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Compressed Sensing in Magnetic Resonance Imaging Using the Multi-step Fresnel Domain Band Split Transformation

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Purpose: We present and demonstrate a new compressed sensing (CS) method to improve the image quality obtained in magnetic resonance CS. The sparsifying function, which transforms the image function to sparsified domain, is very important since it controls the quality of reconstructed image. We investigate the utility of a multi-step directional transform for improving the quality of reconstructed images in CS reconstruction.

Methods: As a sparsifying function, we used the Fresnel domain band split transformation (FREBAS), a method to decompose images with highly directional representation and optional scaling of the decomposition. Our image reconstruction algorithm involved linear and nonlinear operations, such as projection onto a convex set and hard thresholding in the transform domain.

Results: Several numerical experiments demonstrated the acquisition of images of better quality using multi-step successive thresholding and different scaling parameters in the FREBAS domain rather than single-step FREBAS thresholding. Reconstruction experiments showed much more detail of the imaging subject with fewer artifacts in CS images based on the FREBAS transform compared to CS based on the wavelet transform.

Conclusion: The proposed method using multi-step FREBAS as the sparsifying transformation function is suitable for CS magnetic resonance imaging.

Keywords: *compressed sensing, fast imaging, FREBAS, L1-norm, sampling theorem*

Introduction

Reduction of acquisition time is a major issue in magnetic resonance (MR) imaging. Compressed sensing (CS), a recently proposed^{1,2} theory applied to MR image reconstruction (CS-MRI),³ states that a sparsely represented signal can be reconstructed from much fewer measurements than previously suggested by the conventional Nyquist sampling theory. Compressed sensing (CS) has recently been proposed^{1,2} for the reconstruction of MR images (CS-MRI)³; the method involves the reconstruction of a sparsely represented signal using many fewer measurements than previously suggested by the conventional Nyquist sampling theory. In general, most MR images do not exhibit exact sparsity, so CS-MRI requires a function that transforms MR images in the space of sparse signal, that is, a spar-

sifying function.

Successful reconstruction from an undersampled MR signal using CS requires satisfaction of 2 criteria—incoherency between the sampling matrix and the basis of the sparsifying transformation function and the sparsity introduced by that function.² Reconstructed images depend greatly on the sparsifying transform function, and most work in CS-MRI has employed the discrete cosine transform or wavelet transform for sparsification.

Use of the wavelet transform offers good sparsity in the transformed domain but may yield less detail by failing to recover edges and curves,^{4,5} so a more effective sparsifying transform is needed. In this paper, we propose adopting the Fresnel domain band split transformation (FREBAS)^{6,7} for sparsification in CS-MRI reconstruction. The 2 different Fresnel transformation algorithms of this transform allow multi-resolution image decomposition with highly directional representation that contributes to superior images using the proposed CS

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reconstruction. FREBAS algorithm, which is consist of 3 fast Fourier transforms and 3 quadratic phase modulations, is more easily calculated than the wavelet transform and is therefore implemented more easily in the CS procedure. In addition, FREBAS allows the choice of an optional scaling parameter in the image decomposition. To improve incoherency between the sampling matrix and the basis of the sparsifying transform function, we applied successive thresholding in the FREBAS-transformed domain using different scaling parameters. In numerical experiments, we found that the proposed CS reconstruction based on the FREBAS transform increases incoherence and outperforms the equivalent reconstruction using the common wavelet transform.

Material and Methods

Signal restoration using compressed sensing

Consider a length, N_u , a real valued column vector ρ . Suppose that we are allowed to take M ($M \ll N_u$) linear measurement of ρ through the following linear transformation:

$$s = \Phi \rho, \quad (1)$$

where s represents the sampled vector and Φ is an $M \times N_u$ measurement matrix. Because $M \ll N_u$, the reconstruction of ρ from s is generally ill-posed. However, the CS theory is based on the fact that ρ has a sparse representation in a known transform domain, Ψ . In other words, the transform-domain signal, $\tilde{\rho} = \Psi \rho$, can be well approximated using only d ($d < M \ll N_u$) nonzero entries. It was proved in [1,2] that when Φ and Ψ are incoherent, ρ can be well recovered from M measurements of the order $d \log N_u$ through nonlinear optimizations

$$\tilde{\rho} = \arg \min \|\tilde{\rho}\|_1 \quad \text{subj. to} \quad s = \Phi \rho (= \Psi \Phi^{-1} \tilde{\rho}), \quad (2)$$

where $\|\cdot\|_1$ indicates the L1 norm, which is defined as $\|y\|_1 = \sum_i |y_i|$. Because s is often contaminated by noise, one needs to minimize $\|\Psi \rho\|_1$ by solving the equation:

$$\tilde{\rho} = \arg \min \|\tilde{\rho}\|_1 \quad \text{subj. to} \quad \|s - \Phi \rho\|_2 < \varepsilon \quad (3)$$

where $\|\cdot\|_2$ indicates the L2 norm defined as $\|y\|_2 = (\sum_i |y_i|^2)^{1/2}$, and the threshold parameter ε is usually set below the expected noise level. Intuitively, the L2 norm heavily penalizes large coefficients, so solutions tend to have many smaller coefficients. On the other hand, in the L1 norm, the penalty tends to be larger with many small coefficients rather than a few large coefficients, so small coefficients are suppressed and solutions are often sparse.

We now describe the details of the process of nonlinear reconstruction appropriate to MR imaging. Let F_u be the undersampled Fourier transform corresponding to the k -space undersampling scheme; then F_u is the measurement matrix.

Minimizing $\|\Psi \rho\|_1$, we use a technique based on projection.⁹ Algorithm of this class form s by successively projecting and thresholding:

$$\tilde{\rho}^{(i)} = \Psi \rho^{(i)} \quad (4)$$

$$\check{\rho}^{(i)} = \tilde{\rho}^{(i)} + \frac{1}{\beta} \Psi F_u^T (s - F_u \rho^{(i)}) \quad (5)$$

$$\tilde{\rho}^{(i+1)} = \begin{cases} \check{\rho}^{(i)} & |\check{\rho}^{(i)}| \geq \tau^{(i)} \\ 0 & \text{else} \end{cases} \quad (6)$$

where F_u^T means an inverse Fourier transform of data that are zero-filled for non-sampled points, β is a coefficient, i is the step of iteration, and $\tau^{(i)}$ is a threshold value that depends on the iteration. As the starting condition of Eq. (4), we used a zero-filled reconstructed image for $\rho^{(0)}$.

We adopted the FREBAS transform^{6,7} as a sparsifying transform function, which we describe in following Section, and we set the parameter, β , in Eq. (5) as 1.0 for all cases.

FREBAS transform

The FREBAS transform consists of 2 different algorithms for the Fresnel transformation, a diffraction equation used with sound or light waves.^{6,7} The first step of the FREBAS transform is the transformation of the input image into a Fresnel signal using a distance parameter; the second is using the Fresnel signal for reconstruction by a different Fresnel transformation algorithm. Because the 2 Fresnel transforms have different scaling properties, the scale of the reconstructed image varies depending on the distance parameter set in the first Fresnel transform. When down-scaling is performed, the alias signal produced when computing the Fresnel transform signal is separated in the reconstructed image domain, and equivalent band splitting of the Fresnel transformed signal is performed in the image domain. This band splitting of the Fresnel transformed signal domain can be interpreted as analysis of the image by convolution of the data with sinc functions having different modulation indices.

Considering the one-dimensional function, $\rho(x)$, for simplicity, having a spatial coordinate, x , a decomposed image of m -th component $\rho_m(x)$ in the FREBAS domain can be described equivalently as a convolution integral with the kernel of a band-pass filter function:

$$\rho_m(x) = [\rho(x)e^{-jcx^2} e^{-j2cmXx}]^* g(x) \quad (7)$$

$$g(x) = \text{sinc}(2cXx) \quad (8)$$

where X is the field of view of the input image; c is a coefficient directly related to the scaling parameter of the FREBAS transform, namely, $D = \pi/(cN\Delta x^2)$; N is the number of image matrices; and m is an index of the higher components of images that correspond to the higher frequency components of the images.

Figure 1 shows examples of FREBAS decomposed images. Figure 1(a) is an original image, (b) is a FREBAS decomposed image with $D=3$, and (c) is a FREBAS decomposed image with $D=5$. The pixel width of the decomposed image becomes $D\Delta x$, and the input images are each divided into D images, each of which has a different band-width in the frequency space. Figure 2 shows the table of basis functions for the case of $D=5$ shown in Fig. 1(c). Each function located in the 5×5 grid gives the real part of a basis function, i.e., “the band-pass filter function” that is used to decompose the image into the FREBAS form. Figure 3 shows the intuitive diagram of FREBAS transform. For simplicity, only the real part of each signal is displayed. Decomposed image components, $\rho(m, x)$, are obtained by convoluting the object function, $\rho(x - mX) \exp\{-jc(x - mX)^2\}$, centered at the position of mX with the sinc function having a phase modulation and the results of convolution is reconstructed in a region that depends on the value of m in the image domain. Because the decomposed image components are located at the space having a period of DX , image components for which $\text{mod}(m/D)$ is equal are reconstructed in the same section as a result of folding effect.

Though the FREBAS transform is similar to the wavelet transform in that it decomposes input images in the image domain, the 2 transforms differ because: (1) the FREBAS transform with scaling

parameter D decomposes the input image into D^2 images having the same scale size, $1/D$; (2) the calculation consists of 3 FFTs and 3 quadratic phase modulations; (3) scaling parameter D can take on not only integer values but also real values; and (4) FREBAS transform is a redundant decomposition. The edges of the input image are well separated radially.

Though FREBAS can be described as a type of convolution integral with a band-pass filter function, the actual calculation executes the convolution in the Fourier domain, so the calculation cost is not so high.

The FREBAS transform consists of 3 Fourier transforms and 3 quadratic phase modulations: (1) Fourier transform (FT); (2) quadratic phase modu-

Fig. 2. Table of basis functions of the (FREBAS) transform for the case of $D=5$. Function location corresponds to the decomposed image location.

Fig. 1. Examples of the (FREBAS) transform: (a) original image; (b) image with scaling parameter $D=3$, and (c) image with scaling parameter $D=5$.

Fig. 3. Intuitive diagram of image decomposition by the (FREBAS) transform.

lation $\exp \{j(D\pi/N)(i_x^2 + i_y^2)\}$; (3) inverse Fourier transform (IFT); (4) quadratic phase modulation $\exp \{j(\pi/(DN))(i_x^2 + i_y^2)\}$; (5) FT; and (6) quadratic phase modulation $\exp \{j(D\pi/N)(i_x^2 + i_y^2)\}$, where i_x and i_y constitute the indices of signal data.

The discrete wavelet transform in the standard dyadic decomposition form is widely used for image compression but is known to be somewhat deficient; specifically, it lacks shift invariance and significant directional selectivity.¹⁰ As shown in Fig. 2, FREBAS has many directional feature-tracking functions, which allow a much higher degree of directional representation than that obtained using traditional wavelet transforms.

Experimental conditions

A 22-year-old healthy male volunteer underwent scanning with a Toshiba 1.5-tesla MR scanner using 3-dimensional (3D) gradient echo sequence and parameters: $\Delta x = \Delta y = 0.8$ mm; repetition time (TR)/echo time (TE), 50 ms/40 ms, flip angle, 20° ; $N = 256 \times 256$ slice; thickness, 1.5 mm $\times$ 50 slices. In the original CS papers,^{1,2} the sampling trajectory in the k -space was completely random to meet

mathematical requirements. Random point sampling in k -space for all dimensions is impractical because hardware and physiological considerations require the sampling trajectories to be smooth. In this paper, we focus on Cartesian grid sampling because it is most widely used. The sampling trajectories are restricted only in terms of the phase-encoding directions and are fully sampled for the read-out direction. Reduction in scanning time is exactly proportional to the undersampling factor. In the numerical experiments, fully sampled echo signal data were calculated, and the echo signals were then randomly selected in the phase-encoding direction to simulate the reduced number sampling.

We set the initial threshold, $\tau^{(0)}$ to the value, $3.5 \sigma_{e(0)}$, where the $\sigma_{e(0)}$ is the standard deviation of reconstruction error due to k -space random undersampling spread over the FREBAS transformed space of $\rho^{(0)}$. We estimated the σ_{error} by calculating the standard deviation of the FREBAS decomposed image located at the corner of the FREBAS transformed space of $\rho^{(0)}$, where the reconstruction error is dominant in that decomposed image and the components of higher frequency of the subject image are contained. The threshold value decreases by a factor of 0.8 as the iteration step increases, this factor value having been determined from a preliminary examination.

Results

The sparsity and selectivity of directional features vary as the FREBAS scaling parameter, D , changes, so we investigated the relationship between the quality of reconstructed images and the D parameter. Figure 4 shows the peak signal-to-noise ratio (PSNR) results using the average of 10 image models as a parameter of the signal reduction factor, which is defined by the ratio of the number of acquired signals to the number of fully scanned signals. PSNR is defined by

$$\text{PSNR} = 20 \log_{10} \left(\frac{\max [\|\rho_{\text{rec}}\|_1]}{\text{MSE}} \right) \quad (11)$$

where ρ_{rec} is the reconstructed image using fully sampled signal and MSE is the root mean square of reconstruction error compared to the fully sampled image. Figure 4 demonstrates the better PSNR obtained when the value of D is 9 for most reduction factors examined.

Next, we took FREBAS multi-step thresholding into consideration to investigate the combinations of D used in 2-step thresholding. The scaling parameter D values used in the first- and second-step FREBAS domain thresholding must be differ-

Fig. 4. Relationship between peak signal-to-noise ratio (PSNR) of the obtained image and the FREBAS scaling parameter, D ; signal reduction factors 20, 25, 30, 35, and 40%.

Fig. 5. Comparison of image peak signal-to-noise ratio (PSNR) as a function of the signal reduction factor as obtained from one-, 2-, and 3-step FREBAS compressed sensing (FREBAS-CS) and wavelet-based CS.

ent to encourage incoherency. Letting the scaling parameter used in the first step be D_1 , in the second step, D_2 , and in the third step, D_3 , we examined

Fig. 6. Relationship between image peak signal-to-noise ratio (PSNR) and the combination of scaling parameters $D(D_1, D_2)$ used in 2-step thresholding in the (FREBAS) domain for the case $D_1=5$; signal reduction factor (a) 25% and (b) 35%.

image quality in multi-step FREBAS thresholding using various combinations of D . Figure 5 shows representative results for the PSNR versus combinations of (D_1, D_2) in 2-step FREBAS thresholding. Figure 6(a) shows the best case for a reduction factor of 25% and 6(b), for a factor of 35%, in which D_1 takes the value 5. The best combination of (D_1, D_2) is (5, 9) for both the 25% and 35% cases, which represents the combination of parameters that show a higher PSNR in one-step FREBAS-CS (Fig. 4). The combination of $D_1=5$ and $D_2=9$ shows the maximum PSNR for other reduction factors as well, including 20, 30, and 40%.

Figure 5 compares one-, 2-, and 3-step FREBAS-CS and wavelet-based CS. Scaling parameters of $D_1=9$ for one-step FREBAS-CS (Fig. 4) and $D_1=5$ and $D_2=9$ for 2-step FREBAS-CS (Fig. 6) exhibited the best performances. The best combination of D_1, D_2 , and D_3 in 3-step FREBAS was found to be (3, 5, 9) by preliminary examinations. For wavelet-

based CS reconstruction, 4-level dyadic decomposition with the Daubechies 4 wavelet was used.

Figure 7 shows images obtained for reduction factors of 35, 30, 25, and 20% in the 2-step FREBAS-CS and wavelet-based CS. Figure 7(a) shows

an example of a sampling trajectory for a reduction factor of 35%. The phase encoding direction is set to the vertical direction, and the sampling trajectory is shown as white lines. Figure 7(b) shows the fully scanned image; 7(c), the corresponding close-

Fig. 7. Comparison of compressed sensing (CS) images: (a) an example of a sampling trajectory for acquiring 35% signal; (b) fully scanned image; (c) close-up images of (b); (d) reconstructed image using a simple Fourier transform; (e)–(h) obtained images in 2-step (FREBAS)-CS; (i)–(l) close-ups of images (e)–(h) (i:e, j:f, k:g, l:h); (m)–(p) obtained images in wavelet-CS; (q)–(t) close-ups of images (m)–(p) (q:m, r:n, s:o, t:p).

Fig. 8. Point-spread function (PSF) after sparsified domain thresholding: (a) point image, (b) PSF after k-space undersampling, (c) PSF after one-step (FREBAS) domain thresholding, (d) PSF after 2-step FREBAS domain thresholding, and (e) PSFT after wavelet domain thresholding.

up image; 7(d), the reconstructed image using a simple Fourier transform. Figures 7(e) to (l) are images obtained in 2-step FREBAS-CS and Fig. 7(m) to (t), images obtained in wavelet-based CS. Close-ups of reconstructed images are shown under each full-size image.

The CS approach requires that the aliasing artifacts due to k -space undersampling be incoherent (noise-like) in the transformed domain. Consideration of a point-spread function (PSF) is helpful for understanding the behavior of the CS algorithm and measuring incoherence. Let e be a point image (having one at some pixel and zeros elsewhere). The point image was Fourier transformed to calculate the signal in k -space, and the k -space signal was then randomly sampled (Fig. 8[a]). The point image (PSF) after inverse Fourier transform is shown in Fig. 8(b). In the example shown, the result resembles random noise added to the point image. Figures 8 show the PSF after transformed domain thresholding in (c) one-step FREBAS-CS ($D=5$);

(d) 2-step FREBAS-CS ($D=5$ and 9); and (e) wavelet-based CS. The threshold value is taken as 1.5% (0.015) of the maximum amplitude of the point image. The noise-like reconstruction error around the point image becomes smaller after transformed domain thresholding in all cases; however, FREBAS domain thresholding presents better results, especially in 2-step FREBAS thresholding, in which most reconstruction error is removed.

Discussion

Figure 5 demonstrates the higher PSNR with FREBAS-based CS than wavelet-based CS. SNR gain is significant in 2-step compared to one-step FREBAS-CS, but small in 3-step compared to 2-step FREBAS-CS. Figure 5 confirmed that 2-step FREBAS-CS is the best choice for multi-step FREBAS-CS in terms of PSNR and calculation cost. The use of multi-step FREBAS domain threshold-

ing can provide higher PSNR images than wavelet-based CS. This gain in PSNR compared to that of wavelet-based CS was introduced from the property that multi-step FREBAS domain thresholding improve the incoherency between the sparsifying transform function and the measurement matrix, F_u . Figure 8 demonstrates the improved incoherency between the measurement matrix, F_u , and the FREBAS transform as a sparsifying transform function. Violation of Nyquist criteria causes artifacts in the linear reconstruction. For example, equispaced sampling results in coherent folding of images and irregular (random) sampling results in incoherent aliasing. Incoherent aliasing caused by random sampling resembles random noise that spreads over the image space. Figure 8(b) shows an example of incoherent aliasing superimposed over the point image. The point image after single FREBAS domain thresholding using $D_1=5$ presents less error than with wavelet domain thresholding. These differences in noise reduction depend on the image decomposition algorithm in

the sparsifying transformed domain. The point image with reconstruction error is decomposed into D^2 smaller images in the FREBAS transform, whereas the input image is down-scaled in a dyadic manner for increases in the decomposition level in the case of the wavelet transform. The noise-like reconstruction error spreads across $D \times N$ pixels in the FREBAS transform domain (Fig. 9[c]) and across $(11/4)N (= 3 \cdot N/2 + 3 \cdot N/2^2 + 4 \cdot N/2^3)$ pixels in the 3-level wavelet transform domain (Fig. 9[d]). When D becomes larger than 3, the noise-like reconstruction error extends over a much greater area (line width) in the FREBAS domain compared to the wavelet transform domain; therefore, the standard deviation of the reconstruction error decreases as the scaling parameter D increases in the FREBAS transformed domain. This is why the reconstruction error around the point image decreases after one-step FREBAS domain thresholding relative to that of wavelet domain thresholding. As the parameter, D , becomes much larger, the FREBAS transform approaches the Fourier trans-

Fig. 9. Comparison of reconstruction error distribution in the sparsified domain: (a) point image in 2-dimensional space, (b) point-spread function (PSF) after k-space undersampling, (c) (FREBAS) transform of PSF image (b), and (d) wavelet transform of PSF image (b).

form and the reconstruction error extends much further in k -space; however, the sparsity in the FREBAS domain decreases. Therefore, there is an adequate value of the FREBAS scaling parameter, and it is nine (Fig. 3).

As the FREBAS scaling parameter changes, the basis function of FREBAS also changes, and the distribution of the reconstruction error varies in the FREBAS domain. Therefore, the reconstruction error can be further reduced by an additional application of FREBAS domain thresholding using a second scaling parameter, D_2 . Strictly speaking, the best FREBAS scaling parameter depends on both the sampling trajectory and the object function imaged because the reconstructed images initially obtained by applying inverse Fourier transform to the acquired signal can be described as the convolution integral of the object function and the modulation transfer function of the sampling trajectory (PSF in Fig. 7[b]). Therefore, the actual best scaling parameter may differ from the value estimated in this paper. However, image quality differs little between the case in which the combination of scaling parameters 5 and 9 is used and the case in which the actual best combination of scaling parameters for the object function and sampling trajectory is used. Therefore, we can expect that the combination of scaling parameters, 5 and 9, can yield almost the best PSNR in CS reconstruction.

Two-step FREBAS-CS offers much greater subject detail than wavelet-based CS for the same amount of signal (Fig. 6). Comparing the area indicated by the A arrow in Fig. 6(c) among CS images, FREBAS-based images have good visual quality and present better directional features compared to wavelet-based CS. Images obtained using wavelet-based CS tend to suffer from oversmoothing and loss of subject detail. The directional selectivity of the wavelet transform involves only 4 components, (high, high), (high, low), (low, high) and (low, low), whereas FREBAS has many directional feature-tracking functions (Fig. 2). This higher degree of directional representation contributes to the superior images in the proposed CS reconstruction.

The proposed algorithm based on iterative thresholding in the FREBAS domain inherits the fast execution speed of the projection-based CS reconstruction. The execution time for 100 iterations is 12.1 s for a single image reconstruction using a 3.06-GHz Intel® Core™ i7 950 processor (California, USA). We have shown that reconstruction time can be drastically reduced to 1.3 s using the general purpose graphic processing unit (GPGPU) NVIDIA® GeForce GTX-580 (California,

USA). The reconstruction time would be further shortened by improving the reconstruction algorithm or upgrading the GPU.

Conclusion

We examined the application of the FREBAS transform as a directional transform for CS in the application of CS to MR imaging and confirmed that 2 characteristics of the FREBAS transform contribute to superior image quality compared to that with standard wavelet-based CS. One is the highly directional representation of the FREBAS transform, which is excellent in extracting image features, and the second is the successive application of FREBAS domain thresholding, which encourages the incoherence of the sampling matrix and the basis of the sparsifying transform function. Reconstruction experiments show that the images obtained for FREBAS-CS demonstrate much more detail of the imaging subject with fewer artifacts than those obtained using CS based on the wavelet transform. These results indicate that the proposed CS using the multi-step FREBAS as the sparsifying transform function is suitable to CS MR imaging. Future work includes the application of this method to actual MR imaging signals with phase variations.

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