Large Eddy Simulation of a Turbulent Channel Flow Using Dynamic SGS Model with GSMAC-FEM

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In Large Eddy Simulation (LES) using the dynamic Smagorinsky model, the test filtering operation is important because the model constant is dependent on the operation. In order to precisely evaluate the numerical results using the dynamic Smagorinsky model, it is necessary to have a clear grasp of the operation. In the present study, we examine the characteristics of the test filtering operation discretized with Finite Element Method (FEM). In order to clarify this characteristics, we apply the Finite Difference Method (FDM) to only the test filtering operation and compare the results obtained respectively by FEM and FDM. The numerical model is a turbulent channel flow, which is a benchmark problem. Based on the numerical results, we clarify the difference in the discretization of the test filtering operation and reveal the influence of different discretization method.

1. INTRODUCTION

The purpose of the present study is to apply a Large Eddy Simulation (LES), using the dynamic Smagorinsky model, to practical problems. In the application of the model, the test filtering operation is important because the model constant depends on the operation. In the application of FDM, the formulation of the operation is usually conducted using the second-order central difference scheme. Several researchers have validated the effectiveness of this method. In contrast, in the application of FEM, the formulation of the problem is rarely discussed. In the case of FEM, it is difficult to understand the result of discretization intuitively compared with FDM. The reason is that the approximation of the physical value is performed after the basic equations are transformed into integral forms by the weighted residual method. In a previous study, we applied the Generalized Simplified Marker and Cell (GSMAC) - FEM to the analysis of a turbulent channel flow. In the present study, FDM is applied only to the test filtering operation for comparison with FEM. As a primary result, the same tendency was confirmed between two methods in the distribution of the turbulent intensities, but the effect of the filter width on the numerical results is more remarkable, i.e., in the case of FDM than that in case of FEM. The reason for the difference must be examined in detail in order to evaluate the numerical results precisely. In the present study, we examined the difference in discretization between FEM and FDM concerning the test filtering operation. Through this comparison, we clarify the difference in the formulation and evaluate the numerical results.
2. GOVERNING EQUATIONS AND DYNAMIC SMAGORINSKY MODEL

For the LES of an incompressible flow, the governing equations consist of the filtered continuity equation and the filtered Navier-Stokes equation. The non-dimensional equations are written as follows:

$$\frac{\partial \bar{u}_i}{\partial x_j} = 0$$  \hspace{1cm} (1)

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\bar{u}_i \bar{u}_j) = - \frac{\partial \bar{P}}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} + \frac{1}{Re \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}}$$  \hspace{1cm} (2)

where $\bar{u}_i$, $\bar{p}$ and $Re$ are the velocity, pressure and Reynolds number, respectively. $\tau_{ij}$ is the subgrid scale (SGS) stress. In order to close the basic equation, it is necessary to model $\tau_{ij}$. In the present study, the dynamic Smagorinsky model is applied.

In the dynamic Smagorinsky model, two filter functions are defined as follows:

- Grid filter function $\bar{G}$: $\bar{u}_i(x) = \int \bar{G}(x - x') u_i(x') dx'$
- Test filter function $\hat{G}$: $\hat{u}_i(x) = \int \hat{G}(x - x') u_i(x') dx'$

In connection with these filter functions, $\bar{\Delta}$ and $\hat{\Delta}$ are defined as the grid filter width and the test filter width respectively. When two different filters, the grid filter and the test filter, are applied consecutively, a series of operations is considered as one filtering operation. In the operation, the filter width is defined as the effective filter width $\bar{\Delta}$.

First, SGS stress is modeled using the Smagorinsky model$^6$.

$$\tau_{ij} = -2 \left(C_s \bar{\Delta}^2\right) \left[\bar{S}_ij\right] \cdot \left[\bar{S}_ij\right] \equiv (2\bar{S}_ij\bar{S}_ij)^{1/2}$$  \hspace{1cm} (3)

where $\bar{S}_{ij}$ is the rate of deformation tensor in the grid scale. In the dynamic Smagorinsky model, the model constant $\left(C_s \bar{\Delta}^2\right)$ in Equation (3) is calculated by the following procedure. The grid and test filtering operations are applied to Equation (2), consecutively. The SGS stresses are derived from these operations, and the Smagorinsky model is applied. The Germano identity $L_{ij}$ is introduced, and each SGS stress is substituted for $L_{ij}$. As a result, the relational expressions are obtained in the following:

$$L_{ij} = -2 \left(C \bar{\Delta}^2\right) M_{ij}$$  \hspace{1cm} (4)

$$L_{ij} = \bar{u}_i \bar{u}_j - \bar{u}_i \bar{u}_j \cdot M_{ij} = \alpha^2 \bar{S}_i \bar{S}_j - \bar{S}_i \bar{S}_j \cdot \alpha = \frac{\bar{\Delta}}{\bar{\Delta}}$$

In the solver of Equation (4), an approximate solution is applied according to the proposal of Lilly$^7$.

$$\frac{C \bar{\Delta}^2}{4M_{ij}} = - \frac{L_{ij}^* M_{ij}}{2 M_{ij} M_{ij}}$$  \hspace{1cm} (5)

From the relational expression, the SGS stress is calculated.

In the present study, the Gaussian filter is applied as both grid filter and test filter in the streamwise direction ($x$ direction) and the spanwise direction ($z$ direction). No explicit test filtering operation is applied in the wall direction ($y$ direction). The parameter $\alpha^2 = 2^{2/3}$ is used. The ratio $\bar{\Delta}/\bar{\Delta}_i$ is assumed to be equal in the streamwise and spanwise directions.

$$\frac{\bar{\Delta}_i}{\bar{\Delta}_i} = 2 \quad (i = x, z)$$
where $\Delta_i$ and $\bar{\Delta}_i$ are the grid filter and effective filter widths in each coordinate direction.

The entire time marching algorithm from step $n$ to step $n+1$ is summarized as below. Before performing the calculation for velocity and pressure, the eddy viscosity coefficient at step $n$ is calculated. Next, the velocity and pressure are calculated using GSMAC-FEM.

### 3. DISCRETIZATION OF THE TEST FILTERING OPERATION

In the present study, the test filtering operation is carried out in the streamwise and spanwise directions. So, the test filtered value of $\bar{\phi}(x, y, z)$ is defined as follows:

$$
\bar{\phi}(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{G}(X') \bar{G}(Z') \bar{\phi}(x + X', y, z + Z') \, dX' \, dZ'
$$

(6)

In Equation (6), $\bar{\phi}(x + X', y, z + Z')$ is approximated with the Taylor series up to the second-order term and the relational expression is arranged by the characteristics of the even functions, $\bar{G}(X')$ and $\bar{G}(Z')$.

$$
\bar{\phi}(x, y, z) = \bar{\phi}(x, y, z) + \frac{1}{2} \frac{\partial^2 \bar{\phi}(x, y, z)}{\partial x^2} \int_{-\infty}^{\infty} X'^2 \bar{G}(X') \, dX' + \frac{1}{2} \frac{\partial^2 \bar{\phi}(x, y, z)}{\partial z^2} \int_{-\infty}^{\infty} Z'^2 \bar{G}(Z') \, dZ'
$$

(7)

In the present study, the Gaussian filter function is adopted. So, the integral value in Equation (7) is calculated as follows:

$$
\int_{-\infty}^{\infty} X'^2 \bar{G}(X') \, dX' = \frac{\bar{\Delta}_x^2}{12} \quad \int_{-\infty}^{\infty} Z'^2 \bar{G}(Z') \, dZ' = \frac{\bar{\Delta}_z^2}{12}
$$

(8)

These integral values are substituted in Equation (7). As a results, the relational expression of the test filtering operation is derived as

$$
\bar{\phi}(x, y, z) = \bar{\phi}(x, y, z) + \frac{\bar{\Delta}_x^2}{24} \frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\bar{\Delta}_z^2}{24} \frac{\partial^2 \bar{\phi}}{\partial z^2}
$$

(9)

In order to discuss the validity of the filtering operation with FEM, the effect of the filter width on the numerical results is discussed. Through this discussion, we clarify the characteristics of FEM as compared to FDM and the effect of difference in the discretizations on the numerical results is confirmed.

In the present study, the FDM is applied only to the filtering operation in order to compare the filtering operation with FEM. By the application of FDM, the problem of the test filtering operation is separated from the problem of conservation.

In the case of FDM, the second derivatives in Equation (9) are formulated using second-order central difference scheme. Concretely, the discretized equation obtained using the second-order difference is formulated as follows:

$$
\bar{\phi}_{l,m,n} = \bar{\phi}_{l,m,n} + \frac{\bar{\Delta}_x}{24} \left( \frac{\phi_{l-1,m,n} - 2\phi_{l,m,n} + \phi_{l+1,m,n}}{h_x^2} \right) + \frac{\bar{\Delta}_z}{24} \left( \frac{\phi_{l,m,n-1} - 2\phi_{l,m,n} + \phi_{l,m,n+1}}{h_z^2} \right)
$$

(10)

where, $l$, $m$ and $n$ are the stencils which specify the location in the $x$, $y$ and $z$ directions.

In the case of FEM, Equation (9) is discretized using the Galerkin method. The formulation is described using the element stiffness matrix:

$$
\overline{M}_{\alpha\beta} \overline{\phi}_{\beta} = M_{\alpha\beta} \overline{\phi}_{\beta} - \frac{\bar{\Delta}_x^2}{24} L_{\alpha\beta,x} \overline{\phi}_{\beta} - \frac{\bar{\Delta}_z^2}{24} L_{\alpha\beta,z} \overline{\phi}_{\beta}
$$

(11)
\[ D_{\alpha\beta} = L_{\alpha\beta,xx} + L_{\alpha\beta,yy} + L_{\alpha\beta,zz} \]  

where, \( M_{\alpha\beta}, \overline{M}_{\alpha\beta} \) and \( D_{\alpha\beta} \) are the mass matrix, the lumped mass matrix and the diffusion matrix, respectively. \( L_{\alpha\beta,xx} \) and \( L_{\alpha\beta,zz} \) are the components of the diffusion matrix, which are shown in the relational expression (12). The subscripts \( xx \) and \( zz \) denote the second derivatives in the \( x \) and \( z \) directions, respectively, and the subscripts \( \alpha \) and \( \beta \) are the local node numbers in each element.

4. NUMERICAL CONDITIONS

A model analysis is conducted for a channel of \( 2\pi \delta \times \delta \times 2/3 \pi \delta \) at \( Re_f = 360 \) based on the friction velocity \( u_r \) and the channel width \( \delta \), which are the typical velocity and also the length scale in this analysis. The periodic boundary condition is imposed in the streamwise and spanwise directions. The non-slip condition is imposed on the wall. Time averaging is conducted for 20 non-dimensional time and spatial averaging is conducted at the last time step. As the spatial resolution of the grid, \( \Delta x^* \), \( \Delta y^* \) and \( \Delta z^* \), are calculated as \( Re_f \cdot h_i (i = x, y, z) \) where \( h_i (i = x, y, z) \) is the grid length in each direction. Other calculation conditions are summarized in Table 1.

![Figure 1. Analysis model](image)

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>( \Delta x^* )</th>
<th>( \Delta y^* )</th>
<th>( \Delta z^* )</th>
<th>( \Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>32 \times 64 \times 32</td>
<td>70.7</td>
<td>0.92 \sim 16.94</td>
<td>23.6</td>
<td>5.00 \times 10^{-4}</td>
</tr>
</tbody>
</table>

Table 1. Calculation conditions

5. RESULTS AND DISCUSSIONS

In this study, DNS database of Turbulent and heat transfer Laboratory at Tokyo University\(^5\) is adopted. The difference in the numerical results is shown in Figure 2 as the mean velocity profile in the streamwise directions. The distribution of the turbulence intensities is also shown in this figure. FEM and FDM in Figure 2 denote the test filtering operation used to perform discretization with each method and the horizontal axis \( y^+ \) indicates the wall coordinates scaled by the local friction velocity \( u_r \) and the kinematic viscosity \( \nu \). Filter 1 and Filter 2 in Figure 2 are the filter widths in Table 2. In the table, \( h_i (i = x, z) \) denotes the grid length in each direction.

<table>
<thead>
<tr>
<th>Filter</th>
<th>( \Delta / h_i )</th>
<th>( \Delta / h_i )</th>
<th>( \Delta / h_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Filter 1</td>
<td>2/\sqrt{3}</td>
<td>2</td>
<td>4/\sqrt{3}</td>
</tr>
<tr>
<td>Filter 2</td>
<td>2</td>
<td>2/\sqrt{3}</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 2. Ratios of the filter width and grid width

From this figure, it is confirmed that the mean velocity and turbulent intensity in the streamwise direction for each method tends to be overestimated as the filter width is made larger. The differences in the results obtained by Filter 1 and Filter 2, respectively, are more remarkable in the case of FDM than in the case of FEM. First, the global coefficient matrix of the test filtering operation is constructed from the following equation

\[ \overline{M}_{ij} \overline{\phi}_j = M_{ij} \overline{\phi}_j - \frac{\Delta x}{24} L_{ij,xx} \phi_j - \frac{\Delta z}{24} L_{ij,zz} \phi_j \]  

(13)
Figure 2. Numerical results

(a) Mean velocity profile in the streamwise direction

(b) Turbulence intensities (streamwise direction)

(c) Turbulence intensities (wall normal and spanwise direction)
where, \(i\) and \(j\) denote the component of the coefficient matrix. The inverse matrix of \(\overline{M}_{ij}\) is derived as follows:

\[
\overline{M}_{ij} = \left( \sum_j M_{ij} \right) \delta_{ij} = \left( \int_V N_i d\Omega \right) \delta_{ij}, \quad \overline{M}_{ij}^{-1} = \left( \int_V N_i d\Omega \right)^{-1} \delta_{ij} \tag{14}
\]

where \(N_i\) is a bilinear shape function at the node \(i\) and \(\delta_{ij}\) is a Kronecker’s delta. The component of the mass matrix is explained in order to clarify the matrix operation. The \(ith\) component is the integral value of the bilinear shape function which is defined at node \(i\). The value of \(N_i\) is one at the node \(i\) and is zero at the other nodes. In the case of using a hexahedral element, only eight elements which include node \(i\) are concerned with the value at node \(i\). From the integral value of \(N_i\), it is possible to identify the area which is related to the calculation at the node. That is to say, the operation of inverse matrix \(\overline{M}_{ij}^{-1}\) means that the value at the node is averaged by the elements which surround node \(i\). By operating the inverse matrix \(\overline{M}_{ij}^{-1}\) on Equation (13), \(\overline{\phi}_i\) which is the value of \(\overline{\phi}\) at the node \(i\) is calculated as the node average value as follows:

\[
\overline{\phi}_i = \frac{\overline{\phi}}{i} + \frac{\Delta^2}{24} \left( \frac{\partial^2 \overline{\phi}}{\partial x^2} \right)_i + \frac{\Delta^2}{24} \left( \frac{\partial^2 \overline{\phi}}{\partial z^2} \right)_i \tag{15}
\]

\[
\left( \overline{\phi} \right)_i \equiv \frac{M_{ij} \overline{\phi}_j}{\int_V N_i d\Omega}, \quad \left( \frac{\partial^2 \overline{\phi}}{\partial x^2} \right)_i \equiv \frac{L_{ij,xx} \overline{\phi}_j}{\int_V N_i d\Omega}, \quad \left( \frac{\partial^2 \overline{\phi}}{\partial z^2} \right)_i \equiv \frac{L_{ij,zz} \overline{\phi}_j}{\int_V N_i d\Omega} \tag{16}
\]

In order to clarify the concept of the above-mentioned node average, the formulation corresponding to FDM is shown in Figure 3, which each element is rectangular. The number at the lower right of the node is the node number. Circled numbers are the element numbers. As using a rectangular element, only eight elements around the node are needed in order to obtain the node value. Thus, it is possible to discuss the calculation of the value at node 14. The second derivatives at node 14 are shown. The value is defined as \(\left( \frac{\partial^2 \overline{\phi}}{\partial x^2} \right)_{14}\).
\[
\begin{align*}
\left( \frac{\partial^2 \phi}{\partial x^2} \right)_{14} &= \frac{S_{2,1}}{S} \left( \frac{\partial \phi}{\partial x} \right)_{\text{element:2}} - \left( \frac{\partial \phi}{\partial x} \right)_{\text{element:1}} + \frac{S_{4,3}}{S} \frac{1}{2} \left( \Delta x_1 + \Delta x_2 \right) - \frac{1}{2} \left( \Delta x_1 + \Delta x_2 \right) \\
&+ \frac{S_{6,5}}{S} \frac{1}{2} \left( \Delta x_1 + \Delta x_2 \right) + \frac{S_{8,7}}{S} \frac{1}{2} \left( \Delta x_1 + \Delta x_2 \right)
\end{align*}
\]

(17)

\( \left( \frac{\partial \phi}{\partial x} \right)_{\text{element:1}} \) denotes a weighted average value of a first derivative in each element \( i \). Concretely, \( \left( \frac{\partial \phi}{\partial x} \right)_{\text{element:1}} \) is shown as follows:

\[
\left( \frac{\partial \phi}{\partial x} \right)_{\text{element:1}} = \frac{1}{9} \left( \frac{\phi_2 - \phi_1}{\Delta x_1} \right) + \frac{2}{9} \left( \frac{\phi_5 - \phi_4}{\Delta x_1} \right) + \frac{2}{3} \left( \frac{\phi_{11} - \phi_{10}}{\Delta x_1} \right) + \frac{4}{9} \left( \frac{\phi_{14} - \phi_{13}}{\Delta x_1} \right)
\]

(18)

In each element, these values are calculated using the first derivative on the sides in \( x \) direction. In the calculation, weight is defined on each side. The weight is larger on the side located near node 14. Considering the comparison to FDM, the second derivative with respect to \( x \) is constructed through two steps.

1. First derivatives are calculated on each side using the same method of FDM, and the weighted averages are defined for each element.

2. The second derivatives are constructed from the first derivatives for each element.

In the case of FDM, the concept of the weighted average in step 1 is not necessary. Thus, in the case of FEM, considered fluctuation of the second derivatives are to be controlled as compared to FDM. Based on the above discussion, the difference between the results of FEM and FDM in Figure 2 is considered again. The difference between FEM and FDM in Figure 2 is the choice of the scheme for the spatial descretization in the formulation of the filtering operation. In the formulation, the filter widths \( \frac{\Delta^2}{24} \) and \( \frac{\Delta^2}{24} \), are related to the second derivative term. The fluctuation of the filtered value in the case of FDM is more remarkable than that in the case of FEM. That is to say, the difference between the grid and test filtered values is larger for FDM than for FEM. In the dynamic procedure, the information of motion between \( \frac{\pi}{\Delta} \) and \( \frac{\pi}{\Delta} \) in the wave number is important. The difference between the grid and test filtered values is related to the Smagorinsky constant. In order to examine the difference, the distribution of the eddy viscosity coefficient is shown in Figure 4. From the figure, it is confirmed that the influence of filter width is more remarkable for FDM than for FEM. From this figure, it is confirmed that different numerical method cause the difference in the amount of information in the wave number. As a result, the eddy viscosity coefficient is affected by the difference. The distribution of the eddy viscosity coefficient reveals the difference to be smaller for FEM than for FDM.

5. CONCLUSIONS

In this study, we examined the test filtering operation that is important in the Dynamic Smagorinsky model since the test filtering operation has a remarkable effect on the numerical results, it is necessary to confirm the influence of the numerical method on the operation. The present study, we revealed the following:
1. The concept of the inverse matrix of the mass matrix is discussed and it is confirmed that the value at the node is calculated as the node average value by the operation of the inverse matrix. In order to clarify the node average value, the formulation in the case of FEM is shown as compared with FDM.

2. In the case of FEM, as compared with FDM, the fluctuation of the second derivatives is controlled by the weighted average of the first derivatives in each element. The characteristics influence the range of motion in the wave number which is concerned with the calculation of the Smagorinsky constant in the dynamic procedure. As a result, the parameter in the case of FEM is less influential in the eddy viscosity coefficient than that in the case of FDM and the same tendencies is confirmed in the mean velocity profile and turbulence intensities.

REFERENCES


8) http://www.thtlab.t.u-tokyo.ac.jp