A Domain Decomposition Method for Large-Scale 3-D Nonlinear Magnetostatic Problems

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An iterative domain decomposition method is applied to numerical analysis of 3-dimensional (3-D) nonlinear magnetostatic problems taking the magnetic vector potential as an unknown function. The nonlinear simultaneous equations are solved with the Picard iteration or the Newton iteration. In Picard iteration, we compute the reluctivity using H-B curves, and in Newton iteration we compute the reluctivity using v-B curves or v-B² curves. The simultaneous linear equations at each step of the nonlinear iteration are solved by the iterative domain decomposition method. The iterative domain decomposition method is combined with the Conjugate Gradient (CG) procedure, and the Hierarchical Domain Decomposition Method (HDDM) which is adopted for the parallel computing. Numerical results show that the iterative procedure converges, and that the computed magnetic flux density is suitable.

1. INTRODUCTION

Until now, we have used a simple Picard iteration and the Newton iteration to solve 3-D nonlinear magnetostatic problems. In practical applications, large-scale models are required in order to get accurate solutions. However, as the scale and complexity of numerical simulation problems escalates, conventional Finite Element Method (FEM) is found to be inefficient because of its excessive computational time as well as the storage cost. Parallel computing, which concurrently uses a number of processors, has been recognized as a remedy to large-scale computer simulation. In fact, parallel approaches are mandatory for large-scale numerical analysis. Many parallel computing techniques have been developed over the past decades. Among them, parallel computing using Domain Decomposition Method (DDM) is one of useful methods. Hence in this work, we apply an iterative DDM to 3-D nonlinear magnetostatic problems. Particularly our present goal is to solve 3-D nonlinear magnetostatic problems with about one million Degrees Of Freedom (DOF).

In this paper, the fundamental unknown function is the magnetic vector potential. The nonlinear simultaneous equations are solved with a method based on the Picard iteration or the Newton iteration. The magnetic reluctivity $v$ is calculated from the B-H curve mainly. For the Picard iteration, we use the H-B curve, for the Newton iteration, we use the v-B curve or v-B² curve. For the simultaneous linear equations at each step of the nonlinear iteration, we have introduced a formulation without the Lagrange multiplier\(^2\).\(^5\). Owing to this formulation, we can introduce the conjugate gradient (CG) procedure for the interface problem. We adopt the Hierarchical Domain Decomposition Method (HDDM) for parallel computing. HDDM has been shown effective for structural problems\(^6\).\(^7\). Numerical results show that the iterative procedure converges, and that the computed magnetic flux density is suitable.
2. FORMULATIONS

Let $\Omega$ be a polyhedral domain with boundary $\partial \Omega$, and let $n$ be a unit normal vector to the boundary. Assume that the boundary $\partial \Omega$ consists of two disjoint parts $\Gamma_E$ and $\Gamma_N$. Then we consider the following magnetostatic problem with the Coulomb gauge condition:

\[
\begin{align*}
\text{rot} \ (\nu \text{rot} \ A) &= J & \text{in} \ \Omega, \\
\text{div} \ A &= 0 & \text{in} \ \Omega, \\
A \times n &= 0 & \text{on} \ \Gamma_E, \\
(\nu \text{rot} \ A) \times n &= 0 & \text{on} \ \Gamma_N, \\
A \cdot n &= 0 & \text{on} \ \Gamma_N.
\end{align*}
\]

where $A=(A_1, A_2, A_3)[\text{Wb/m}]$ is the magnetic vector potential, $J=(J_1, J_2, J_3)[\text{A/m}^2]$ is an electric current density and $\nu[\text{m/H}]$ is the magnetic reluctivity. We assume that $A$ is a positive scalar, which depends on a function of $A$ in the magnetic body, and in other parts is taken as a positive number. Moreover $\text{div} \ A = \partial A_1/\partial x_1 + \partial A_2/\partial x_2 + \partial A_3/\partial x_3$ and $\text{rot} \ A = (\partial A_2/\partial x_3 - \partial A_3/\partial x_2, \partial A_3/\partial x_1 - \partial A_1/\partial x_3, \partial A_1/\partial x_2 - \partial A_2/\partial x_1)$ represents the position vector, and the symbols $\cdot$ and $\times$ represent vector inner and outer products, respectively. Finally, we assume that

\[
\begin{align*}
\text{div} \ J &= 0 \text{ in} \ \Omega, \\
J \cdot n &= 0 \text{ on} \ \Gamma_N.
\end{align*}
\]

Let $L^2(\Omega)$ be a space of functions defined in $\Omega$ and square summable in $\Omega$ with its inner product $(\ , \ )$ and let $H^1(\Omega)$ be a space of functions in $L^2(\Omega)$ with derivatives up to the first order. We define $V$ and $Q$ by

\[
\begin{align*}
V &\equiv \{v \in (L^2(\Omega))^3, \text{rot} \ v \in (L^2(\Omega))^3, v \times n = 0 \text{ on} \ \Gamma_E, \} \\
Q &\equiv \{q \in H^1(\Omega), q = 0 \text{ on} \ \Gamma_E, \}
\end{align*}
\]

Now, a weak formulation $[P]$ of (1) is constructed by the introduction of the Lagrange multiplier $p$:

\[
\begin{align*}
\text{Find} \ (A, p) \in V \times Q \text{ such that} \\
(\nu \text{rot} A, \text{rot} A') + (A^*, \text{grad} p) &= (J, A') \text{ for any } A' \in V, \quad (3a) \\
(A, \text{grad} p^*) &= 0 \text{ for any } p^* \in Q. \quad (3b)
\end{align*}
\]

where grad $p=(\partial p/\partial x_1, \partial p/\partial x_2, \partial p/\partial x_3)$. It is easy to see $p=0$ in $\Omega$, because the electric current density is divergence free.

Let us decompose $\Omega$ into a union of tetrahedral elements. We approximate $A$ by the Nedelec elements of simplex type, and $p$ by the conventional piecewise linear tetrahedral elements. Denoting finite element spaces corresponding to $V$ and $Q$ by $V_h$ and $Q_h$, respectively, we have the following finite element approximation $[P_h]$:

\[
[P_h] \left\{ \begin{array}{l}
\text{Find} \ (A_h, p_h) \in V_h \times Q_h \text{ such that} \\
(v_h \text{rot} A_h, \text{rot} A'_h) + (A^*_h, \text{grad} p_h) &= (J_h, A'_h) \text{ for any } A'_h \in V_h, \\
(A_h, \text{grad} p^*_h) &= 0 \text{ for any } p^*_h \in Q_h
\end{array} \right.
\]

where $v_h$ is assumed to be constant in each of the tetrahedral elements and $J_h$ is a corrected electric current density with consideration of the continuity.

By elimination of the Lagrange multiplier $p_h$ formally $1)$, $[P_h]$ can be written as follows:

\[
[P_h'] \left\{ \begin{array}{l}
\text{Find} \ A_h \in V_h \text{ such that} \\
v_h \text{rot} A_h, \text{rot} A'_h &= (J_h, A'_h) \text{ for any } A'_h \in V_h
\end{array} \right.
\]

3. NONLINEAR SIMULTANEOUS EQUATIONS

To solve the nonlinear equation $[P_h']$, we adopt the following Picard iteration $1,2)$:

Given $v_2^h$, solve $A_1^h \in V_h$ such that

\[
(v_2^h \text{rot} A_1^h, \text{rot} A'_1) = (J_h, A'_1) \quad \text{for any } A'_1 \in V_h,
\]

then, Find $A_2^{n+1} \in V_h \ (n=1, 2, \ldots)$ such that

\[
(v_2^h \text{rot} A_2^{n+1}, \text{rot} A'_2) = (J_h, A'_2) \quad \text{for any } A'_2 \in V_h,
\]
where \( \nu_n^* \) is computed from \( A_n^0 \).

Similarly, we can also solve \([P_n^*]\) by the Newton iteration\(^5\):

\[
\begin{align*}
\nu_n^* \cdot \text{rot} A_n^{n+1} \cdot \text{rot} A_n^* + \left( \frac{\partial \nu}{\partial A} \right) A_n^{n+1} \cdot \text{rot} A_n^* + \text{rot} A_n^* &= (J_n, A_n^*) + \left( \frac{\partial \nu}{\partial A} \right) A_n^n \cdot \text{rot} A_n^* \cdot \text{rot} A_n^* \\
\text{for any } A_n^* \in V_n^*,
\end{align*}
\]

where \( \frac{\partial \nu}{\partial A} \) is the Frechet derivative at \( A_n^n \). In Newton iteration, \( \nu_n^* \) is calculated from the nonlinear curves once \( A_n^0 \) is given. But a suitable initial value \( \nu_n^* \) is often being taken for the faster convergence.

4. INTERFACE PROBLEM

In this chapter, we assume that \( \nu \) is given for simplicity. Then, the problem (5) approximates the following problem:

\[
\begin{align*}
\text{rot}(\nu \text{ rot } A) &= J & \text{in } \Omega, \\
A \times n &= 0 & \text{in } \Gamma_E, \\
(\nu \text{ rot } A) \times n &= 0 & \text{on } \Gamma_N.
\end{align*}
\]

(9a) (9b) (9c)

For simplicity, assume that the domain \( \Omega \) is partitioned into two non-overlapping subdomains \( \Omega_1 \) and \( \Omega_2 \), and that the boundary \( \Gamma_E \) (or \( \Gamma_N \)) into \( \Gamma_{E_1} \) and \( \Gamma_{E_2} \) (or \( \Gamma_{N_1} \) and \( \Gamma_{N_2} \)), and set \( \gamma_{12} = \partial \Omega_1 \cap \partial \Omega_2 \). Let \( A_1, A_2, J_1 \) and \( J_2 \) be the restrictions of the magnetic vector potential and the electric current density into each subdomain.

The magnetic vector potentials \( A_1 \) and \( A_2 \), which are assumed to be smooth enough, satisfy

\[
\begin{align*}
&\begin{cases}
\text{rot}(\nu \text{ rot } A_j) = J_j & \text{in } \Omega_j, \\
A_j \times n = 0 & \text{on } \Gamma_{E_j}, \\
(\nu \text{ rot } A_j) \times n = 0 & \text{on } \Gamma_{N_j}. 
\end{cases} \\
&\begin{cases}
A_1 \times n = A_2 \times n & \text{on } \gamma_{12}, \\
(\nu \text{ rot } A_1) \times n = (\nu \text{ rot } A_2) \times n & \text{on } \gamma_{12}, \\
A_1 \times n = 0 & \text{on } \Gamma_{E_2}, \\
(\nu \text{ rot } A_2) \times n = 0 & \text{on } \Gamma_{N_2}. 
\end{cases}
\end{align*}
\]

(10a) (10b) (10c) (10d) (10e) (10f) (10g) (10h)

We define a function space \( W \) that consists of functions from \( \gamma_{12} \) to 3-D Euclidean space \( \mathbb{R}^3 \):

\[
W = \{ w : \gamma_{12} \rightarrow \mathbb{R}^3 ; w = (\nu \times n)_{|_{\gamma_{12}}}, \nu \in V \},
\]

and let \( W' \) denote the corresponding dual space. We denote the norm and the inner product by \( \| . \|_w \) and \( ( . , . )_w \), respectively. For the characterization of \( W' \), see Alonso and Valli\(^5\). Let \( \mathcal{A} \) be a linear operator from \( W \) to \( W' \) defined by

\[
< \mathcal{A} \lambda, \lambda' > = \sum_{i=1}^2 \left\{ (\nu \text{ rot } \tilde{A}_i, \text{rot} \tilde{A}_i')_{|_{\gamma_{12}}} \right\} \quad \text{for } \lambda, \lambda' \in W.
\]

(11)

where \( \tilde{A}_i \) and \( \tilde{A}_i' \) are functions defined by \( \tilde{A}_i \equiv \varepsilon_i (0, \lambda) \) and \( \tilde{A}_i' \equiv \varepsilon_i (0, \lambda') \), respectively, \( \varepsilon_i \) is an operator \( \varepsilon_i(J_i, g) = u_i \) from \( J_i \) and \( g \) to \( u_i \) that is the solution of the equation

\[
\begin{align*}
&\begin{cases}
\text{rot}(\nu \text{ rot } u_i) = J_i & \text{in } \Omega_i, \\
u_i \times n = 0 & \text{on } \Gamma_{E_i}, \\
(\nu \text{ rot } u_i) \times n = 0 & \text{on } \Gamma_{N_i}. 
\end{cases} \\
&\begin{cases}
u_i \times n = g & \text{on } \gamma_{12}, \\
\text{ro}\text{t} \tilde{A}_i & \text{on } \gamma_{12}.
\end{cases}
\end{align*}
\]

(12a) (12b) (12c) (12d)

and \( ( . , . )_{|_{\gamma_{12}}} \) is \( L^2 \)-inner product over \( \Omega \).

For the solution of (10), let us set \( \lambda \in W' \) by

\[
\lambda = (A_1 \times n)_{|_{\gamma_{12}}} = (A_2 \times n)_{|_{\gamma_{12}}}.
\]

Then, the interface problem

\[
< \mathcal{A} \lambda, \lambda' > = \sum_{i=1}^2 \left\{ (J_i, \tilde{A}_i')_{|_{\gamma_{12}}} - (\nu \text{ rot } \tilde{A}_i, \text{rot} \tilde{A}_i')_{|_{\gamma_{12}}} \right\} \quad \text{for } \lambda' \in W
\]

(13)

is satisfied with \( \tilde{A}_i \equiv \varepsilon_i (J_i, 0) \).

Conversely, setting the solution \( \lambda \) of (13) as the boundary condition, we solve the system (10) except (10d) and
(10e) in each subdomain. This procedure leads to get the solution of (9) in the whole domain.

5. DISCRETIZED ITERATIVE DOMAIN DECOMPOSITION METHOD

Each subdomain $\Omega_i$ is decomposed into a union of tetrahedral elements. Let $X_h$ be a finite element space consisting of the Nedelec elements of simple type. Set

$$V_h = X_h \cap V,$$

$$V_{ih} = \{ v_h | v_h \in V_h, v_h |_{\partial \Omega_i} = 0 \},$$

$$W_h = \{ (v_h \times n) |_{\partial \Omega_i}; v_h \in V_h \},$$

$$U_{ih} (\lambda_h) = \{ v_h \in V_{ih}; (v_h \times n) |_{\partial \Omega_i} = \lambda_h \} \quad \text{for } \lambda_h \in W_h,$$

$$U_{ih} = U_{ih}(0).$$

As in Glowinski et al.\(^9\), we can now describe the following conjugate gradient method for finite element approximation of (13):

Choose $\lambda_h^0 \in W_h$;
Compute $g_h^0$ by (14);
Set $w_h^0 = g_h^0$;

for $n = 0, 1, \ldots$;

$$\rho_h^n = \frac{(g_h^n, g_h^n)_W}{(w_h^n, w_h^n)_W};$$

$$\lambda_h^{n+1} = \lambda_h^n - \rho_h^n w_h^n;$$

$$g_h^{n+1} = g_h^n - \rho_h^n \mathcal{A} w_h^n;$$

If $\frac{\| g_h^{n+1} \|_W}{\| g_h^n \|_W} < \delta$, break;

$$\gamma_h^n = \frac{(g_h^{n+1}, g_h^{n+1})_W}{(g_h^n, g_h^n)_W};$$

$$w_h^{n+1} = g_h^{n+1} + \gamma_h^n w_h^n;$$

end.

where $\delta$ is a positive constant. For $\lambda_h^n \in W_h$, the initial residual $g_h^0$ is computed by

$$<g_h^0, \lambda_h^n> = \sum_{i=1}^r \left( (\nu \text{ rot } A_h^n, \text{ rot } \tilde{A}_h^n)_{\Omega_i} - (\tilde{J}_h^n, \tilde{A}_h^n)_{\Omega_i} \right)$$

(14)

where $A_h^n \in U_{ih}(\lambda_h^n)$ satisfies that, for $A_h^n \in U_{ih}$,

$$<\nu \text{ rot } A_h^n, \text{ rot } \tilde{A}_h^n>_{\Omega_i} = (\tilde{J}_h^n, A_h^n)_{\Omega_i}$$

(15)

and $\tilde{A}_h^n$ is a function approximating $\tilde{A}_h^n$. Here, $\tilde{J}_ih$ is the corrected electric current density. For $\lambda_h^n \in W_h$, $\mathcal{A} w_h^n$ is computed by

$$<\mathcal{A} w_h^n, \lambda_h^n> = \sum_{i=1}^r \left( (\nu \text{ rot } \tilde{A}_h^n, \text{ rot } \tilde{A}_h^n)_{\Omega_i} \right)$$

(16)

where $\tilde{A}_h^n \in U_{ih}(w_h^n)$ satisfies that, for $\tilde{A}_h^n \in U_{ih}$,

$$<\nu \text{ rot } \tilde{A}_h^n, \text{ rot } \tilde{A}_h^n>_{\Omega_i} = 0$$

(17)

The computations (14) and (16) imply that we do not need to construct $\mathcal{A}$ explicitly. Moreover, because those are independent of each subdomain at every iterative step, the CG procedure seems to be suitable for the parallel computing.

6. HIERARCHICAL DOMAIN DECOMPOSITION METHOD

The original domain is hierarchically divided into parts, which are further decomposed into smaller domains called subdomains. This hierarchically structured DDM classifies processors into 3 groups, 'Grand Parent', 'Parent' and 'Child'. This is called the Hierarchical Domain Decomposition Method (HDDM). One of the processors is
assigned as Grand Parent, a few as Parent, and others as Child. The number of processors assigned as Parent is the same as that of parts. The number of Child processors can be varied; and it affects the parallel performance.

The role of Grand Parent is to organize all processor communications (i.e. message passing) which occur between all processors. Parents prepare mesh data, manage FEA (Finite Element Analysis) results, and coordinate the CG iteration, including convergence decision for CG iteration. Parents send data to Child processors, where FEA is performed in parallel. After the FEA, Child processors send the results to Parents. This computation will be repeated until the CG iteration is convergent. After the linear computation is convergent, it goes into Picard iteration or Newton iteration to compute the next step’s magnetic reluctivity until the relative change rate of the unknown degrees of freedom becomes small.

7. NUMERICAL RESULTS

We computed an axi-symmetric problem\(^2\), the TEAM Problem 13\(^3\), a transformer 1/6 problem and a transformer full problem\(^9\). In this section, we will introduce the transformer full problem, which is the most large-scale model among them.

**A transformer full model**

We consider a transformer model, which consists of a tank, shields, a core, coils, and the air; see Figure 1. The transformer model is originally a 3-D eddy current model, and here it is set up for a nonlinear magnetostatic model again. The height of the tank is 3.845[mm] and thickness is 15[mm].

We decompose the transformer model into a union of tetrahedral elements; see Figure 2. The total numbers of elements, degrees of freedom are 414,128 and 574,055, respectively. The computational domain is decomposed into 2 parts and 2,000 subdomains. The electric currents in the coils are shown in Table 1, and the positive direction of the electric currents is the anti-clockwise in view of the Figure 1. Because the electric currents exist in the coils only, we correct the electric current density in the coil regions only. The magnetic reluctivity is a positive constant in each element such that in the region of air and coil, the value is \(1/(4\pi)\times 10^7 [m/H]\). Table 2 shows the relative permeability in each part of the transformer. The B-H curve as Figure 3 is used for the v-B curve in the core. Boundary conditions are described in (1c).

The Newton iteration is stopped by \(\| x^n - x^{n-1} \|/\| x^n \| < 1.0 \times 10^{-5} \), where \(x^n\) is the solution vector concerned with \(Ax^n\) of the n step simultaneous linear equations, and \(\| \cdot \|\) denotes the Euclidean norm, respectively. In the initial computation of the Newton iteration, the magnetic reluctivity of the core is \(7.59 \times 10^7 [m/H]\).

In each subdomain, the resultant linear equations are solved by the shifted ICCG method\(^1\) (the shift value is 1.1), and the ICCG method is stopped when the residual norm \(\| M^{-1}(Ax-b) \|/\| M^{-1}b \|\) becomes less than \(10^{-11}\). Here, \(A\) denotes the resultant coefficient matrix, \(x\) the solution vector, \(b\) the resultant given vector and \(M\) the preconditioner.

Computation was performed by 10 CPUs using Pentium 4 2.0 GHz. The total computational time is about 17 hours. Figure 4 shows the convergence history of the Newton iteration, and Figures 5 and 6 show the computed magnetic flux density vectors. The flow of the magnetic flux density in Figures 5 and 6 is as same as the stream lines in the 3-D eddy current analysis\(^10\), so we can conclude that our results are suitable qualitatively.

8. CONCLUDING REMARKS

Using an iterative domain decomposition method, a hierarchical parallel computing module has been developed and successfully applied to 3-D nonlinear magnetostatic problems. Also this module has been adopted by ADVVENTURE (ADVanced ENgineering analysis Tool for Ultra large REAL world) project\(^3\), which tries a development of computational mechanics system for large-scale analysis and design (one research for the future program of the Japan Society for the Promotion of Science (JSPS)). In future research, we will analyze larger scale 3-D nonlinear magnetostatic problems. Also, we will analyze the convergence character for the same model by changing the degrees of freedom of the model.
Table 1: Electric currents (same for three coils).

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>inner</td>
<td>$-101.4 \times 10^3$ (A)</td>
</tr>
<tr>
<td>middle</td>
<td>$-193.3 \times 10^3$ (A)</td>
</tr>
<tr>
<td>outer</td>
<td>$294.7 \times 10^3$ (A)</td>
</tr>
</tbody>
</table>

Table 2: Relative permeability.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<tbody>
<tr>
<td>tank</td>
<td>600</td>
</tr>
<tr>
<td>shields</td>
<td>10,000</td>
</tr>
<tr>
<td>coils/air</td>
<td>1</td>
</tr>
</tbody>
</table>

Fig. 1: A transformer full model.

Fig. 2: A mesh of the transformer full model.

Fig. 3: B-H curve.

Fig. 4: Convergence history of the Newton iteration.
Fig. 5: The magnetic flux density vector (1).

Fig. 6: The magnetic flux density vector (2).
REFERENCES

12) ADVENTURE Project Home Page, http://adventure.q.t.u-tokyo.ac.jp/