10^6 \\
LE & \approx 1.5 \\
\min \sqrt{\frac{D_1 D_2 D_3}{D_{12} D_{23}}} & = \min \frac{\sqrt{D_1 D_2 D_3}}{\sqrt{D_1} - \sqrt{D_3}} \approx \sqrt{D_3} \approx \sqrt{2} \approx 1.4.
\end{align*}

In the observed atmosphere such a short period sometimes accompanies the energy exchange between the zonal flow and the eddy components of disturbances, and the period of non-linear exchange of kinetic energy among eddy components may be supposed to be far longer than the above (because $\lambda$ is nearer to $D_2$). To see this fact further from other points of view we have integrated the barotropic vorticity equation (1) or (2) by assuming that a non-linear term is constant over time increment $\Delta t=6$ hours to obtain the analytical solution [Appendix] and compared with the results obtained by the conventional method with the use of time increment $\Delta t=3$ hours. Table 1 indicates the differences between the above two kinds of computations for each component, which seems to verify the conclusion that the period of variation of a non-linear term is relatively long.
1961 Some Characters of the Barotropic Atmosphere with Respect

Table 1. A-COMPONENTS

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*) the first line; initial values,
the second line; forecast values with the assumption of constant non-linear term,
the third line; forecast values with \( dt=3/8 \) hours and later with \( dt=3/4, dt=3/2, dt=3, dt=6 \) hours successively.

Let us here scrutinize into the detailed behaviour of the period, the limiting case of which is to be expressed by the formula, if \( \lambda > D_2 \) or \( D_1 D_{23} |\phi_3| < D_1 D_{12} |\phi_1| \),

\[
T = 4K/L \sqrt{p_{21}} = \frac{4K}{LE} \sqrt{\frac{D_1 D_2 D_3}{D_{12} (\lambda - D_3)}},
\]
with modulus

\[ k^2 = \frac{D_{23}}{D_{12}} \frac{D_1 - \lambda}{\lambda - D_3}, \]

and if \( \lambda < D_2 \) or \( D_2 D_{23} |\phi_3|^2 > D_1 D_{12} |\phi_1|^2 \),

\[ T = \frac{4 K}{L} \sqrt{\frac{D_1 D_2 D_3}{D_{23}(D_1 - \lambda)}}, \]

with modulus

\[ k^2 = \frac{D_{12}}{D_{23}} \frac{\lambda - D_3}{\lambda - D_1}, \]

And these types of period have two sides, one of which is proper to the respective simplified system of motion and can be determined only by the interaction coefficient \( L \) and wave numbers \( D_1, D_2, D_3 \), and the other depends completely upon the initial spectral distribution of kinetic energies in that system. Parameters \( \lambda \) and \( E \) belong to the latter side. The side proper to the system is of immense interest to us, but it is very difficult to treat in general because \( L \) is a complex function of wave number \( D \). Therefore, we shall here take up only the latter side of this problem, and study the nature of the period of variation under the specified parameters \( L, E, D_1, D_2 \) and \( D_3 \).

(case I) Small-scale motion predominates, \( i.e., |\phi_1|^2 \) is very large.

This case belongs to \( \lambda > D_2 \), and from equation (42) it is seen that the higher the degree of predominance of \( |\phi_1|^2 \) is, the nearer the value of \( \lambda \) approaches to \( D_1 \) and the period of variation becomes the shorter. Moreover, that inclination is increased in such a system that the middle-scale \( D_2 \) is located near the smaller-scale \( D_1 \).

(case II) Large-scale motion predominates, \( i.e., |\phi_3|^2 \) is very large.

This case belongs to \( \lambda < D_2 \), and from equation (43) it can be seen that the higher the degree of predominance of \( |\phi_3|^2 \) is, the nearer the value of \( \lambda \) approaches to \( D_3 \) and the period of variation becomes shorter. Moreover, such inclination is increased in such a system that the middle-scale \( D_2 \) is located near the larger-scale \( D_3 \).

To synthesize this, the period of variation becomes longer when a disturbance of middle scale predominates, and the nearer the scale of that middle-scale disturbance to each of other two components, the longer the period of variation becomes.

And the period is proportional to the reverse of total kinetic energy, the reason for which is not clear to us, but is supposed to be the magnification effect of kinetic energy when their spectral balance has been broken.

4. Remarks

Hitherto it has been a subject of discussion whether the origin of periodicity observable in the so-called index cycle phenomena could be found in the barotropic variation in the atmosphere or in the baroclinic variation or else in other forces outside of our system. However, so far as this study has been concerned, it seems
to be impossible to decide ultimately, because it is possible in principle to contain all periods longer than three days even in the barotropic variation. However, it may be concluded from the observed data that the so-called index-cycle can be more reasonably explained by the baroclinic process.

In the present study it has been made pretty clear that the influence of small-scale disturbances upon large-scale ones is small and its period is very long and our system of $m \leq 12$, $n - m \leq 15$ is supposed to be enough for long-range forecasting with our method. One more reason supporting this standpoint is that there seem to be no observations showing the concentration of kinetic energies in the domain of wave numbers higher than $m = 12$, $n - m = 15$, and this system cut off the infinite growth of kinetic energies in the small-scale disturbances due to its feed-back mechanism. Namely, instability in terms of linear theory can not exist in this system.

Finally, the advantageous point stated here that time change of non-linear effect must be tested also for the baroclinic model in the future. However, it seems that this rule also has been undisturbed even in the baroclinic model, which is now under investigation by M. Aihara.

References

Kubota, S., M. Hirose, Y. Kimura and N. Kurihara, 1961: Barotropic forecasting with the use of surface spherical harmonics. (to be published)

Appendix If the quadratic relation with respect to time is assumed for the dynamic equation of the barotropic atmosphere,

$$\begin{align*}
\dot{A} &= R\dot{B} + S_A t + S_{A1} t^2 + S_{A2} t^3, \\
\dot{B} &= -R\dot{A} + S_B t + S_{B1} t^2 + S_{B2} t^3,
\end{align*}$$

the solutions of which are given by

$$\begin{align*}
A &= a_0 + a_1 t + a_2 t^2 + \varepsilon \sin R(t - t_0), \\
B &= b_0 + b_1 t + b_2 t^2 + \varepsilon \cos R(t - t_0),
\end{align*}$$

with

$$\begin{align*}
a_0 &= -\frac{1}{R^2} (2S_{B2} - RS_{A1} - R^2 S_B), \\
b_0 &= \frac{1}{R^2} (2S_{A2} + RS_{B1} - R^2 S_A), \\
a_1 &= \frac{1}{R^2} (2S_{A2} + RS_{B1}), \\
b_1 &= \frac{1}{R^2} (2S_{B2} - RS_{A1}), \\
a_2 &= \frac{1}{R} S_{B2},
\end{align*}$$
\[ b_2 = -\frac{1}{R} S_{A_2} \]

and

\[ \varepsilon = + \sqrt{\alpha^2 + \beta^2} \quad \text{for } \beta > 0 \]
\[ = - \sqrt{\alpha^2 + \beta^2} \quad \text{for } \beta < 0, \]

and

\[ t_0 R_0 = -\frac{\alpha}{\beta} \quad \text{or} \quad \sin R_0 = -\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \]

where

\[ -\varepsilon \sin R_0 = A_0 - a_0 \equiv \alpha, \]
\[ \varepsilon \cos R_0 = B_0 - b_0 \equiv \beta. \]

And if one uses the computed values of non-linear term \( A_{N_1}, B_{N_1} \) at \( t=0 \), \( A_{N_2}, B_{N_2} \) at \( t=\Delta t \), and \( A_{N_3}, B_{N_3} \) at \( t=2\Delta t \), the coefficients in the preceding treatments will be given as follows:

\[ S_A = A_{N_1}, \quad S_B = B_{N_1} \]
\[ S_{A_1} = -\frac{1}{2\Delta t} (3A_A - 4A_{N_2} + A_{N_3}), \]
\[ S_{B_1} = -\frac{1}{2\Delta t} (3B_B - 4B_{N_2} + B_{N_3}), \]
\[ S_{A_2} = \frac{1}{2(\Delta t)^2} (A_A - 2A_{N_2} + A_{N_3}), \]
\[ S_{B_2} = -\frac{1}{2(\Delta t)^2} (B_B - 2B_{N_2} + B_{N_3}). \]

スケールに関連した順圧大気の諸性質

宮 田 重 八

3つないし6つのじょう乱だけから成りたつ順圧大気方程式は第1種および第3種の機能函数を用いて解析的に解くことができる。これから、

1）順圧大気中の変動の週期は最低5日位で安定であるが、一般流とのやりとりからわかる週期は短い。
2）週期はシステムの全エネルギーに逆比例するから、一般流の強いものいききょうは大きい。
3）代表的スケールが最中のスケールに近いほど週期は長く、しかも、その真のじょう乱のスケールが両側のいずれかに片寄っているとその程度は大きい。
4）運動エネルギーのやりとりは、小さいスケールが互に近いと、その間だけのやりとりが大きく、大きいスケールをなめえききょうは小さい。
5）非線型項の時間変動の週期は長い。
6）観測値を使った積分では2万個くらいのinteractionの係数を使ったところで最高の相関係数を示す。小さいスケール同士の寄与がさらにむしろ実際の計算ではエラーの源になっていることを推測させる。