Statistical Method of Estimating Isolines

by

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Abstract

In this paper, it is pointed out firstly that the so-called objective method of drawing isolines which is based on the method of least squares can be generalized but that it is disadvantageous in practice. As an alternative, a method based on the simultaneous confidence region of true value at each station is proposed. What has the objective meaning is the confidence region by which the existence region of isolines is provided. Especially, the confidence interval of proportional division point of two adjacent stations can be easily found by a graphical method. A definite set of isolines may be drawn arbitrarily in the restricted limit, by considering several points in each local area successively. The method is developed for two cases; the one concerns the distribution of mean values and the other the directly observed values. In the former the only assumption is that every set of random fluctuations of simultaneously observed values submits to the same multivariate normal probability distribution law, and in the latter the assumption that the covariances of observed values are known must be added.

1. Introduction

Scientific conclusions on the synoptic meteorology must be based on an objective analysis, and in the synoptic weather analysis isolines of various quantities play an important role. As an objective method of drawing isolines, H. A. PANOFSKY [1] adopted the method of least squares. In this paper, after criticizing PANOFSKY's method (Section 2), we shall give an alternative method of estimating isolines.

Let $a=f(\xi, \eta)$ be a function of two independent variables ξ, η . Then a set of isolines which is an intuitional representation of the function is given by

$$(1) \quad f(\xi, \eta) = c_j \quad (j=0, \pm 1, \pm 2, \dots),$$

where c_j ($j=0, \pm 1, \dots$) are constants and usually taken as $c_j = c_0 + j\Delta c$. In research

works, such as the above mentioned drawing of weather chart, we must estimate the function or its isolines by the observations at a finite number of points (ξ_i, η_i) ($i=1, 2, \dots, k$)

$$(2) \quad x_i = a_i + z_i \quad (i=1, 2, \dots, k),$$

where $a_i = f(\xi_i, \eta_i)$ and z_i is the error of observation which is due to measuring instruments, method of observation and other complicated causes.

In the above case we are concerned with directly observed values. However, in mean weather chart and climatic chart, for instance, we treat the distribution of mean values. In this case we may assume that a function $a = f(\xi, \eta; t)$ is a stationary stochastic process with time parameter t and the mean value function to be estimated is that of the process, i. e.

$$\bar{a} = \bar{f}(\xi, \eta) = Ef(\xi, \eta; t).$$

The estimation of a set of isolines

$$(3) \quad \bar{f}(\xi, \eta) = c_j \quad (j=0, \pm 1, \pm 2, \dots)$$

should be done, in this case, by the observations

$$(4) \quad x_{it} = f(\xi_i, \eta_i; t) + z_{it} = \bar{f}(\xi_i, \eta_i) + y_{it} + z_{it}, \quad (t=1, 2, \dots, n)$$

$$(5) \quad \bar{x}_i = \bar{a}_i + z_i^* \quad (i=1, 2, \dots, k),$$

where $\bar{a}_i = \bar{f}(\xi_i, \eta_i)$, $\bar{x}_i = \sum_{t=1}^n x_{it}/n$, $z_i^* = \sum_{t=1}^n y_{it}/n + \sum_{t=1}^n z_{it}/n$ and where y_{it} is the random fluctuation of the process (the real phenomena) itself and z_{it} is the error of observation.

In general, z_i ($i=1, 2, \dots, k$) in (2) as well as z_i^* ($i=1, 2, \dots, k$) in (5) are correlated to each other. In Sections 3 and 4 of this paper we shall treat the case (5) and in Section 5 the case (2).

2. On the so-called objective method of drawing isolines

As an objective method, H. A. PANOFSKY [1] adopted a method of least squares by which a function $a = f(\xi, \eta)$ is directly estimated. In his method, firstly,

I. functional form of $f(\xi, \eta; \theta_1, \theta_2, \dots)$ must be assumed, where θ_i ($i=1, 2, \dots$) are unknown parameters which are to be estimated by the observed data. For instance, let us suppose that we wish to estimate the height of constant pressure surface $f(\xi, \eta; \theta_1, \theta_2, \dots)$. For the radiosonde stations (ξ_i, η_i) ($i=1, 2, \dots, k_0$) and the pilot-balloon stations (ξ'_j, η'_j) ($j=k_0+1, \dots, k$) we put

$$\begin{aligned} f_i &= f(\xi_i, \eta_i; \theta_1, \theta_2, \dots) & i=1, 2, \dots, k_0 \\ &= 0 & i=k_0+1, \dots, k \end{aligned}$$

$$\text{and} \quad f'_j = f(\xi'_j, \eta'_j; \theta_1, \theta_2, \dots) \quad j=k_0+1, \dots, k$$

respectively. Let x_i be the observed height at (ξ_i, η_i) ($i=1, 2, \dots, k_0$) and let

$$x_j = v_j \cdot \nabla f'_j \quad (j=k_0+1, \dots, k),$$

where v_j is the wind velocity vector observed at (ξ'_j, η'_j) . Then the unknown parameters $\theta_1, \theta_2, \dots$ may be estimated, by the method of least squares, so as to minimize the sum

$$S = \sum_{i=1}^k w_i (x_i - f_i)^2,$$

where w_i is the weight of observed value x_i . Thus, in this method, the following second assumption is necessary:

II. The weight w_i 's ($\sum_{i=1}^k w_i = 1$) or the ratios of variances $\sigma_i = \sigma^2 / w_i$ ($i = 1, 2, \dots, k$) are known, where σ^2 is a proportional constant.

In the above formulation as well as in PANOFSKY's method it is assumed that the errors of x_i ($i = 1, 2, \dots, k$) are independent from each other, but if

II'. the ratios of covariances $r_{ij} \sigma_i \sigma_j$ of x_i and x_j ($i, j = 1, \dots, k$) are known, the method of least squares is likewise applicable; that is, the sum which should be minimized is written as

$$S = \sum_{i,j=1}^k w_{ij} (x_i - f_i) (x_j - f_j),$$

where $(w_{ij}) = \sigma^2 (r_{ij} \sigma_i \sigma_j)^{-1}$ (σ^2 is a proportional constant).

Further, even if the position (ξ_i, η_i) of observation has a random error, the method of least squares is also applicable, provided that

I₁. the type of functional relation $a = f(\alpha, \beta; \theta_1, \theta_2, \dots)$ is known, where α and β are the expected values of ξ and η respectively, that is, (α, β) is the true position corresponding to (ξ, η) ,

and that

II₁. for three dimensional variables (x_i, ξ_i, η_i) an assumption similar to that of the above II or II' holds.

If the probability distribution of errors is Gaussian (or normal), the estimation by the above method has a probabilistic meaning. When the functional form is not linear with respect to unknown parameters, the estimation procedures may be carried out by DEMING's method [2]. For the estimation of mean value function $\bar{f}(\xi, \eta)$ a similar formulation holds.

Thus the method of least squares offers a tool of fairly general utility for our purpose. However, a serious disadvantage of this method lies, we think, in the assumption I. For instance, let us consider the case when we take a polynomial

$$a = f(\xi, \eta) = \sum_{i,j} \theta_{ij} \xi^i \eta^j.$$

In order that this function should satisfactorily represent the real state, the order of this polynomial must be so large that $x_i - a_i$ ($i = 1, 2, \dots, k$), where $a_i = f(\xi_i, \eta_i)$, are all considered to be random error, that is, the k -dimensional point $(a_1, \dots, a_k)$ must lie in the confidence ellipsoid with suitably selected confidence coefficient. In general, in order to estimate faithfully the real state of distribution, it may be necessary to assume a functional form $f(\xi, \eta)$ with a fairly large number of unknown parameters the estimation procedure of which requires a great deal of labour and may be almost prohibitable for routine work.

The so-called objective analysis based on the method of least squares may be a sort of point estimation of isolines. We shall propose an alternative method in the

following sections, which may be said to be a method of interval estimation of isolines.

3. Isolines for mean values

If we assign a functional form $f(\xi, \eta)$, the most probable (maximum likelihood) estimate of a set of isolines is uniquely determined. However, in general, there may be an innumerable set of isolines each of which is admissible as a representation of true distribution so long as the information is limited in the observed data. We wish to know the existence region of such admissible isolines.

Firstly, let us consider the distribution of mean value. Let the observed values at the i th station be x_{it} ($t=1, 2, \dots, n$), $i=1, 2, \dots, k$, then the sample mean and the sample variance are

$$\bar{x}_i = \sum_{t=1}^n x_{it}/n \text{ and } s_i^2 = \sum_{t=1}^n (x_{it} - \bar{x}_i)^2/n \quad (i=1, 2, \dots, k)$$

respectively. Now, we may assume about the probability distribution of the random factor $y_{it} + z_{it}$ in (4) as follows.

Assumption: For every t , x_{it} ($i=1, 2, \dots, k$) are independently distributed according to the same k -dimensional normal distribution

$$(6) \quad (2\pi)^{-k/2} A^{-1/2} \exp \left[-\frac{1}{2} \sum_{i,j=1}^k A_{ij} (x_{it} - a_i) (x_{jt} - a_j) \right],$$

where $a_i = f(\xi_i, \eta_i)$ is the expected value (true value) of x_{it} , (A_{ij}) is the inverse matrix of the covariance matrix of x_{it} ($i=1, 2, \dots, k$) and $A = \det(A_{ij})$. (In what follows we need not assume anything about the true value of A_{ij} .)

Then the confidence region for $(a_1, a_2, \dots, a_k)$ with confidence coefficient $1-\alpha$ is given by the interior of hyper-ellipsoid

$$(7) \quad \frac{(n-k)n}{k} \sum_{i,j=1}^k s^{ij} (\bar{x}_i - a_i) (\bar{x}_j - a_j) = F_{n-k}^k(\alpha) \quad (k < n),$$

where

$$(8) \quad (s^{ij}) = (nr_{ij} s_i s_j)^{-1},$$

$$r_{ij} = \frac{1}{n} \sum_{t=1}^n (x_{it} - \bar{x}_i) (x_{jt} - \bar{x}_j) / s_i s_j$$

and $F_{n-k}^k(\alpha)$ is the $100\alpha\%$ point of F -distribution with k and $n-k$ degrees of freedom [3]*. The equation (7) shows that to what extent the differences $\bar{x}_i - a_i$ can be allowed to get, where a_i is the value at the i th station which is obtained by interpolation under a supposed set of isolines. But the formula (7) may not be penetrating and the restriction $k < n$ seems to limit its applicability. In connection with these points, we put

$$(9) \quad u = |\bar{x}_i - a_i| \sqrt{n}/s_i \quad (i=1, 2, \dots, k)$$

and observe the following special cases.

* The formula (7) is easily derived from the distribution of Hotelling's generalized Student ratio. M. MASUYAMA used this formula in 1943.

i) The case where $r_{ij} \equiv r$ ($i, j=1, 2, \dots, k$)*. In this case (3. 2) becomes as follows,

$$(10) \quad \sum_i \frac{(\bar{x}_i - a_i)^2}{s_i^2} - \frac{2r}{1 + (k-2)r} \sum_{i < j} \frac{(\bar{x}_i - a_i)(\bar{x}_j - a_j)}{s_i s_j} = \frac{k(1-r)[1+(k-1)r]}{(n-k)[1+(k-2)r]} F_{n-k}^k(\alpha)$$

Substituting (9) into this equation, we have

$$(11) \quad u^2 = \frac{nk(1-r)[1+(k-1)r]}{(n-k)[k+(k^2-2k+2h)r]} F_{n-k}^k(\alpha),$$

where

$$h = (\text{the number of negative } (\bar{x}_i - a_i)(\bar{x}_j - a_j)) \\ - (\text{the number of positive } (\bar{x}_i - a_i)(\bar{x}_j - a_j)).$$

If $k \rightarrow \infty$ and if then $k/n \rightarrow p < 1$ and $h/k \rightarrow 0$,

$$(12) \quad u^2 \rightarrow \frac{1-r}{1-p}.$$

ii) The case where $r_{ij} = r^{|i-j|}$ ($0 < r < 1$). In this case

$$(13) \quad \frac{(\bar{x}_1 - a_1)^2}{s_1^2} + (1+r^2) \sum_{i=2}^{k-1} \frac{(\bar{x}_i - a_i)^2}{s_i^2} + \frac{(\bar{x}_k - a_k)^2}{s_k^2} \\ - 2r \sum_{i=1}^{k-1} \frac{(\bar{x}_i - a_i)(\bar{x}_{i+1} - a_{i+1})}{s_i s_{i+1}} = \frac{(1-r^2)k}{n-k} F_{n-k}^k(\alpha)$$

and we have

$$(14) \quad u^2 = \frac{nk(1-r)}{(n-k)[k+(k-2)r^2+2hr]} F_{n-k}^k(\alpha).$$

If $k \rightarrow \infty$ and if then $k/n \rightarrow p < 1$ and $h/k \rightarrow q$,

$$(15) \quad u^2 \rightarrow \frac{1-r^2}{(1+r^2+2qr)(1-p)}.$$

Table 1 shows some values of u for (11) and (14). The means of h for the cases of $k=2$ and $k=3$ are illustrated in Fig. 1.

In Table 1 we observe that the value of u does not remarkably decrease as k increases barring the case where h is negative and the correlation r is large. u rather increases when k/n approaches to one, which fact is also seen for large n from (12) and (15). Therefore, we see that, in general, even if we take many points (stations) simultaneously into consideration the precision of the estimation of isolines does not necessarily increase; thus it may be sufficient to consider two or three points in each local area successively. For comparison, if we individually estimate the true value a_i at each point, its confidence limits with confidence coefficient $1-\alpha$ is given by

* This is not a restriction on the sample variables (observed values), but we suppose the case where we got a special sample satisfying this condition.

Table 1. Values of $u = |\bar{x}_i - \bar{a}_i| \sqrt{n/s_i}$ ($i=1, 2, \dots, k$); $1-\alpha=95\%$.(1) The case where $r_{ij} \equiv r$.

$n \backslash k$		$r=0$				$r=0.5$				$r=0.9$			
		2	3	9	$n-5$	2	3	9	$n-5$	2	3	9	$n-5$
$h=1$	10	2.36	2.49	49.09	3.18	1.67	1.84	36.08	2.37	0.75	0.84	16.28	1.08
	30	1.89	1.81	1.84	5.21	1.34	1.34	1.35	3.75	0.61	0.61	0.61	1.68
	50	1.82	1.71	1.61	6.67	1.29	1.28	1.18	4.77	0.57	0.57	0.53	2.13
$h=-1$	10	2.36				2.89				3.26			
	30	1.89				2.32				2.65			
	50	1.82				2.23				2.51			
$h=3$	10			49.09	3.18			35.40	2.21			15.52	0.98
	30			1.84	5.21			1.33	3.74			0.58	1.68
	50			1.61	6.67			1.16	4.76			0.51	2.13
$h=-3$	10		2.49	49.09	3.18		3.53	38.00	2.82		4.25	17.01	1.33
	30		1.81	1.84	5.21		2.56	1.43	3.78		3.09	0.64	1.69
	50		1.73	1.61	6.67		2.42	1.24	4.78		2.91	0.56	2.14

(2) The case where $r_{ij} = r^{|k-j|}$.

$n \backslash k$		$r=0$				$r=0.5$				$r=0.9$			
			3	9	$n-5$		3	9	$n-5$		3	9	$n-5$
$h=0$	10		2.49	49.09	3.18		2.07	38.97	2.56		0.97	17.01	1.15
	30		1.81	1.84	5.21		1.50	1.46	4.07		0.70	0.64	1.72
	50		1.71	1.61	6.67		1.42	1.28	5.19		0.66	0.56	2.18
$h=2$	10		2.49	49.09	3.18		1.64	35.74	2.21		0.69	15.05	0.93
	30		1.81	1.84	5.21		1.19	1.34	3.95		0.50	0.56	1.65
	50		1.71	1.61	6.67		1.12	1.17	5.10		0.47	0.49	2.14
$h=-2$	10		2.49	49.09	3.18		3.35	43.08	3.18		4.10	19.01	1.59
	30		1.81	1.84	5.21		2.43	1.62	4.21		2.98	0.71	1.79
	50		1.71	1.61	6.67		2.29	1.41	5.29		2.81	0.62	2.23

Fig. 1.

The relation between observed values and the corresponding isolines restricted by h . Full line (x_i) is the isoline with the observed value at the i th point. Dotted line (a_i) is the isoline through the i th point.

$$(16) \quad \bar{x}_i \pm \sqrt{\frac{n}{n-1}} t_{n-1}(\alpha) s_i,$$

where $t_{n-1}(\alpha)$ is the positive $\alpha/2$ point of the t -distribution with $n-1$ degrees of freedom. The values of the coefficient $\sqrt{n/(n-1)} \cdot t_{n-1}(\alpha)$ are given in Table 2.

An interval estimation by confidence region does not assign a specially fixed set of isolines, but in our research work we need a set of definitely drawn isolines. A practical procedure of drawing a probable isoline may be formulated as follows:

- i) Plot $\bar{x}_i \pm s_i/\sqrt{n}$ for each point. If possible, it will be desirable to write the correlation coefficient r_{ij} between adjacent points on the line connecting them.
- ii) Draw a trial set of isolines by considering two or three adjacent points and by referring to the interval $\bar{x}_i \pm u s_i/\sqrt{n}$, for successive local regions, where the value of u can be selected in the column $k=2$ or $k=3$ in Table 1 (if we take $1-\alpha=95\%$) according to the type of isolines illustrated in Fig. 1.
- iii) Read the value a_i for the i th point by the above drawn trial set of isolines and test if $|\bar{x}_i - a_i| \leq u s_i/\sqrt{n}$ or not. If not, some correction will be necessary.

Table 3 is an example of the test of propriety of an assumed set of isolines in the western part of Chiba prefecture in Fig. 3.

Table 3. An example of the test of propriety of an assumed set of isolines for the monthly mean temperature for January in the Western part of Chiba prefecture (1916~25).

$n=10, k=3, 1-\alpha=95\%$ and we assumed that $r=0.9$

Station	$\bar{x}_i$ Observed mean value	$\frac{s_i}{\sqrt{n}}$	a_i Read by isolines	$\bar{x}_i - a_i$	$u s_i/\sqrt{n}$	
					The case of $r_{ij} \equiv r$	The case of $r_{ij} = r^{ i-j }$
Anegasaki	3.0°C	0.41	3.3°C	-0.3	0.34	0.40
Yawata	3.3	0.76	3.4	-0.1	0.64	0.74
Chiba	3.8	0.61	3.7	0.1	0.51	0.59
Yawata	3.3	0.76	3.4	-0.1	0.64	0.52
Chiba	3.8	0.61	3.7	0.1	0.51	0.42
Miyako	3.0	0.62	3.5	-0.5	0.52	0.43

If we wish to adopt a more rigorous procedure, we must use the equation (7) itself. For $k=3$, it is reduced to

$$(17) \quad \sum_{i,j=1}^3 a_{ij} t_i t_j = R^2,$$

where $t_i = (\bar{x}_i - a_i) \sqrt{n}/s_i$ and

$$\begin{aligned} a_{11} &= (1 - r_{23}^2)/\Delta, & a_{22} &= (1 - r_{13}^2)/\Delta, & a_{33} &= (1 - r_{12}^2)/\Delta, \\ a_{12} &= (r_{31} r_{32} - r_{12})/\Delta, & a_{23} &= (r_{12} r_{13} - r_{23})/\Delta, & a_{31} &= (r_{23} r_{21} - r_{31})/\Delta, \\ \Delta &= 1 + 2r_{12} r_{23} r_{31} - r_{12}^2 - r_{23}^2 - r_{31}^2, \\ R^2 &= \frac{3n}{n-3} F_{n-3}^3(\alpha) \end{aligned}$$

and if we put $u = |t_i|$ ($i=1, 2, 3$)

$$(18) \quad u = R \left[\sum_{i,j=1}^3 a_{ij} \right]^{-1/2}$$

The foregoing method is comparatively simple but somewhat technical. If it is admitted to draw an isoline by considering only two adjacent points, we can derive a more definite method which will be stated in the following section.

4. Confidence interval of proportional division point

Let the observed value at two stations P_1 and P_2 be x_1 and x_2 respectively. As a simple method of drawing an isoline $f(\xi, \eta) = c$, a linear inverse interpolation (or extrapolation) has been generally used by which the line is so drawn that it passes through the point P_{12} dividing the segment $\overline{P_1 P_2}$ in proportion to the ratio $(c - x_1) : (x_2 - c)$. If the observed values x_1 and x_2 have any random error the division point is also a random variable.

Again let us consider the case of isolines for mean values and derive the confidence interval of the division point under the assumption (6) ($k=2$). Let P_1 be origin and $\overrightarrow{P_1 P_2}$ be axis of abscissa and let $P_1 P_2 = 1$. Since the expected values (true value) of $\bar{x}_1$ and $\bar{x}_2$ are a_1 and a_2 respectively, the abscissa ξ of true division point is obtained from the equation of straight line

$$(19) \quad y = a_1 + (a_2 - a_1) \xi,$$

by putting $y = c$;

$$(20) \quad \xi = (c - a_1)/(a_2 - a_1).$$

Now, according to (7), the confidence region of (a_1, a_2) with confidence coefficient $1 - \alpha$ is given by the ellipse

$$(21) \quad A(a_1 - \bar{x}_1)^2 + 2H(a_1 - \bar{x}_1)(a_2 - \bar{x}_2) + B(a_2 - \bar{x}_2)^2 = R^2,$$

where

$$(22) \quad \left. \begin{aligned} A &= 1/s_1^2, \quad B = 1/s_2^2, \quad H = -r/s_1 s_2, \\ R^2 &= \frac{2(1-r^2)}{n-2} F_{n-2}^2(\alpha), \end{aligned} \right\}$$

provided that r is the sample correlation coefficient between x_{1t} and x_{2t} ($t=1, 2, \dots, n$). When the point (a_1, a_2) moves along the circumference of ellipse (21), the envelope of straight line (19) is obtained as a hyperbola:

$$(23) \quad y = \bar{x}_1 + (\bar{x}_2 - \bar{x}_1) \xi \pm R \sqrt{\frac{A \xi^2 - 2H \xi (1 - \xi) + B(1 - \xi)^2}{AB - H^2}}$$

$$(23') \quad = \bar{x}_1 + (\bar{x}_2 - \bar{x}_1) \xi \pm \sqrt{\frac{2}{n-2} F_{n-2}^2(\alpha) \sqrt{s_2^2 \xi^2 + 2r s_1 s_2 \xi (1 - \xi) + s_1^2 (1 - \xi)^2}}.$$

Using this we can get the confidence interval of ξ as follows.

First Method. Draw the hyperbola (23). Let the intersection of straight line $y=c$ with the hyperbola be (ξ_1, ξ_2) (ξ_1 may be $-\infty$ and ξ_2 may be $+\infty$). Then the (ξ_1, ξ_2) is the required confidence interval.

The abscissas ξ_1 and ξ_2 can be also obtained as follows. Let the point (a_1, a_2) move along the circumference of ellipse (21), then ξ gets its extreme value when

$$(24) \quad \begin{cases} a_1 = \bar{x}_1 \pm R \sqrt{Q^2 / (AQ^2 - 2HPQ + BP^2)}, \\ a_2 = \bar{x}_2 \mp R \sqrt{P^2 / (AQ^2 - 2HPQ + BP^2)}, \end{cases}$$

where

$$(25) \quad \begin{cases} P = A(c - \bar{x}_1) + H(c - \bar{x}_2), \\ Q = H(c - \bar{x}_1) + B(c - \bar{x}_2). \end{cases}$$

Accordingly,

$$(26) \quad (a_2 - \bar{x}_2) / (a_1 - \bar{x}_1) = -\sqrt{P^2} / \sqrt{Q^2}.$$

While, from (21) we have

$$\frac{da_2}{da_1} = -\frac{A(a_1 - \bar{x}_1) + H(a_2 - \bar{x}_2)}{H(a_1 - \bar{x}_1) + B(a_2 - \bar{x}_2)}.$$

Hence,

$$(27) \quad \frac{a_2 - \bar{x}_2}{a_1 - \bar{x}_1} = \left(\frac{da_2}{da_1} \right)_{a_1 = a_2 = c}.$$

Thus we may go through the following procedure:

Second Method: (i) Draw the ellipse (21). (ii) Draw a tangent to the ellipse at the intersection point of the ellipse with the straight line connecting two points (c, c) and $(\bar{x}_1, \bar{x}_2)$. (iii) Let the coordinates of both endpoints of diameter of the ellipse parallel to the tangent be (a_1, a_2) and (a_1', a_2') . (iv) Let the abscissa of intersection of the straight line $y=c$ with another straight line connecting two points $(0, a_1)$ and $(1, a_2)$ be ξ_1 and let the abscissa of intersection of $y=c$ with the straight line connecting $(0, a_1')$ and $(1, a_2')$ be ξ_2 . Then ξ_1 and ξ_2 are the required confidence limits of division point ξ with confidence coefficient $1-\alpha$.

The confidence interval thus obtained will be easily transferred on the straight line in question; for instance, in a mean weather chart, by proportional relation. Two examples are given in Fig. 2. In drawing an isoline $f(\xi, \eta) = c$, we may select, in general, pairs of stations (P_i, P_j) such that they are as near as possible to each other and $\bar{x}_i < c < \bar{x}_j$ or $\bar{x}_i > c > \bar{x}_j$. It is easily seen that, if the two stations P_i and P_j were far apart from each other, the confidence interval of the division point would become very broad and the assumption that the function $f(\xi, \eta)$ is linear in such a wide area could not be admitted.

Fig. 2. Confidence interval of proportional division point.
Confidence coefficient is 80%.

Fig. 3 is an illustration in which the above method was applied to some part (cf. Fig. 2) and to the other part (except several stations) the method stated in the previous section was applied (cf. Table 3). Owing to the shortage of observed years, the reliability of isolines in Fig. 3 is very low. The correlation coefficients of representative pairs of stations in Fig. 3 are shown in Table 4.

When the period of observation varies with stations and the observations are not always simultaneous, to the estimation of isolines for mean values, as a rule, the method for the case of direct observation, which will be stated in the next section, must be applied. In fact, this is done for several stations in Fig. 3, since those stations have some lack of observations.

Fig. 3. Mean temperature distribution for January in Chiba prefecture (1901~1930). Observed period is for most stations 1916~1925. It varies in 1901~1930 with several stations in that period the fluctuation of monthly mean temperature can be considered to have no trend in the district.

Table 4. Spatial correlation of monthly mean temperature for January (1916~25).

Stations (Number of observed years)	Correlation coefficient	Confidence limits with conf. coef. 90%
Tōkyō-Matsudo (8)	0.39	-0.27, 0.78
-Fusa (7)	0.86	0.44, 0.96
-Urayasu (9)	0.84	0.50, 0.95
-Kisarazu (10)	0.85	0.55, 0.94
-Ohara (8)	0.98	0.90, 0.99
Asahi-Fuma (5)	0.76	-0.10, 0.95
Yōkaichiba-Sanrizuka (9)	0.73	0.25, 0.91
Tōgane-Yachimata (10)	0.97	0.89, 0.99
Mobara-Ichinomiya (9)	0.84	0.48, 0.95
Ohara-Tsurumai (8)	0.98	0.90, 0.99

When the periods of observation at two stations P_1 and P_2 are different, let the observed values be

$$\begin{aligned} x_{11}, x_{12}, \dots, x_{1m}, x'_{1m+1}, \dots, x'_{1n_2} & \quad (\text{at } P_1), \\ x_{21}, x_{22}, \dots, x_{2m}, x'_{2m+1}, \dots, x'_{2n_1} & \quad (\text{at } P_2), \end{aligned}$$

where x_{1t} and x_{2t} are simultaneous observations ($t=1, 2, \dots, m$) but there is no temporal correspondence between any x'_{1u} and x'_{2v} ($u=m+1, \dots, n_1$; $v=m+1, \dots, n_2$). Then the correlation coefficient r' between average values $\bar{x}_1$ and $\bar{x}_2$ may be estimated by

$$(28) \quad r' = \frac{m}{\sqrt{n_1 n_2}} r,$$

where r is the correlation coefficient of the pair (x_1, x_2) and is estimated by their sample (x_{1t}, x_{2t}) ($t=1, \dots, m$), provided that x_{1t} ($t=1, 2, \dots$) as well as x_{2t} ($t=1, 2, \dots$) are independent from each other.

5. Isolines for directly observed values

Let x_i ($i=1, 2, \dots, k$) be the observed value of $a_i=f(\xi_i, \eta_i)$ ($i=1, 2, \dots, k$) respectively. In regard to the random errors $z_i=x_i-a_i$ ($i=1, 2, \dots, k$) we assume, firstly, the same assumption as (6);

Assumption I: $(x_1, \dots, x_k)$ is distributed according to the k -dimensional normal probability distribution law

$$(29) \quad (2\pi)^{-k/2} A^{-1/2} \exp\left[-\frac{1}{2} \sum_{i,j=1}^k A_{ij} (x_i - a_i)(x_j - a_j)\right],$$

where $(A_{ij})=(\rho_{ij}\sigma_i\sigma_j)^{-1}$ and $A=\det(A_{ij})$, provided σ_i^2 is the variance of x_i and ρ_{ij} is the correlation coefficient between x_i and x_j . If we add another

Assumption II: σ_i and ρ_{ij} ($i, j=1, 2, \dots, k$) are known,

then the confidence region of $(a_1, \dots, a_k)$ with confidence coefficient $1-\alpha$ is given by the interior of a hyperellipsoid

$$(30) \quad \sum_{i,j=1}^k A_{ij} (a_i - x_i)(a_j - x_j) = \chi_k^2(\alpha),$$

where $\chi_k^2(\alpha)$ is the $100\alpha\%$ point of χ^2 -distribution with k degrees of freedom. The equation (30) corresponds to (7) by a translation

$$(31) \quad s_i/\sqrt{n} \rightarrow \sigma_i, \quad r_{ij} \rightarrow \rho_{ij}, \quad \frac{nk}{n-k} F_{n-k}^k(\alpha) \rightarrow \chi_k^2(\alpha).$$

Therefore, we see that the arguments in sections 3 and 4 hold for the present case also. If we put, in the equation (30),

$$(32) \quad v = |a_i - x_i|/\sigma_i \quad (i=1, 2, \dots, k),$$

we have, as an example, the following Table 5.

We may draw an isoline as we consider that, for several adjacent points, each true value lies, with $100(1-\alpha)\%$ certainty, in the interval

Table 5. Values of $v = |a_i - x_i| / \sigma_i$ ($1 - \alpha = 95\%$)

$k=2, \rho_{12}=\rho$				$k=3, \rho_{ij}=\rho$				$k=3, \rho_{ij}=\rho^{ i-j }$			
$h \backslash \rho$	0	0.5	0.9	$h \backslash \rho$	0	0.5	0.9	$h \backslash \rho$	0	0.5	0.9
1	1.73	1.22	0.55	1	1.61	1.19	0.54	0	1.61	1.34	0.62
-1	1.73	2.12	2.39	-3	1.61	2.28	2.70	2	1.61	1.06	0.45
								-2	1.61	2.17	2.66

$$(33) \quad x_i \pm v \sigma_i,$$

where the value of v may be chosen from such a table as Table 5. The method of test for propriety of an assumed set of isolines and other details are similar to those in the foregoing sections.

When $k=2$, the equation (30) is reduced to

$$(34) \quad A(a_1 - x_1)^2 + 2H(a_1 - x_1)(a_2 - x_2) + B(a_2 - x_2)^2 = R^2,$$

where

$$(35) \quad A = 1/\sigma_1^2, \quad B = 1/\sigma_2^2, \quad H = -\rho/\sigma_1\sigma_2, \quad R^2 = (1 - \rho^2)\chi_2^2(\alpha),$$

provided $\rho = \rho_{12}$. The envelope of the straight line connecting two points $(0, a_1)$ and $(1, a_2)$

$$(36) \quad y = a_1 + (a_2 - a_1)\xi$$

is a hyperbola

$$(37) \quad y = x_1 + (x_2 - x_1)\xi \pm R \sqrt{\frac{A\xi^2 - 2H\xi(1-\xi) + B(1-\xi)^2}{AB - H^2}}$$

$$(37') \quad = x_1 + (x_2 - x_1)\xi \pm \sqrt{\chi_2^2(\alpha)\{\sigma_1^2(1-\xi)^2 + 2\rho\sigma_1\sigma_2\xi(1-\xi) + \sigma_2^2\xi^2\}}.$$

In the case where we take only two adjacent points into consideration, we can geometrically draw the confidence interval of their proportional division point by the ellipse (34) or hyperbola (37).

As an example, let us estimate iso-geopotentialmeter-lines of 500 mb pressure surface at 12 00 (L.M.T.) Feb. 3, 1950, in the neighbourhood of Japan. We might assume that

$$\sigma_1 = \sigma_2 = \sigma = 20 \text{ gpm and } \rho = 0.$$

Then the confidence ellipse of (a_1, a_2) is reduced to a circle

$$(38) \quad (a_1 - x_1)^2 + (a_2 - x_2)^2 = \sigma^2 \chi_2^2(\alpha).$$

Two examples of graphical solution for confidence interval of proportional division point are shown in Fig. 4. (cf. The Second Method in section 4 and Fig. 7. Of course the First Method can be applied also.) A trial set of isolines is drawn with the confidence intervals thus obtained in view. (Fig. 7)

Next, let us test by the observations of wind (at 5 km height), the propriety of the iso-gpm-lines thus drawn. In regard to wind direction we might assume that

Fig. 4. Confidence interval of constant geopotential-meter point on the 500 mb constant pressure surface. Confidence coefficient is 80%.

$$\sigma_1 = \sigma_2 = \sigma = 10 \text{ (degree) and } \rho = 0.$$

Then the hyperbola for this case is reduced to

$$(39) \quad y = x_1 + (x_2 - x_1) \xi \pm R' \sqrt{\xi^2 + (1 - \xi)^2},$$

where $R' = \sqrt{\chi^2_2(\alpha)} \cdot \sigma$. Let the ordinates of the hyperbola at the intersection of an iso-gpm-line with the segment connecting the two adjacent points where the observations of wind are performed be θ_1 and θ_2 . (Fig. 5) If the direction of an iso-gpm-line lies inside of the interval (θ_1, θ_2) , we may regard that the iso-gpm-line is adapted for the wind direction locally. (Fig. 7)

Fig. 5. Confidence region of wind direction on an iso-gpm-line. Confidence coefficient is 80%. ξ is a point on the line through two stations intersected by an iso-gpm-line.

$$\text{Hyperbola: } y = x_1 + (x_2 - x_1) \xi \pm R' \sqrt{\xi^2 + (1 - \xi)^2},$$

$$R' = \sqrt{\chi^2_{\alpha}(\alpha)} \cdot \sigma, \quad \alpha = 20\%, \quad \sigma = 10^\circ.$$

As to the wind velocity, in addition to the assumption $\rho = 0$, we assume that σ_i is equal to the integral part of $(x_i/10)$ m/sec, where x_i (m/sec) is the observed wind velocity at (ξ_i, η_i) . Then the equation of hyperbola concerning two points (ξ_1, η_1) and (ξ_2, η_2) is

$$(40) \quad y = x_1 + (x_2 - x_1) \xi \pm \sqrt{\chi^2_{\alpha}(\alpha)} \cdot \sqrt{\sigma_1^2(1 - \xi)^2 + \sigma_2^2 \xi^2}.$$

By the same method as the above, we can find the confidence limits (v_1, v_2) of true wind velocity v at the middle point of the segment which is the portion of the straight line connecting two stations that lies between two adjacent iso-gpm-lines in question. (Fig. 6) Now, the confidence limits (d_1, d_2) of the distance d of two adjacent iso-gpm-lines are derived from that of wind velocity by the following equation:

Fig. 6. Confidence interval of wind velocity at an intermediate point between two adjacent iso-gpm-lines. Confidence coefficient is 80%. ξ is the middle point of the segment of a line through Sendai and Tateno between two iso-gpm-lines.

Hyperbola:

$$y = x_1 + (x_2 - x_1) \xi \pm \sqrt{\chi^2_{\alpha}(\alpha)} \{ \sigma_1^2(1 - \xi)^2 + \sigma_2^2 \xi^2 \},$$

$$\alpha = 20\%, \quad \sigma_1 = 4 \text{ m/sec}, \quad \sigma_2 = 3 \text{ m/sec}.$$

Fig. 7. Iso-gpm-lines on the 500 mb constant pressure surface over the neighbourhood of Japan. Feb. 3, 1200 (L.M.T.), 1950.

$$(41) \quad d_j = \Delta\phi / (2\omega \sin\phi \cdot v_j) \text{ (meter)} \quad (j=1, 2),$$

where, in our case, $\Delta\phi = 100 \text{ gpm} = 1000 [m^2/\text{sec}^2]$. Diameters of concentric circles in Fig. 7 are d_1 and d_2 . As, in this example, the two iso-gpm-lines pass through the ring-like region between the concentric circles, the distance of the iso-gpm-lines may be regarded to be pertinent, in the local area, to the wind velocity.

In the above method we treated individually the height of constant pressure surface, wind direction and wind velocity. But, in a more rigorous method, they should be simultaneously considered. However, such a method would be extremely complicated since we must then deal with a hyper-ellipsoid as their confidence region.

6. Additional remarks

By means of finite number of observations with random errors, we cannot find definitely the true spatial distribution of a quantity or its isolines. In our method of estimation of isolines, what is essential is the confidence region of a set of true values which is the object of our observation at a finite number of stations. A de-

finite set of isolines, such as shown in Fig. 3 and Fig. 7, which was drawn for the benefit of an intuitional grasp, is only an admissible candidate for the unknown true set. Owing to the finiteness of the number of stations, we cannot find the confidence region of isolines themselves, but admissible isolines may have, in general, a fairly wide sphere and it should be emphasized that the difference between two admissible sets of isolines is not significant. Accordingly, it may be inevitable and admissible that the method of drawing a representative set of isolines depends on a somewhat subjective procedure.

We could reasonably apply the method of least squares if we, in the first place, estimate the confidence region of a set of true values by our method and then we assume a functional form of spatial distribution on the results of the first step. However, it may be also unavoidable that the functional form itself will be subjectively chosen, so long as no regulations are provided.

In our method of proportional division point, we assumed the local linearity of spatial distribution $f(\xi, \eta)$ in regard to the region for which we have no observed data. In the case where we have no information except the observed data, such assumption may be unavoidable. However, if we have a priori some information on the local distribution, we should assume a non-linear function $f(\xi, \eta)$ suitable for the local peculiarity.

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