On Perturbation Expansions of the Classical Limit of Yang-Yang’s Integral Equation for Sutherland Lattice

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The classical limit of the Yang and Yang’s integral equation for the Sutherland lattice (the sinh^{-2} x potential) is investigated. Two kinds of perturbation expansion of the integral equation are proposed. One is from the δ-potential lattice and the other from the Toda lattice. The merits of these methods are discussed.

1 Brief history

In early 70’s, Sutherland made a remarkable suggestion summarized as follows. Consider a one-dimensional quantum lattice whose wavefunction is given by Bethe’s ansatz asymptotically. Then, the thermodynamics of the system at given temperature β^{-1} can be given by the Yang–Yang’s integral equation

\[ \frac{1}{2}p^2 - \mu - \epsilon(p) + \frac{1}{2\pi\hbar^2} \int_{-\infty}^{\infty} \theta'(p - p') \ln(1 + e^{-\beta\epsilon(p')}) dp' = 0. \] (1)

\( p \) is the independent variable and denotes the quasimomenta of a particle. \( \theta \) is the two-body phase shift which is given by the corresponding model. The quantity \( e^{-\beta\epsilon(p)} \) is the dependent variable and gives the ratio of the numbers of occupied and omitted quasimomenta. \( \mu \) is the chemical potential. If we give the pressure \( P \), (1) has to be solved under the condition that

\[ \beta P = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \ln(1 + e^{-\beta\epsilon(p)}). \] (2)

Later, Opper considered the classical limit of (1) and (2) given by

\[ \frac{1}{2}p^2 - \mu - \epsilon(p) + \frac{1}{\beta} \int_{-\infty}^{\infty} \delta(p - p')e^{-\beta\epsilon(p')} dp' = 0, \] (3)

\[ \beta P = \int_{-\infty}^{\infty} e^{-\beta\epsilon(p)} dp, \] (4)
where \( \theta_d \) is the classical two body phase shift. By setting
\[
x = \sqrt{\beta p},
\]
\[
\phi(x) = \frac{1}{\sqrt{\beta}} e^{-\beta \kappa(p)},
\]
\[
\rho = \beta \mathcal{P} \left( \int_{-\infty}^{\infty} \phi(x) dx \right),
\]
we have
\[
\beta \mu = \beta \left( \frac{1}{2} x^2 + \frac{1}{2} \ln \beta + \ln \phi(x) + \int_{-\infty}^{\infty} \theta_d \left( \frac{x - x'}{\sqrt{\beta}} \right) \phi(x') dx' \right). \tag{8}
\]
Differentiating by \( x \), (8) becomes the following nonlinear integro-differential equation.
\[
\frac{d \phi(x)}{dx} + x \phi(x) = \phi(x) \int_{-\infty}^{\infty} \frac{-1}{\sqrt{\beta}} \delta_d \left( \frac{x - \xi}{\sqrt{\beta}} \right) \phi(\xi) d\xi. \tag{9}
\]
Opper solved (9) for the Toda lattice. By Substituting the resulting \( \phi \) into the R.H.S. of (8), the Gibbs free energy \( \beta \mu \) was obtained\(^3\).

## 2 Integral equation for the Sutherland lattice

In this work, we discuss two kinds of perturbation expansions of (9) for the one-dimensional Sutherland lattice (the \( \sinh^{-1} x \) potential), whose classical thermodynamics is remained unclear. The classical two-body phase shift of the Sutherland lattice is given by\(^3\)
\[
\theta_d(p) = 2S \arctan \left( \frac{p}{2S} \right) + \frac{p}{2} \ln \left( 1 + \frac{4S^2}{p^2} \right). \tag{10}
\]
Depending on the positive parameter \( S \), the system reduces to the Toda lattice as \( S \to \infty \) and the \( \delta \)-potential lattice as \( S \to 0 \). Setting \( a = 2S \sqrt{\beta} \), substituting (10) into (9), we have the equation
\[
\frac{d \phi(x)}{dx} + x \phi(x) = \phi(x) \int_{-\infty}^{\infty} \frac{a^2}{(x - \xi)^3 + a^2(x - \xi)} \phi(\xi) d\xi. \tag{11}
\]

## 3 Perturbation from the \( \delta \)-potential lattice

We expand \( \phi \) around the point \( a = 0 \).
\[
\phi(x) = \phi_0(x) + \phi_1(x) a + \phi_2(x) a^2 + \cdots. \tag{12}
\]
To handle the last term in the R.H.S of (11), we introduce \( u_n \) by
\[
u_n(z) = \int_{-\infty}^{\infty} \frac{\phi_n(\xi)}{z - \xi} d\xi, \tag{13}
\]
which is analytic in the complex \( z \)-plane except for the real axis. We also introduce
\[
u_n^+(x) := \lim \limits_{\text{Im} z \to 0^+} u_n(z) = T[\phi_n(x)] - i\pi \phi_n(x), \tag{14}
\]
\[
u_n^-(x) := \lim \limits_{\text{Im} z \to 0^-} u_n(z) = T[\phi_n(x)] + i\pi \phi_n(x), \tag{15}
\]
where $T$ is a Hilbert type transform defined by

$$T[\phi_n(x)] := \int_{-\infty}^{\infty} \frac{\phi_n(\xi)}{x - \xi} \, d\xi.$$  \hspace{1cm} (16)

Note that the change of variables (14) and (15) was developed by Satsuma. (See ref. 4 for example.) Using $u_n^\pm$ and its Taylor expansion, we rewrite the last term in the R.H.S. of (11) as

$$\int_{-\infty}^{\infty} \frac{a^2}{(x - \xi)^3 + a^2(x - \xi)} \phi(\xi) \, d\xi
= T[\phi] - \frac{1}{2} \int_{-\infty}^{\infty} \left( \frac{1}{x + ai - \xi} + \frac{1}{x - ai - \xi} \right) \phi(\xi) \, d\xi
= T[\phi] - \frac{1}{2} \sum_{n=0}^{\infty} a^n \left( u_n^+(x + ai) + u_n^-(x - ai) \right)$$ \hspace{1cm} (17)

$$= T[\phi] - \frac{1}{2} \sum_{n=0}^{\infty} a^n \left( \sum_{k=0}^{\infty} (ai)^k \frac{d^k u_n^+}{dx^k} + (-ai)^k \frac{d^k u_n^-}{dx^k} \right)
= -a \pi \phi_0' - a^2 (\pi \phi_1' - \frac{1}{2} T[\phi_0'']) - \cdots.$$ 

From (14), (15) and (17), (11) leads the following equations.

$$\phi_0' + x \phi_0 = 0,$$ \hspace{1cm} (18)
$$\phi_1' + x \phi_1 = -\pi \phi_0 \phi_0',$$ \hspace{1cm} (19)
$$\cdots,$$
$$\phi_n' + x \phi_n = \text{a differential polynomial of } \phi_j \text{ and } T[\phi_j] \hspace{1cm} (j \leq n - 1),$$ \hspace{1cm} (20)
$$\cdots.$$ 

The first few $\phi_j$'s are explicitly given by

$$\phi_0 = Ae^{-\frac{1}{2}x^2},$$ \hspace{1cm} (21)
$$\phi_1 = -\pi A^2 e^{-x^2},$$ \hspace{1cm} (22)
$$\phi_2 = \frac{3}{2} \pi^2 A^3 e^{-\frac{3}{2}x^2} - \frac{1}{2} \pi A^2 x e^{-x^2} \text{Erf}(\frac{x}{\sqrt{2}}),$$ \hspace{1cm} (23)
$$\cdots.$$ 

The constant $A$ is determined by the given pressure (7). Substituting (21), (22), (23) into the R.H.S. of (8), we can deduce the Gibbs free energy perturbatively.

4 Perturbation from the Toda lattice

The perturbation expansion around the point $a = \infty$ is more complicated. Setting $\epsilon = 1/a^2$, we expand $\phi$ as

$$\phi(x) = \psi_0(x) + \psi_1(x) \epsilon + \psi_2(x) \epsilon^2 + \cdots.$$ \hspace{1cm} (24)

We substitute (24) into (11). The equation of the lowest order becomes

$$\frac{d\psi_0}{dx} + x \psi_0 = \psi_0 T[\psi_0],$$ \hspace{1cm} (25)
which was solved by Opper\textsuperscript{23} with the solution
\begin{equation}
\psi_0 = \frac{2}{\pi} \text{Im} \left[ \frac{d}{dx} \ln f(x) \right],
\end{equation}
(26)
\begin{equation}
f(x) = \int_0^\infty e^{ix\xi - \frac{z^2}{2}\xi^2} \xi^{m_0 - 1} d\xi.
\end{equation}
(27)

The constant $\rho_0$ is determined by (7). The equation of higher order becomes
\begin{equation}
\frac{d\psi_n(x)}{dx} + x\psi_n(x) = \sum_{k=0}^{n-1} \psi_k(x)\alpha_{n-k-1}(x)
\end{equation}
(28)

\[ + \sum_{k=0}^{n} \psi_k(x)T[\psi_{n-k}(x)]. \]
\begin{equation}
\alpha_m(x) := \sum_{l=0}^{m} \int_0^\infty (-1)^{l+1} (x - \xi)^{2l+1} \psi_{m-l}(\xi) d\xi.
\end{equation}
(29)

It should be noticed that $\alpha_m(x)$ is a polynomial of degree $2m + 1$. Then we substitute
\begin{equation}
v_n(z) := \int_{-\infty}^{\infty} \frac{\psi_n(\xi)}{z - \xi} d\xi,
\end{equation}
(30)
\begin{equation}
v_n^+(x) := \lim_{\text{Im}x \to 0^+} v_n(z) = T[\psi_n(x)] - i\pi \psi_n(x),
\end{equation}
(31)
\begin{equation}
v_n^-(x) := \lim_{\text{Im}x \to 0^-} v_n(z) = T[\psi_n(x)] + i\pi \psi_n(x),
\end{equation}
(32)

into (28). The terms like $v_k^-v_{n-k}^+$ offset each other and we have
\begin{equation}
\frac{dv_n^-}{dx} + xv_n^- - \frac{1}{2} \sum_{k=0}^{n} v_k^-v_{n-k}^- + \sum_{k=0}^{n-1} \alpha_{n-k-1} v_k^-
= \frac{dv_n^+}{dx} + xv_n^+ - \frac{1}{2} \sum_{k=0}^{n} v_k^+v_{n-k}^+ + \sum_{k=0}^{n-1} \alpha_{n-k-1} v_k^+.
\end{equation}
(33)

This implies that the function
\begin{equation}
F_n(z) := \frac{dv_n}{dz} + zv_n - \frac{1}{2} \sum_{k=0}^{n} v_k v_{n-k} + \sum_{k=0}^{n-1} \alpha_{n-k-1} v_k
\end{equation}
(34)
is analytic everywhere including the real axis. Furthermore, since
\begin{equation}
v_n(z) \to \frac{1}{x} \int_{-\infty}^{\infty} \psi_n(\xi) d\xi \text{ as } |z| \to \infty,
\end{equation}
(35)
we have
\begin{equation}
\lim_{|z| \to \infty} F(z) = (-1)^n \rho_0 z^{2n-2}.
\end{equation}
(36)

Thus we find that
\begin{equation}
F_n(z) = (\text{a polynomial of degree } 2n - 2),
\end{equation}
(37)
which is the linear first order differential equation. For example, $\psi_1$ is obtained as

$$
\psi_1 = \frac{v_1^- - v_1^+}{2\pi i}
$$

$$
= -\frac{1}{\pi} \text{Im} \left[ e^{-\frac{1}{2}x^2} f \int e^{\frac{1}{2}\xi^2} f (\rho_1 f - 2\rho_0 f'') d\xi \right],
$$

where $f$ is defined in (27). The constants $r, \rho_1$ are determined by

$$
\int_{-\infty}^{\infty} e^{\frac{1}{2}x^2} f (\rho_1 f - 2\rho_0 f'') d\xi = 0,
$$

$$
\int_{-\infty}^{\infty} \psi_1(x) dx = \rho_1.
$$

5 Conclusion

We presented the two kinds of perturbation expansion of the classical Yang–Yang’s integral equation for the Sutherland lattice. The key of our investigation is to expand the integral transform in the R.H.S. of (11) around the Hilbert transforms, which we believe to be also useful to other general integral transforms. As to the perturbation from the $\delta$-potential lattice, it was shown that we could obtain the solutions of each order recursively. The perturbation from the Toda lattice is a certain analogue of the Oppen’s method$^{(3)}$. In this case, we obtained the solution of the first order. For the solutions of higher order, we still have a task to determine the polynomial in the R.H.S. of (37).

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References


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