Flow Pattern Formation in a Two-Dimensional Flow on the Rotating Hemisphere Bounded by the Meridional Line

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The time evolution of a flow field on a rotating hemisphere bounded by the meridional line is investigated by a numerical model with the two-dimensional incompressible Navier-Stokes equations. We observed a westward propagation of flow patterns, taking a smooth initial condition expressed by streamlines of coaxial circles. The westward flow pattern may be relevant to Stommel’s westward intensification on the β-plane.

1. INTRODUCTION

The two-dimensional (2D) incompressible Navier-Stokes flow on a rotating sphere has been extensively studied. Williams (1978)\(^1\) performed a numerical experiment of 2D turbulence on a rotating sphere with a stochastic vorticity forcing, and reported that a band structure as found on Jupiter emerges in the flow field in the course of time development. However, his numerical experiments assumed a strong symmetry of the flow in order to reduce memory and CPU time. Therefore, it was left unsolved whether the band structure really emerges under no assumption of the flow symmetry. Later in 1993, Yoden and Yamada\(^2\) studied the 2D decaying turbulence on a rotating sphere numerically with no assumption of the symmetry. They reported that there occur westward circumpolar jets in the polar regions and a band structure in mid-latitude region, when the rotation rate of the sphere is sufficiently large. Nozawa and Yoden\(^3\) showed in 1997 that the band structure is also observed when a stochastic vorticity forcing is applied to the flow field. Recently, Ishioka, Yamada, Hayashi and Yoden\(^4\),\(^5\),\(^6\), employing a variety of initial conditions, showed that the emergence of the westward circumpolar jets in polar regions is commonly observed in the 2D non-divergent decaying turbulence, if the rotation rate of the sphere is sufficiently large.

On the other hand, time development of the 2D turbulence on a rotating sphere has not been well-understood when the flow region is restricted to a part of the sphere with a rigid boundary. Goldsborough (1933) discussed the free linear oscillations of a shallow water in a partial area on a rotating sphere\(^7\). The configuration may remind us of the ocean dynamics, where we observe, for example, the westward intensification of ocean currents which has no counterpart in 2D flows on a full rotating sphere. In this paper we report a
Fig. 1: The flow domain is a hemisphere (non-hatched one) the boundary of which is orthogonal to the equator. A domain of oblique line is the flow region. The rotating rate is $\Omega$.

numerical experiment of a 2D decaying non-divergent flow on a rotating hemisphere with the boundary along the meridional line being orthogonal to the equator (Fig. 1). We take this flow configuration, because the circumpolar jets appearing commonly in the polar regions are inhibited in this case, and a fundamentally different flow pattern is expected to be observed.

2. NUMERICAL PROCEDURE

Our numerical procedure consists of a stereographic projection of the hemisphere to a plane disk, and an application of Fourier-Chebyshev expansion method. The flow region is then mapped onto a unit circular plane disk (Fig. 2), where the equator is mapped to a diameter of the unit circle. The vorticity equation on the disk then becomes

$$
\frac{\partial \Delta_{sphere} \Psi}{\partial t} + \frac{(1 + r^2)}{4} \frac{1}{r} \frac{\partial (\Psi, \Delta_{sphere} \Psi)}{\partial (r, \psi)} - \Omega \left\{ (1 + r^2) \sin \psi \frac{\partial \Psi}{\partial r} 
+ \frac{1 - r^2}{r} \cos \psi \frac{\partial \Psi}{\partial \psi} \right\} = \nu (\Delta_{sphere} + 2) \Delta_{sphere} \Psi.
$$

where $\Psi$ is a stream function of the flow, $r$ and $\psi$ are the radial coordinate and the azimuthal angle on the disk, $\nu$ is the kinetic viscosity coefficient, and $\Omega$ is the rotation rate of the sphere. Note that the boundary $r = 1$ corresponds to the boundary meridian on the hemisphere, and $(r, \psi) = (1, 0), (1, \pi)$ to North- and South-Poles, respectively. The
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Fig. 2: The mapping onto a unit circular plane disk. The left and right figures show the views from above and from the side, respectively.

The Laplacian on the hemisphere is expressed in the plane polar coordinate as

$$\Delta_{\text{sphere}} = \frac{(1 + r^2)^2}{4} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \psi^2} \right).$$

We employ the rigid boundary condition at the boundary, $\Psi = \frac{\partial \Psi}{\partial r} = 0$ ($r = 1$). It should be noted that an advantage of this numerical method is that the nonlinear term and the Laplacian term are transformed in a similar manner; simple multiplication of the factor $(1 + r^2)^2/4$, and the transformed equation is similar to the usual fluid equation on the plane disk. This similarity permits us to make use of numerical techniques devised for the 2D plane flow.

We now expand $\Psi$ as

$$\Psi(r, \psi, t) = \sum_n R_n(r, t) e^{in\psi},$$

where $e^{in\psi}$ is the angular mode expanded by Fourier series and $R_n$ is the radius mode expanded by the Chebyshev polynomials. The functions $R_n(r, t)$ are written as

$$R_0(r, t) = (x - 1)^2 (x + 2) a_0^{(0)}(t) + (x - 1)^2 (x + 1)^2 \sum_{m=1}^{M} a_m^{(0)}(t) T_{m-1}(x),$$

$$R_1(r, t) = (x - 1)^2 (x + 1) (x + 2) a_0^{(1)}(t) + (x - 1)^2 (x + 1)^3 \sum_{m=1}^{M} a_m^{(1)}(t) T_{m-1}(x),$$

$$R_n(r, t) = (x - 1)^2 (x + 1)^2 \sum_{m=0}^{M} a_m^{(n)}(t) T_m(x), \quad n \geq 2,$$

where $x = 2r - 1$. These expansions of $R_n$ are adopted because they satisfy the boundary conditions, and they belong to class $C^1$ even at the center of the disk. We take $n = 64$ and
Fig. 3: Temporal evolution of the stream function. The time step is 0.003. We plot the contour lines of the stream function. The solid and dotted lines show positive contour lines and negative ones, respectively. The period is 0.025.

$M = 62$. We calculate the time marching of the expansion coefficients by the collocation method. The second-order Runge-Kutta method and Crank-Nicolson method are used for time integration with the time step of $6.0 \times 10^{-6}$.

3. RESULTS

Streamlines of the initial condition are coaxial circles in the hemisphere expressed as

$$\Psi(r, \psi, t)|_{t=0} = 5(1 - r^2)^2.$$  

We adopted the viscosity coefficient $\nu = 10^{-2}$ and rotation rate of the sphere $\Omega = 400$. In this report the scales of length and time are non-dimensionalized using the radius of the sphere and an appropriate time interval $T$, respectively. The sphere rotates with a non-dimensional time period $2\pi/400$. The mean velocity of the initial flow field is $U = 2.88$, with which the fluid goes around the sphere in $2\pi/2.88$ non-dimensionalized time.

In the course of time development, a westward propagation of a flow pattern is observed as shown in Figure 3. This propagation looks periodic with the period roughly equal to 0.025. We continued the numerical integration and confirmed that the seemingly periodic solution continues at least up to $t=6.0$, which strongly suggests that the solution is really time periodic. During the time development, the flow pattern is kept symmetric with respect to the equator. The northward (southward) flow on the west side of negative (positive) vortex is intensified periodically at the western boundary of the hemisphere.
Fig. 4: The above figures show the time variation of the $L^1$ norms for the diffusion term, nonlinear term and Coriolis term. The left figure shows the $L^1$ norms in the interior region, while the right in the boundary region.

This flow pattern reminds us of the so-called westward intensification discussed by Stommel(1948) who proved the existence of the steady solution of the linearized flow equation on the $\beta$-plane with a forcing term which generates a negative vortex. Stommel's solution clearly demonstrates the westward intensification of the generated current, and has been considered to be a model of ocean currents which are intensified at the eastern coast of a continent as Kuroshio and Gulf Stream. We also note that Lighthill(1968) showed the westward propagation of both barotropic and baroclinic wave energy on the $\beta$-plane in linearized theory of ocean response to monsoon\(^8\).

4. DISCUSSION

In order to see if the periodic solution in our experiment is basically a linear solution, we evaluated the $L^1$ norm of the nonlinear term, diffusion term and Coriolis term in an interior region of the flow field ($0 \leq r \leq 1$), and in a boundary region ($0.978 < r < 1.0$). Figure 4 depicts the time evolution of the norms, showing that the diffusion term is dominant only near the boundary, and the Coriolis term is dominant in the interior part of the domain, while the nonlinear term is weak all over the regions. This result implies that the periodic solution is actually a linear solution without the dissipation in the interior region, and also that the linear periodic solution is stable to the week nonlinearity.

Non-forced inviscid cases are easily analyzed in the framework of the $\beta$-plane approximation. The one-dimensional model equation is written as

\[
\frac{\partial}{\partial t} \Delta \Psi + \beta \frac{\partial \Psi}{\partial x} = 0,
\]

\[
\Delta = \frac{\partial^2}{\partial x^2},
\]

with the Dirichlet boundary condition that $\Psi = 0$ at $x = 0$ and $\pi$. Then the fundamental
solution is
\[ \Psi = \exp \left( ik \left( x + \frac{\beta}{2k^2} t \right) \right) \sin kx, \quad k \in \mathbb{Z} \]

which gives the "phase velocity" of the pattern \( c = -\frac{\beta}{2k^2} \). Note that without the Dirichlet boundary condition at \( x = 0, \pi \), the fundamental solution is \( \Psi = e^{ik(x+\frac{\beta}{k^2}t)} (k \in \mathbb{R}) \) where \( k \) is the wavenumber, and the phase velocity becomes \( c = -\frac{\beta}{k^2} \), which is twice that in the above case.

A similar analysis is possible also in the two-dimensional \( \beta \)-plane. The fluid equation is
\[
\frac{\partial}{\partial t} \Delta \Psi + \beta \frac{\partial \Psi}{\partial x} = 0,
\]
\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]
with the Dirichlet boundary condition at \( x = 0, \pi \) and \( y = 0, \pi \). Then the fundamental solution is
\[ \Psi = e^{ik(x-ct)} \sin(mx) \sin(ny) \]

where \( m^2 + n^2 = k^2 (m, n \in \mathbb{Z}, k \in \mathbb{R}) \) and the "phase velocity" is \( c = -\frac{\beta}{2k^2} \). If the Dirichlet condition is not imposed, the solution is \( \Psi = e^{ik(x-ct)} \) with \( c = -\frac{\beta}{k^2} \), which is again twice that in the Dirichlet case.

Unfortunately, in the case of the hemisphere, an analytic expression for a solution of the similar type to the vorticity equation
\[
\frac{\partial \Delta \Psi}{\partial t} + 2\Omega \frac{\partial \Psi}{\partial \varphi} = 0,
\]
\[
\Delta = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}
\]
with Dirichlet boundary condition at \( \varphi = 0, \pi \), has not yet been obtained. It is also still unknown whether there exists a time periodic solution to the above equation. However, our numerical result strongly suggests that there is a time periodic solution which has similar properties to the solutions in the one-dimensional and two-dimensional \( \beta \)-plane cases. The existence of the time-periodic solution may be interesting because the oscillation have occurred coherently over the hemisphere, although in the \( \beta \)-plane the frequency depends on \( \beta \) which changes latitude by latitude.\(^9\)

We remark that the analytical results in the \( \beta \)-plane case also suggest that the phase velocity in the case of Dirichlet boundary is smaller than that in non-boundary case. Actually, we obtained numerically \( c = 251 \) on the rotating hemisphere, which is smaller than \( c = 400 \) on rotating full sphere.
5. CONCLUSIONS

In this paper we have reported that there is a periodic solution with a coherent spatial pattern propagating westward on a rotating hemisphere, which may correspond to Stommel's westward intensification in a forcing case. We have not yet succeeded to obtain an analytical expression for the periodic solution, nor a proof of the existence of a periodic solution in the hemisphere case. Lastly we remark that the flow pattern of the periodic solution will be drastically changed by a formation of the circumpolar jets when the flow region includes the polar regions. The interaction between the periodic solution and the circumpolar jets is an interesting subject, which will be reported elsewhere.

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REFERENCES


