Barotropic Forecasting with the Use of Surface Spherical Harmonic Representations

by

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Abstract

The method to integrate the vorticity equation with the use of surface spherical harmonics has several advantages over the conventional method that uses the grid points. Namely, 1) no relaxation is required to solve the poisson equation, 2) we are not worried by the troubles accompanying the boundaries, 3) it has no such malignant truncation error as can be seen in the grid point method, which was already pointed out by Phillips, 4) square velocity and square vorticity are conserved. So, several points are explained here for preparations for the actual integration of barotropic equation.

First, selection rules of interaction coefficients to determine the combinations of disturbances which contribute to the variation of a certain disturbance and the method to calculate them are investigated. And the method how to expand the meteorological elements into the series of surface spherical harmonics, the individual points necessary for the actual integration are also presented, with one example.

1. Introduction

The spatial truncation errors, accompanying the conventional method of numerical integration of equations of motion or vorticity equation, have been considered to be unavoidable. Meanwhile, the method of expansion of meteorological quantities into the power series of some kind of functions has been considered to be advantageous in that they can avoid those kinds of errors, while how to determine the limit of the wave number of utilized waves has been left to be resolved. The latter question, we suppose, should be discussed in considering the observational results of spectrum distributions of atmospheric disturbances. Concerning this point, the surface spherical harmonic representations are supposed to be advantageous as compared with others, because the convergency of this representation has been found to be superior to others.

2. Calculation of interaction coefficients

We can expand the stream function on a certain isobaric surface into spherical harmonics as follows:

$$(1) \quad \phi = a^2 \Omega \sum_{n_1=|m_1|+1}^{n_1'} \sum_{m_1=-m_1'}^{m_1'} \phi_{n_1}^{m_1} Y_{n_1}^{m_1},$$

where a is the radius of the earth, Ω the angular velocity of the earth's rotation and $\phi_{n_1}^{m_1}$ the harmonic coefficient in complex, Y_n^m is a surface spherical harmonic and is defined by the associated Legendre function of the first kind of order m and degree n ,

$$Y_n^m = e^{im\lambda} P_n^m.$$

Here, P_n^m is normalized to unity,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_n^m P_s^m \cos \varphi d\varphi = \delta_{ns},$$

$$\delta_{ns} = 1 \quad \text{if } n=s$$

$$= 0 \quad \text{if } n \neq s,$$

and satisfies the following relation:

$$P_n^{-m} = (-1)^m P_n^m.$$

In our case, m_1' in (1) is twelve. And $n_1 - m_1$ is an odd integer, as the stream function is assumed to be an odd function with respect to the equator. The maximum of $(n_1 - m_1)$, is 15.

Then, we can obtain the time change of $\phi_{n_1}^{m_1}$ by the following equation, which is derived by substituting (1) into the vorticity equation for the barotropic model:

$$(2) \quad \dot{\phi}_{n_1}^{m_1} = \frac{2i\Omega m_1}{n_1(n_1+1)} \phi_{n_1}^{m_1} + \frac{i\Omega}{2} \sum_{n_2=|m_2|+1}^{n_2'} \sum_{m_2=-m_2'}^{m_2'} \sum_{n_3=|m_3|+1}^{n_3'} \sum_{m_3=-m_3'}^{m_3'} \phi_{n_2}^{m_2} \phi_{n_3}^{m_3} H^{*m_1 m_2 m_3}_{n_1 n_2 n_3},$$

where $n_2 - |m_2|$ and $n_3 - |m_3|$ are odd integers and $m_2' = m_3' = 12$, $n_2' = |m_2| + 15$ and $n_3' = |m_3| + 15$ in our case. $H^{*m_1 m_2 m_3}_{n_1 n_2 n_3}$ is a coefficient which represents to what extent the coupling between $\phi_{n_2}^{m_2}$ and $\phi_{n_3}^{m_3}$ ($m_3 = m_1 - m_2$) contributes to the change of $\phi_{n_1}^{m_1}$. $H^{*m_1 m_2 m_3}_{n_1 n_2 n_3}$ is written as

$$(3) \quad H^{*m_1 m_2 m_3}_{n_1 n_2 n_3} = \frac{n_2(n_2+1) - n_3(n_3+1)}{n_1(n_1+1)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_1}^{m_1} \left\{ m_2 P_{n_2}^{m_2} \frac{\partial P_{n_3}^{m_3}}{\partial \varphi} - m_3 P_{n_3}^{m_3} \frac{\partial P_{n_2}^{m_2}}{\partial \varphi} \right\} d\varphi,$$

and has the relations

$$H^{*m_1 m_2 m_3}_{n_1 n_2 n_3} = H^{*m_1 m_3 m_2}_{n_1 n_3 n_2},$$

$$H^{*m_1 m_2 m_3}_{n_1 n_2 n_3} = 0, \quad \text{if } n_2 = n_3, \text{ or } m_1 \neq m_2 + m_3.$$

Consequently, the summation in (2) can be simplified as follows:

$$(4) \quad \sum_{n_2=|m_2|+1}^{n_2'} \sum_{m_2=-m_2'}^{m_2'} \sum_{n_3=|m_3|+1}^{n_3'} \sum_{m_3=-m_3'}^{m_3'} \phi_{n_2}^{m_2} \phi_{n_3}^{m_3} H^{*m_1 m_2 m_3}_{n_1 n_2 n_3}$$

$$= \sum_{\substack{n_2' \\ n_2 = |m_2| + 1, \\ \text{and } n_2 > n_3}} \sum_{\substack{m_2' \\ m_2 = -m_2'}} \sum_{\substack{n_3' \\ n_3 = |m_3| + 1}} \sum_{\substack{m_3' \\ m_3 = -m_3}} \phi_{n_2}^{m_2} \phi_{n_3}^{m_3} H_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3},$$

where

$$H_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3} = 2H_{n_1 \ n_2 \ n_3}^{*m_1 \ m_2 \ m_3}.$$

Hereafter, $H_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3}$ will be called the interaction coefficients.

In our study, calculation of H was performed by the method given by SILBERMAN. Let

$$(5) \quad L_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_1}^{m_1} \left(m_2 P_{n_2}^{m_2} \frac{\partial P_{n_3}^{m_3}}{\partial \varphi} - m_3 \frac{\partial P_{n_2}^{m_2}}{\partial \varphi} P_{n_3}^{m_3} \right) d\varphi,$$

then,

$$(6) \quad H_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3} = 2 \frac{n_2(n_2+1) - n_3(n_3+1)}{n_1(n_1+1)} L_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3}.$$

In the following, the evaluation of L will be made with the assumption that m_1 , m_2 and m_3 are not negative. This does not lose generality at all, as

$$\begin{aligned} L_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3} &= (-1)^{m_2} L_{n_3 \ n_2 \ n_1}^{m_3 \ |m_2| \ m_1}, & \text{if } m_2 < 0 \\ L_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3} &= (-1)^{m_3} L_{n_2 \ n_1 \ n_3}^{m_2 \ m_1 \ |m_3|}, & \text{if } m_3 < 0. \end{aligned}$$

L can be expressed by the following formula:

$$(7) \quad L_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3} = -E_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3} + E_{n_1 \ n_3 \ n_2}^{m_1 \ m_3 \ m_2},$$

where

$$\begin{aligned} (8) \quad E_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3} &= m_3 [(n_2 + m_2 + 1)(n_2 - m_2)]^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_1}^{m_1} P_{n_2}^{m_2+1} P_{n_3}^{m_3} d\varphi \\ &= m_3 \left[\frac{(2n_2+1)(n_2-m_2)!}{(n_2+m_2)!} \right]^{\frac{1}{2}} \\ &\quad \times \sum_q \left[\frac{(2q+1)(q+m_2)!}{(q-m_2)!} \right]^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_1}^{m_1} P_q^{m_2} P_{n_3}^{m_3} \cos \varphi d\varphi. \end{aligned}$$

In (8), the values of q to be used for summation are

$$q = n_2 - 1, n_2 - 3, \dots, m_2 + 1 \text{ or } m_2.$$

As has been shown by INFELD and HULL (1951), the integral in (8), for the case $m_1 = m_2 + m_3$, becomes

$$\begin{aligned} (9) \quad &\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_{n_1}^{m_1} P_q^{m_2} P_{n_3}^{m_3} \cos \varphi d\varphi \\ &= \frac{(n_3 + q - n_1 - 1)!! [(2n_1+1)(2q+1)(2n_3+1)]^{\frac{1}{2}}}{(n_3 + n_1 - q)!! (q + n_1 - n_3)!! (n_1 + q + n_3 + 1)!!} \end{aligned}$$

$$\times \left[\frac{(n_1+m_1)!(n_1-m_1)!(q-m_2)!(n_3-m_3)!}{2(q+m_2)!(n_3+m_3)!} \right]^{\frac{1}{2}} \\ \times \sum_{h=0}^{n_1-m_1} \frac{(-1)^{(n_3-q+n_1)/2+m_3+h} (n_3+m_3+h)!(q+n_1-m_3-h)!}{(n_1-m_1-h)!h!(n_3-m_3-h)!(q-n_1+m_3+h)!},$$

where $n!! = n(n-2)(n-4)\cdots 2$ or 1 , $0!! = (-1)!! = 1$. This integral vanishes unless

$$(10) \quad \begin{cases} n_1+q+n_3 = \text{even integer} \\ |n_3-q| \leq n_1 \leq n_3+q. \end{cases}$$

By making use of the the above-mentioned formulae and conditions, (5) through (10), the interaction coefficients $H_{n_1 n_2 n_3}^{m_1 m_2 m_3}$ can be calculated.

We used IBM 704 electronic computer for the calculation of H . The ordering of indices and the selection rules which we adopted at the machine computation are listed in Table 1.

Table 1. Ordering of indices and selection rules.

A	B	C
$m_1=0$ (1) 12		
$n_1=m_1+1$ (2) m_1+15		
$m_2=m_1-12$ (1) 12		
$m_3=m_1-m_2$		
$n_3= m_3 +1$ (2) $ m_3 +15$		
$n_2= m_2 +1$ (2) $ m_2 +15$	$n_2 > n_3$	$ n_1-n_3 < n_2 < n_1+n_3$

Column *A* shows the ordering and the conditions on indices. Here, $m=A(B)C$ means that m takes the values in sequence from A to C with the interval of B , i.e. $m=A, A+B, A+2B, \dots, C$. The ordering rule is that the first index, e.g. m_1 in our case, will be varied after all other indices are varied, and so on. As mentioned above, $n_i-|m_i|$ must be an odd integer as the stream function is an odd function with respect to the equator in our case. Of course, m_2+m_3 must be equal to m_1 . The numerical limits for m_i , i.e. $0 \leq |m_i| \leq 12$, and n_i , i.e. $1 \leq n_i-|m_i| \leq 15$ are imposed by us. Columns *B* and *C* show the selection rule. If the indices do not satisfy these conditions, the calculation of H for such indices is unnecessary. Then the next indices in the ordering rule are taken up and it is examined whether they satisfy the selection rule. The condition in Column *B* results directly from the condition of summation on the right hand-side of the formula (4). The condition *C* are derived from the condition (10) and the formulae (7), (8) and (9). From (10), it follows also that $H_{n_1 n_2 n_3}^{m_1 m_2 m_3}$ is zero unless $n_1+n_2+n_3 =$ an odd integer. But this condition is not included in the selection rule, as $(n_1+n_2+n_3)$ is always an odd integer when the indices satisfy the condition in Column *A*.

At the beginning of the machine computation, $\log n$ for $n=1$ (1) 100, $\log(n-1)!$ for $n=1$ (1) 90 and $\log(n-2)!!$ for $n=1$ (1) 120 were computed and stored as the fundamental table. Computations for the formulae (8) and (9) were mostly performed by using this table. At the summation in the right-hand side of (9), $(k+1)$ -th term was obtained by multiplying k -th term, for which $h=k-1$, by

$$(-1, 0) \times \frac{(n_3 + m_3 + k)(n_1 - m_1 - k + 1)(n_3 - m_3 - k + 1)}{(q + n_1 - m_3 - k + 1) \cdot k \cdot (q - n_1 + m_3 + k)},$$

if k -th term was not zero.

As $m_3 = m_1 + m_2$, five indices are dependent variables. So, for each fixed value of m_1 and n_1 , the computed values of $H_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3}$ were temporarily stored in the three-dimensional data field, *i.e.* H , in the machine. The location in which $H_{n_1 \ n_2 \ n_3}^{m_1 \ m_2 \ m_3}$ was stored was $H(N2\ T, N3\ T, M2\ T)$,

where

$$N2\ T = \frac{n_2 - |m_2| + 1}{2},$$

$$N3\ T = \frac{n_3 - |m_3| + 1}{2},$$

and

$$M2\ T = m_2 + 13.$$

$N2\ T$, $N3\ T$ and $M2\ T$ are integers, and have the value between 1 and 8, between 1 and 8 and between 1 and 25, respectively. Therefore, the total number of H -field was 1600. When the indices did not satisfy the selection rule which was mentioned above, the content of the storage for the concerned indices remained clear. And before changing n_1 or m_1 , all 1600 values were written on a magnetic tape as one record. As the range of m_1 was 13, *i.e.* 0 (1) 12, and that of n_1 was 8, *i.e.* $m_1 + 1$ (2) $m_1 + 15$, 104 records in a tape constituted one file of the interaction coefficients. Table 2 shows the number of coefficients which were computed as indices satisfied the selection rule. Total number is 43772.

Table 2. Number of interaction coefficient.

$n_1 - m_1 \backslash m_1$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	0	180	161	288	255	340	298	352	306	344	305	336	301
3	175	335	298	407	358	427	371	411	358	388	342	365	323
5	324	464	410	501	437	490	421	453	394	416	364	379	332
7	446	566	496	569	491	528	454	478	414	428	371	379	329
9	540	640	555	610	519	548	470	486	418	424	364	366	315
11	605	685	586	623	529	550	469	477	406	405	344	341	291
13	640	700	588	617	521	534	451	451	379	372	312	305	258
15	644	684	571	592	495	500	416	409	338	326	269	259	217
total	3,374	4,254	3,665	4,207	3,605	3,917	3,350	3,517	3,013	3,103	2,671	2,730	2,366

grand total=43,772.

The interaction coefficients have been computed by Baer. Some comparisons between his result and ours are represented in Table 3. As a slight difference exists in the normalization factor of P_n^m and so on, our result is multiplied by $1/\sqrt{2}$ for the sake of comparison.

In order to calculate the interaction coefficient, we used the formulae (5) through

Table 3. Some comparisons between Baer's coefficient and ours.

m_1	n_1	m_2	n_2	m_3	n_3	$H_{n_1}^{m_1} H_{n_2}^{m_2} H_{n_3}^{m_3} / \sqrt{2}$	Baer's result
3	4	2	3	1	2	0.67082	0.67082
3	4	3	4	0	3	1.73176	1.73176
7	8	5	8	2	3	-13.53664	-13.53663
7	12	5	10	2	3	-6.58583	-6.58584
14	17	2	19	12	17	-23.31074	-23.31081
16	17	0	19	16	17	-2.74619	-2.74603
16	17	16	19	0	3	23.60126	23.60140

(9). But, it may be obtained also by the numerical integration of the right-hand side of (5). We carried out the test computation for the case of $m_1=0$, $n_1=5$, $m_2=-1$, $n_2=4$, $m_3=1$ and $n_3=2$, by using unnormalized value of P_n^m . The results of numerical integration by Simpson's law with the use of the values at every one degree latitude, two degrees and three degrees were $L_5^0 - L_4^{-1} L_2^1 = 7.2451$, 7.0099 and 7.0955, respectively. While the accurate value is 7.2727.

3. Evaluation of the surface spherical harmonic coefficients

In order to perform the numerical prediction with the use of surface spherical harmonic representation of basic equations, the evaluation of expansion coefficients of the quantities given initially (500 mb height in our case) is required.

The height of 500 mb surface z can be expanded with the surface spherical harmonic series as follows:

$$(11) \quad z = \frac{1}{2} \sum_{n=0}^N A_n^0 P_n^0(\sin \varphi) + \sum_{m=1}^M \sum_{n=m}^{m+N} (A_n^m \cos m\lambda + B_n^m \sin m\lambda) P_n^m(\sin \varphi),$$

where

$$(12) \quad A_n^m = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} z \cos m\lambda P_n^m(\sin \varphi) \cos \varphi d\varphi d\lambda, \quad (m \geq 0)$$

$$(13) \quad B_n^m = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} z \sin m\lambda P_n^m(\sin \varphi) \cos \varphi d\varphi d\lambda, \quad (m \geq 1)$$

λ is longitude, φ is latitude and $P_n^m(\sin \varphi)$ is the normalized associated Legendre function. If it can be assumed that 500 mb height is symmetric with respect to the equator, the harmonic coefficients may be evaluated only from the observed data covering the whole northern hemisphere. In general, however, it is difficult to obtain the data over the whole northern hemisphere, e.g., the lack of the data can be seen usually in the region near the equator. For this, two methods will be considered in order to evaluate the harmonic coefficients. One is the least square method and the other is to estimate the integrals in (12) and (13) after extrapolating the values of height to cover the region having few observations.

In the first place, we tried the evaluation of the harmonic coefficients by the least square method.

If $C_m(\varphi)$ and $S_m(\varphi)$ satisfy the following relations:

$$(14) \quad C_m(\varphi) = \sum_{n=m}^{m+N} A_n^m P_n^m(\sin \varphi) \quad (m \geq 0)$$

and

$$(15) \quad S_m(\varphi) = \sum_{n=m}^{m+N} B_n^m P_n^m(\sin \varphi), \quad (m \geq 1)$$

equation (1) can be written in the form

$$z = \frac{1}{2} C_0(\varphi) + \sum_{m=1}^M [C_m(\varphi) \cos m\lambda + S_m(\varphi) \sin m\lambda].$$

The values of $C_m(\varphi)$'s and $S_m(\varphi)$'s may be obtained by the usual methods of evaluation of Fourier coefficients at every latitude with observed data. Thus the problem is how to determine the coefficients A_n^m 's to minimize the value of F , which is expressed by

$$F = \sum_{i=1}^I \left[\sum_{n=m}^{m+N} A_n^m P_n^m(\sin \varphi_i) - C_m(\varphi_i) \right]^2,$$

φ_i 's denoting the latitudes where the values of height will be read. It is same with coefficients B_n^m 's. This procedure will be equivalent to solving the following matrix equations for A_m and B_m at each m from $m=0$ to $m=M$:

$$(16) \quad E_m A_m = C_m,$$

and

$$(17) \quad E_m B_m = S_m,$$

where

$$A_m = \begin{bmatrix} A_m^m \\ A_{m+1}^m \\ \vdots \\ A_{m+N}^m \end{bmatrix}, \quad B_m = \begin{bmatrix} B_m^m \\ B_{m+2}^m \\ \vdots \\ B_{m+N}^m \end{bmatrix}, \quad C_m = \begin{bmatrix} C_m^m \\ C_{m+1}^m \\ \vdots \\ C_{m+N}^m \end{bmatrix}, \quad S_m = \begin{bmatrix} S_m^m \\ S_{m+1}^m \\ \vdots \\ S_{m+N}^m \end{bmatrix},$$

$$E_m = \begin{bmatrix} E_{m \ m}^m & E_{m+1 \ m}^m & \cdots & E_{m+N \ m}^m \\ E_{m \ m+1}^m & E_{m+1 \ m+1}^m & \cdots & E_{m+N \ m+1}^m \\ \vdots & \vdots & \ddots & \vdots \\ E_{m \ m+N}^m & E_{m+1 \ m+N}^m & \cdots & E_{m+N \ m+N}^m \end{bmatrix},$$

with

$$C_l^m = \sum_{i=1}^I C_m(\varphi_i) P_l^m(\sin \varphi_i),$$

$$S_l^m = \sum_{i=1}^I S_m(\varphi_i) P_l^m(\sin \varphi_i),$$

and

$$E_{nl}^m = \sum_{i=1}^I P_n^m(\sin \varphi_i) P_l^m(\sin \varphi_i).$$

If E_m^{-1} is the inverse matrix of E_m , the solutions of (16) and (17) will be given by

$$A_m = E_m^{-1} C_m,$$

and

$$B_m = E_m^{-1} S_m.$$

When the latitudes where the values of height can be read are fixed, these E_m^{-1} 's will express constant matrices, the elements of which are to be evaluated from the associated Legendre function. Therefore, these matrices can be calculated once for all.

The coefficients A_n^m 's and B_n^m 's of 500 mb height of 14th September 1958 were computed by the above method for $M=18$ and $N=16$ from the values, north of 15°N , read at every 5 degrees of longitude and latitude. The results are shown in Table 4. The coefficients with $n-m$ of an odd integer vanish because of the assumption of symmetry of 500 mb height with respect to the equator. It is seen from the table that the values of coefficients are on the whole large, and increase with the increase of the wave number m , in spite of the expectation of an opposite tendency.

Table 4. The surface spherical harmonic coefficients (A_n^m) computed by the least square method.

$n-m$ m	0	2	4	6	8	10	12	14	16
0	8,127.4	-233.0	-83.6	29.2	-7.0	-11.6	-3.6	9.5	-0.1
1	79.8	-42.1	33.8	-71.3	31.2	-22.2	29.1	-18.8	13.5
2	41.4	12.1	39.3	9.6	5.8	-11.0	7.7	-2.3	0.6
3	23.7	15.6	55.7	8.7	-0.3	-8.2	9.2	2.6	3.0
4	-70.2	59.0	-25.2	57.1	-18.4	31.2	-6.2	9.0	-10.2
5	-20.5	20.2	1.6	20.6	-0.2	12.3	-5.4	0.2	-4.1
6	-14.4	3.7	-27.6	-15.3	-8.6	12.0	5.1	5.5	2.5
7	139.4	-100.6	57.9	-65.2	40.8	-22.9	23.4	-9.4	4.8
8	198.8	-111.7	114.7	-62.6	47.8	-40.5	17.8	-16.8	7.0
9	164.0	-100.2	86.5	-47.2	42.9	-23.8	13.6	-10.9	1.6
10	-272.7	176.3	-119.9	89.0	-57.8	37.8	-24.3	11.5	-6.3
11	-758.9	461.8	-318.6	230.8	-155.7	91.8	-51.6	25.2	-8.8
12	383.5	-228.0	157.1	-112.7	65.1	-45.7	21.4	-10.3	3.7
13	-300.9	179.1	-118.7	80.2	-53.2	30.4	-17.2	6.4	-3.6
14	-138.3	77.1	-47.2	30.9	-14.5	9.3	0.3	-0.9	1.0
15	261.3	-141.4	90.8	-57.1	34.3	-14.4	6.1	-0.6	-0.3
16	-114.4	54.5	-31.0	12.3	-3.2	-3.4	6.2	-4.7	5.5
17	-1,315.0	710.4	-447.1	273.7	-160.4	81.2	-36.9	12.5	-2.8
18	4,085.3	-2,186.1	1,355.0	-825.8	470.9	-244.7	108.4	-39.5	8.6

Then, we considered about the second method, that is, extrapolation and integration. The extrapolation of 500 mb height into the region from 15°N to the equator was made so that on the equator the mean zonal height might take its normal value. In the estimation of the integrals in (12) and (13), Simpson's one third rule was

employed for the same data as used in Table 4 under the same assumption of symmetry. The results, giving reasonable values, are shown in Table 5.

Table 5. The surface spherical harmonic coefficients (A_n^m) computed by the method of extrapolation and integration.

$m \backslash n-m$	0	2	4	6	8	10	12	14	16
0	5,856.5	-241.8	-76.4	22.4	-1.1	-16.6	-0.2	6.8	2.2
1	15.7	14.8	-18.4	-26.2	-8.3	8.4	6.2	-2.8	2.6
2	22.5	27.7	25.8	21.2	-4.5	-4.1	2.5	1.1	-0.8
3	5.9	28.9	44.1	18.5	-7.9	-2.7	5.0	4.1	1.4
4	2.4	5.8	18.7	20.8	9.7	10.6	8.8	-1.0	-7.0
5	0.1	5.7	13.8	11.3	7.6	5.8	-0.7	-2.4	-1.1
6	0.5	-6.1	-20.3	-21.9	-4.6	9.0	7.3	4.7	4.0
7	-1.4	-7.0	-13.8	-10.5	0.1	5.4	4.4	0.5	0.8
8	4.5	14.8	19.1	8.8	-4.1	-4.7	-4.2	-3.7	0.9
9	-0.6	4.0	10.3	8.7	4.5	1.9	-0.9	-3.0	-1.9
10	1.9	5.5	3.0	1.0	2.3	-0.9	-1.7	0.1	-1.4
11	0.1	1.3	5.8	4.1	-3.7	-1.8	2.0	-1.0	-0.5
12	-0.7	0.3	-0.4	-4.7	-5.2	-3.1	-1.1	0.2	-0.4
13	0.8	2.1	1.5	-0.3	-1.4	-1.0	-0.4	-1.1	-1.5
14	0.1	0.7	1.2	1.7	2.0	2.0	1.4	-0.2	-0.1
15	1.7	2.4	-0.2	-0.6	1.7	1.8	-0.5	0.1	0.8
16	-0.2	-1.2	-1.4	-0.8	-1.0	-0.3	1.2	1.5	2.0
17	-1.8	-2.4	-0.4	-0.5	-1.8	-2.4	-0.4	0.7	-0.7
18	0.7	1.8	0.8	-1.3	-0.9	0.5	-1.0	-1.1	0.8

There is little difference between the heights reproduced from the coefficients obtained by those two methods. Therefore, the coefficients evaluated by the least square method are supposed to be suitable for the reproduction of the original field, but their values themselves do not seem to be physically significant. Of course, if the data covering the region from equator to pole are used, the significant coefficients may also be obtained by the least square method.

It is certainly a difficult point to decide how many terms should be taken into account in the series for practical use. However, we tried to see to what extent the height field can be reproduced accurately by composing the wave components assuming the various values of M and N which denote the maximum wave numbers along a latitude circle and between poles respectively. Table 6 shows the root mean square of the difference between the reproduced field and the original one. With the increase of M and N , the root mean square decreases rapidly within small M and N , but become slow with large M and N beyond 12 and 14 respec-

Table 6. The root mean square of difference between original and reproduced fields for the various maximum values of m and n .

$N-M \backslash M$	10	12	14	16
12		20.6		
14	19.5	17.5	16.9	16.4
16		16.3		
18		15.4		

tively. Thus we have determined to adopt the values of $M=12$ and $N=14$ in the following calculations.

4. Geostrophic relation and time interval

To integrate the barotropic equation, we must determine the stream function from the geopotential of 500-mb surface. For this, we have adopted the relations,

$$(18) \quad \begin{aligned} fV^2\phi &= \nabla^2\phi, \\ \phi &= a^2\Omega \sum_{n=|m|+1}^{n'} \sum_{m=-m'}^{m'} \psi_n^m Y_n^m, \\ \phi &= gZ = a^2\Omega^2 \sum_{n=|m|}^{n'} \sum_{m=-m'}^{m'} \Phi_n^m Y_n^m, \end{aligned}$$

which will be found to be equivalent to the equation,

$$\psi_{n+1}^m = \frac{n}{n+2} \left[\frac{4(n+1)^2-1}{(n+1)^2-m^2} \right]^{\frac{1}{2}} \left\{ \frac{1}{2} \Phi_n^m - \frac{n-1}{n+1} \left[\frac{n^2-m^2}{4n^2-1} \right]^{\frac{1}{2}} \psi_{n-1}^m \right\},$$

and the following time scheme is used for the integration:

$$\begin{aligned} \alpha_1 &= \alpha_0 + \frac{1}{4} \delta t \cdot \left(\frac{\partial \alpha}{\partial t} \right)_0, \\ \alpha_2 &= \alpha_0 + \frac{1}{2} \delta t \cdot \left(\frac{\partial \alpha}{\partial t} \right)_1, \\ \alpha_3 &= \alpha_0 + \delta t \cdot \left(\frac{\partial \alpha}{\partial t} \right)_2, \\ \alpha_4 &= \alpha_0 + 2 \delta t \cdot \left(\frac{\partial \alpha}{\partial t} \right)_3, \\ \alpha_{\tau+1} &= \alpha_{\tau-1} + 2 \delta t \cdot \left(\frac{\partial \alpha}{\partial t} \right)_\tau. \end{aligned} \quad \text{for } \tau \geq 4$$

In this case, the number of forward extrapolations at the beginning of the integration may be supposed to be too large, but we dared to adopt the above scheme for $\delta t \equiv 3$ hours.

In this calculation, most of the machine time was found to have been spent for reading and rewinding the interaction functions in the tape. And at the beginning of our work, 70 minutes or more were presumed to be required for one day integration, but it was found later that it can be shortened to about seventeen minutes. The main reasons for this shortening are that 1) the indices to be used for the selection of coefficients are calculated in advance, 2) two tapes storing the interaction functions are used and the simultaneous performance of reading and rewinding has become possible, 3) and at last the necessary of tapes itself became very short by canning four words (one interaction function, three indices necessary for selection of disturbances) into one memory.

5. One example

To investigate the above scheme of calculation, we will present here one result.

Fig. 1. Chart (Z) for 1958, Sept. 14, 500 mb.

Fig. 2. Chart (ψ) for 1958, Sept. 14, 500 mb.

OBSERVED CHANGE (Z)

Fig. 3. 24-hour time change of Z-pattern for 1958, Sept. 14 ~15th.

OBS CHANGES DATE 14

Fig. 4. The same (observed) for ϕ -pattern with Fig. 3.

Fig. 5. Computed 24-hour time change of ψ -pattern.

Namely, we integrated the vorticity equation, assuming the barotropic atmosphere for 500-mb level over the northern hemisphere.

At first, we investigated the degree of resemblance of the pattern expressed by ψ (Fig. 1) used in our method to that by height Z (Fig. 2). From a comparison of these two figures it may be found that the patterns of contours resemble each other very much, except for small discrepancies especially in the lower latitudes.

Next, we compared the observed time changes of these patterns (Figs. 3 and 4), and it was noticed that the general behaviour of those patterns is similar to each other, but the magnitude of changes, as can be imagined easily from the relation (18), is diminished for ψ -pattern in the higher latitudes and exaggerated in the lower latitudes as compared with Z -pattern.

Fig. 5 shows the computed time change of ψ -pattern in twenty-four hours, and we calculated the correlation coefficients between the observed and the computed 24-hour time changes of stream function ψ as tabulated in Table 7, by use of such grid points as shown in Fig. 6.

In this table the numerals indicating the longitude denote the initial longitudes of various areas in which the coefficients were calculated and those areas are indicated by the closed heavy fan-shaped lines* in Figs. 4 and 5. In those calcula-

* These areas are incorrectly displaced towards east by five degree for the west longitude.

Table 7.

INIT. LONG.	CORR. COEF.	STAND. DEV. OBS. VALUES	STAND. DEV. COMP. VALUES
I 110E	0.531 (0.555)	33.435 (45.955)	24.336 (23.099)
II 140W	0.677 (0.723)	38.453 (44.641)	37.805 (31.436)
III 95W	0.671 (0.880)	45.921 (57.344)	35.364 (49.037)
IV 25W	0.685 (0.789)	45.660 (34.563)	31.380 (33.448)
V 60E	0.625 (0.713)	37.762 (41.547)	28.840 (42.118)
VI 40E	0.735 (0.703)	43.799 (43.230)	40.583 (40.672)
VII 150E	0.476 (0.695)	37.015 (48.282)	23.117 (31.605)
VIII 130W	0.617 (0.741)	40.170 (44.747)	36.658 (43.622)
IX 60W	0.724 (0.757)	48.923 (44.584)	37.261 (44.463)

Fig. 6. Grid points used in calculations of correlation coefficients.

tions of correlation coefficients the values north of 20° degree latitude circle are contained, and the values of correlation coefficients can be improved when we use the data north of 30° latitude circle, because our data cover only the area north of 15° latitude circle, as shown by the numerals in parentheses in Table 7.

These values inform us that 1) as is generally expected from the nature of surface spherical harmonic functions, the results are worse in the higher and the lower latitudes because the magnitudes of computed values were far smaller than the observed ones in those areas and relatively good in the middle latitudes, 2) the result near Japan is worst because this calculation could not indicate the deepening of the low north of Japan and the southward displacement of the developing anti-cyclone east of Japan. These, together with the existence of the typhoon south of Formosa, could not explain the actual change in the behaviour of the atmosphere.

Next, let us see the spectral distributions of observed and computed changes in the stream function ϕ , which are shown in Table 8.

Table 8.

(a) Spectral distribution of the observed changes.

$N-M$		1	3	5	7	9	11	13	15
M	A	0	7.87	-11.88	15.79	-20.49	11.23	-4.67	8.53
	B	0	-0	0	-0	0	-0	0	-0
1	A	1.28	0.53	-0.63	10.68	5.97	6.15	-0.30	-8.86
	B	1.96	-6.40	10.73	14.14	-2.47	1.65	-13.93	-0.14
2	A	-5.83	7.25	-1.86	2.84	10.36	-7.28	8.52	0.37
	B	-5.04	-12.09	-3.97	10.06	16.12	13.09	3.66	-4.78
3	A	-12.89	-0.71	4.59	-1.03	7.74	-3.69	0.56	-0.57
	B	9.53	3.61	-0.09	2.09	-1.40	1.54	0.62	7.80
4	A	-13.31	10.28	3.21	-4.29	6.84	-5.73	3.96	-2.92
	B	-2.76	-5.35	3.89	5.13	3.51	6.03	2.58	2.67
5	A	-6.44	4.50	-10.62	-12.70	-2.28	-5.59	4.26	-2.17
	B	-2.28	1.61	3.68	-0.67	2.68	-3.01	-7.55	2.04
6	A	-15.35	7.44	8.39	-0.95	-3.37	-9.70	4.50	-1.87
	B	-4.63	-3.85	-2.86	-0.50	2.67	0.71	1.38	1.29

7	A	-8.92	3.64	5.03	2.43	3.23	-4.41	-0.97	-1.26
	B	2.65	-13.49	-15.96	-7.56	0.42	4.96	-1.62	-1.05
8	A	-1.13	-17.99	-16.16	-6.23	-0.97	4.82	1.90	-0.05
	B	-0.10	13.37	8.22	1.36	0.61	-3.22	0.97	0.60
9	A	7.20	-2.01	-3.79	-1.41	-2.95	-1.41	0.30	3.36
	B	0.84	2.36	2.68	1.06	0.18	-1.00	-1.72	-0.64
10	A	-1.98	-1.83	0.57	-4.20	-3.35	3.25	0.75	-0.67
	B	6.07	5.34	8.13	5.01	-1.15	-1.17	-1.36	1.30
11	A	-3.07	-1.47	-3.11	0.54	2.75	-1.85	-1.20	-0.21
	B	-2.47	-0.11	1.52	1.13	0.77	1.29	0.86	-0.42
12	A	-6.06	1.57	0.26	2.28	5.01	-0.54	-0.43	0.34
	B	-3.54	-0.51	-4.01	-5.54	-2.23	-1.84	-0.47	0.27

(b) Spectral distributions of the computed changes.

$N-M$	1	3	5	7	9	11	13	15
0	-0.00 0.00	-2.11 0.00	2.99 -0.00	2.49 -0.00	1.09 -0.00	0.36 0.00	-0.57 0.00	1.85 0.01
1	-12.29 5.89	-8.43 1.18	0.95 12.36	4.70 8.26	6.12 3.22	1.43 1.54	0.23 -1.98	-3.51 -2.56
2	-20.65 -10.44	0.71 -13.10	-2.27 -4.50	0.56 3.15	0.73 11.11	-2.19 3.70	4.87 4.30	2.84 -0.55
3	-9.87 10.14	-3.46 2.01	3.69 -4.42	0.68 -1.45	-3.48 -0.01	-3.82 1.83	-2.18 1.43	1.01 -0.69
4	0.83 5.61	7.37 10.45	2.44 -0.88	0.07 2.52	-2.37 7.14	-1.04 4.83	2.62 2.24	2.58 0.07
5	1.87 0.37	-2.26 -2.58	-6.23 -2.93	-5.29 2.78	-5.56 2.61	-2.19 -2.58	3.18 -0.99	1.82 1.19
6	-12.57 0.67	-0.56 -10.07	5.48 -7.79	-0.27 -1.47	-5.02 2.60	-3.80 3.00	0.41 2.69	1.97 2.30
7	-3.22 -4.13	0.37 -11.32	2.67 -10.10	0.20 -6.01	-2.50 -0.58	-1.97 2.08	-0.57 -0.25	-0.14 -0.80
8	-1.09 -7.09	-9.40 0.82	-11.32 0.68	-5.03 0.13	-0.80 0.21	3.45 -1.91	3.30 0.07	0.62 -1.15
9	-1.73 5.51	-1.56 4.97	-3.09 7.34	-4.31 5.54	-4.75 -0.08	-1.27 -1.56	2.46 0.09	2.64 1.41
10	-6.04 0.56	-0.51 -3.58	-0.66 4.89	-4.92 4.88	-2.44 0.29	0.26 0.11	0.79 -2.72	-0.11 -0.25
11	5.02 -4.03	-0.38 4.32	0.70 2.51	3.35 2.67	1.72 2.99	1.66 -1.99	1.05 -1.50	-0.24 -1.79
12	0.71 1.33	-2.51 -4.68	3.38 -7.18	6.14 -6.14	2.89 -3.89	2.97 -1.13	0.94 1.55	0.06 0.44

Then it will be noticed that the change in the zonal component can not be computed by this procedure, but the general behaviour of changes in other components of smaller wave numbers can be explained.

We suppose that the fact that the behaviour of the zonal flow can not be computed may be attributed to the variable Coriolis parameter in the conversion of height Z into the stream function ψ . Namely, the error of observations near the equator exerts a great influence over our estimation of the behaviour of the zonal current. On the contrary, the magnitude of computed changes in the stream functions ψ is found to be very little in the lower latitudes. Why?

In these calculations, changes in total kinetic energies expressed by the following relation :

$$\bar{k} = \sum_{n=|m|+1}^{n'} \sum_{m=-m'}^{m'} n(n+1) |\Psi_n^m|^2,$$

could be observed only to a slight degree, amounting only to 0.18% for one day, as shown in Fig. 7.

This suggests that this scheme will prove fruitful for long-range weather forecasting, because it is supposed to be advantageous in avoiding the spatial truncation errors and further troubles accompanying the boundary conditions in the usual grid method. Further, the total kinetic energies were remarkably increased during this period up to $179.66 \text{ m}^2 \text{ sec}^{-1}/\text{cm}$.

Fig. 7. Time change in total kinetic energies.

Appendix

Computation of the Associated Normalized Function

$$P_n^m(x) = \sqrt{\frac{2n+1}{2} \frac{(n+m)!}{4^m (m!)^2 (n-m)!}} (1-x^2)^{\frac{m}{2}} F(\alpha, \beta, \gamma; z),$$

here,

$$F(\alpha, \beta, \gamma; z) = 1 + \sum \frac{\alpha(\alpha+1) \cdots (\alpha+K-1) \beta(\beta+1) \cdots (\beta+K-1)}{K! \gamma(\gamma+1) \cdots (\gamma+K-1)} z^K,$$

$$\alpha = -(n-m),$$

$$\beta = n+m+1,$$

$$\gamma = m+1,$$

$$z = (1-x)/z,$$

$$x = \sin x.$$

The direct computation using the above expansion formula was found to be accompanied with much trouble because the factorial terms are very large in higher orders of P_n^m , which must minimize the precision of the calculated value.

The following is the computation program we used:

i) P_n^m for $m=n$ was calculated from

$$P_m^m = \sqrt{\frac{2m+1}{2m}} \cos x P_{m-1}^{m-1},$$

with

$$P_0^0 = \sqrt{1/2}$$

$$P_1^1 = \sqrt{3/4} \cos x$$

ii) And the recurrence formula,

$$P_{m+1}^m = \sqrt{2m+3} \sin x P_m^m.$$

was used for calculation of P_n^m .

iii) Namely, with P_n^m and P_{m+1}^m that we have got so far, we can compute all values successively from the recurrence formula,

$$P_n^m = \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}} \sin x P_{n-1}^m - \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}} P_{n-2}^m.$$

Further, we didn't use the double precision method in our computation, but the values could be considered reliable as far as the digit of 10^{-6} at least.

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球函数表示を用いた順圧予報

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球函数表示を用いると、格子点による方法に伴うような空間誤差は存在しないので、将来、長期予報に役立つと思う。もちろん、それはそれなりに厄介な問題もあるが、ここでは一応、北半球順圧予報方式の技術的説明、誤差分布からみて妥当と思われる波数の決定、2つの波が干渉して1つの波を作るとき用いられる Interaction Functions の計算法および球函数への分解法を呈出し、あわせて1つの予報例に基づき、 $fP^2\phi = P^2Z$ の変換の問題などを見た。

なお、相関係数は 0.73 程度、全運動エネルギーの時間変動は 0.18% に過ぎなかつた。