Identification of Chaos with Wavelet Transform Technique

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This study is concerned with the identification of chaotic vibration by continuous wavelet transform technique. The time histories to be examined are from numerical simulation of dynamic response of a nonlinear seven degrees of freedom full-vehicle model. The results of continuous wavelet transform for the time histories with five mother wavelets, Haar, Mexican Hat, Meyer, Morlet and Daubechies are discussed. It is found that the selection of mother wavelet in the identification is necessary. It is shown that the chaotic motion and almost-periodic motion can be distinguished with presented method when power spectra analysis fails to do so. As a qualitative technique, the continuous wavelet transform could be used in searching for the boundaries of chaotic and non-chaotic responses.

1. INTRODUCTION

Since it is usually difficult to find the analytical solution of a nonlinear system, the numerical analysis of chaotic response is the basic way of investigating critical states of the system. Therefore, the method for detecting chaotic motion from a sampled time history is very important and it is a research topic for many researchers. The chaotic response of a system may be identified by power spectra analysis, phase portraits, Poincaré maps, or bifurcation diagrams, but the existence of chaos is proved conclusively by a laborious, time-consuming quantitative analysis of the Lyapunov exponents. The wavelet transform (WT) is a powerful technique to decompose time series in time-frequency domain and to isolate relevant characteristics. Since WT is particularly suitable for the description of irregular states and the chaotic response of a system is usually a random-like motion, the wavelet transform could be an alternative to the above-mentioned qualitative identification techniques.

In recent years, it is found that, the random-like property of chaotic response can be observed by WT for a set of short recorded response data. Jubran et al. explored wavelet analysis as a tool to detect the chaotic behavior of flow induced vibration with modulated Gaussian basic wavelet. Wong and Chen investigated the dynamic behavior of a five degree-of-freedom Duffing’s structural system by modulus and phase of WT for sampled response and it was found that the random property of chaotic time histories in the time-frequency domain could be observed by the transform. Glabisz used discrete wavelet analysis in its classic and packet versions to search for the boundaries of chaotic and non-chaotic solutions of an one degree-of-freedom model under a dynamic non-conservative load. Although these researches are concentrated on the specific system and the reason for using specific mother wavelet is not given, their results show that chaos could be detected by WT.
Since algorithm of continuous wavelet transform (CWT) is relatively simple and time history which is to be identified needs to be sampled with much smaller sampling period than that for power spectra analysis\(^9\),\(^1\)\(^0\), the CWT can be used practically for detecting chaos. Therefore in this study, only the continuous transform is considered. It is known that the effectiveness of the WT is influenced by the selection of mother wavele\(^9\),\(^1\)\(^0\). Because the characteristics of a chaotic signal are rarely known in advance, the determination of the optimal mother wavelet is usually difficult. The unified criterion for selecting mother wavelet in detecting chaos is not yet established. The aim of this paper is to demonstrate the effectiveness of the CWT which ensures qualitative identification of chaotic states of the system and show the importance of selection of mother wavelet. This is done by a case study in which the time history is obtained from numerical simulation of a nonlinear vehicle model with seven degree-of-freedom. In the present paper, a brief introduction to the CWT is given first. Then results of numerical analysis are presented which show that the CWT is effective in detecting chaotic response and in distinguishing chaotic and almost periodic motions. At last, the selection of mother wavelet for detecting chaos is discussed.

2. CONTINUOUS WAVELET TRANSFORM AND THE NONLINEAR VEHICLE MODEL

The continuous wavelet transform is a time-scale analysis that consists of expanding signals in terms of wavelets constructed from a single function, the mother wavelet \(\psi(t)\), by means of dilations and translations\(^1\)\(^1\). The CWT of the time function \(f(t)\) is defined as

\[
W_f(a, b) = \langle f, \psi \rangle = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} \, dt \quad a > 0
\]

where \(f(t)\) is an arbitrary time function, \(W_f(a, b)\) or \(\langle f, \psi \rangle\) denotes the CWT of the function \(f(t)\), \(\overline{\psi\left(\frac{t-b}{a}\right)}\) is the conjugate form of \(\psi\left(\frac{t-b}{a}\right)\). The function \(\psi(t)\) is used as tick marks to measure signal \(f(t)\) and it is defined as

\[
\psi(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)
\]

Because all tick marks are simply dilated and translated versions of a single prototype function \(\psi(t)\), traditionally \(\psi(t)\) is called the mother wavelet. The parameter \(a\) represents the scale index, determining the center frequency of the function \(\psi\left(\frac{t-b}{a}\right)\). The parameter \(b\) indicates the time shifting or translation. The CWT in Eq. (1) takes \(f(t)\), a member of the set of square integrable functions of one real variable \(t\) in \(L^2(R)\), and transforms it to \(W_f(a, b)\), a member of the set of functions of two real variables \((a, b)\). By definition of CWT in Eq. (1), the low scale \(a\) is corresponding to high frequency of \(f(t)\) and the high scale \(a\) is corresponding to its low frequency.

In computation of the CWT, the mother wavelet \(\psi(t)\) needs to be selected. There are different kinds of mother wavelet whose qualities vary according to several criteria. It is possible that, for a same signal \(f(t)\), different results of CWT could be produced due to the selection of mother wavelets. Therefore, the most suitable mother wavelet should be decided before making the computation. In choosing the mother wavelet, there are several factors which should be considered\(^1\)\(^0\). The first is the selection of orthogonal or nonorthogonal mother wavelet. Since the orthogonality is not utilized in calculation of CWT, it will not affect the result of the transform\(^9\). According to its shape, the mother wavelet can be divided into a symmetric type, a skewsymmetric type and an asymmetric type by the symmetric property of the function. In this study, these three types of mother wavelet are used and the results for identification of chaotic responses are discussed.
Fig. 1(a)–1(e) show the wave form of Haar, Mexican Hat, Meyer, Morlet and Daubechies wavelet (db6) which are used in the analysis of present study. The detailed description of these mother wavelets can be found in reference\textsuperscript{13} and related papers. Here only the brief definitions are given. Some properties of these mother wavelets are listed in Table 1. It should be noticed that, Haar wavelet is the skewsymmetry type, Mexican Hat, Meyer and Morlet wavelet are the symmetric type, Daubechies wavelet is the asymmetric type, respectively. The only common property of the five mother wavelets is that they can be implemented in continuous wavelet transform.

\textit{Haar wavelet}

\[ \psi(t) = \begin{cases} 
1 & \text{if } -1/2 \leq t < 0 \\
-1 & \text{if } 0 \leq t < 1/2 \\
0 & \text{otherwise} 
\end{cases} \]  

\textit{Daubechies wavelet}

The Daubechies family wavelets are written dbN, where N is the order. These wavelets have no explicit expression except for db1, which is the Haar wavelet. The detailed description about this mother wavelet family can be found in the book by Daubechies\textsuperscript{14}. Most Daubechies wavelets are not symmetrical. For some, the asymmetry is very pronounced. The regularity of the shape of the
Table 1. Summary of properties of the mother wavelets

<table>
<thead>
<tr>
<th>Property</th>
<th>Haar</th>
<th>Mexican Hat</th>
<th>Meyer</th>
<th>Morlet</th>
<th>Daubechies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetry</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
</tr>
<tr>
<td>Asymmetry</td>
<td></td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
</tr>
<tr>
<td>Skewsymmetry</td>
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<td>●</td>
<td></td>
<td>●</td>
<td>●</td>
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<tr>
<td>Orthogonal analysis</td>
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<td>●</td>
<td></td>
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<td>●</td>
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<tr>
<td>Biorthogonal analysis</td>
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<td>●</td>
<td></td>
<td>●</td>
<td>●</td>
</tr>
<tr>
<td>Compactly supported orthogonal</td>
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<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
</tr>
<tr>
<td>Compactly supported biorthogonal</td>
<td></td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
</tr>
<tr>
<td>Explicit expression</td>
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<td>●</td>
<td>●</td>
<td>●</td>
<td>For splines</td>
</tr>
<tr>
<td>Continuous transform</td>
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<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
</tr>
<tr>
<td>Discrete transform</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
</tr>
</tbody>
</table>

function increases with the order \( N \). As an example, db6 is shown in Fig. 1(e).

**Mexican Hat wavelet**

\[
\psi(t) = \left( \frac{2}{\sqrt{3}} \pi^{-\frac{1}{4}} \right) (1 - t^2) e^{-\frac{2}{3} t^2}
\]

**Meyer wavelet**

\[
\psi(\omega) = \begin{cases} 
\sqrt{\frac{e^{i\omega}}{2\pi}} \sin \left( \frac{\pi}{2} \left( \frac{3}{2\pi} |\omega| - 1 \right) \right) & \text{if } \frac{2\pi}{3} \leq |\omega| \leq \frac{4\pi}{3} \\
\sqrt{\frac{e^{i\omega}}{2\pi}} \cos \left( \frac{\pi}{2} \left( \frac{3}{4\pi} |\omega| - 1 \right) \right) & \text{if } \frac{4\pi}{3} \leq |\omega| \leq \frac{8\pi}{3} \\
0 & \text{otherwise}
\end{cases}
\]

where \( v(a) = a^4 \left( 35 - 84a + 70a^2 - 20a^3 \right) \quad a \in [0, 1].

**Morlet wavelet**

\[
\psi(t) = Ce^{-\frac{t^2}{2}} \cos (5t)
\]

where constant C is used for normalization in view of reconstruction. The nonlinear full-vehicle model is shown in Fig. 2. It consists of a single sprung mass \( m_s \) connected to four unsprung masses (front-left, front-right, rear-left, rear-right wheels) at each corner. The sprung mass is free to heave \( z(t) \), pitch \( \phi(t) \) and roll \( \theta(t) \) while the unsprung masses are free to bounce vertically with respect to the sprung mass \( z_{ul}(t), z_{ulr}(t), z_{ur}(t), z_{urr}(t) \). Therefore, the mechanical model has seven degree-of-freedom. The suspensions between the sprung mass and unsprung masses are modeled as nonlinear spring elements and nonlinear dampers, while the tires are modeled as nonlinear springs with viscous damping. The sinusoid forcing function is used to describe the excitations \( \phi(t), \theta(t), z_{ul}(t), z_{ulr}(t), z_{ur}(t), z_{urr}(t) \) which are caused by bumpy road surface.

3. NUMERICAL RESULTS

For a certain choice of parameters, the full-vehicle model shown in Fig. 2 behaves chaotically. The chaotic motion is confirmed quantitatively by calculation of the dominant Lyapunov exponent.
Fig. 3(a) and Fig. 4(a) show the chaotic response of the sprung mass $m_s$ in the time domain and the frequency domain, respectively. Here, the sampling period is $T = 0.018$ s and the dominant Lyapunov exponent is 0.111 in the unit of bits/s. On the other hand, the time history of almost-periodic response of $m_s$ and corresponding power spectral density are shown in Fig. 3(b) and Fig. 4(b), respectively. The power spectral density in Fig. 4(b) looks like broad band. However, there are several strong frequency components which indicate the harmonic response. In this case, one cannot distinguish whether the response is chaotic or not by time history or power spectral density.

The results of continuous wavelet transform are shown in Figs. 5~11. In these figures, the horizontal axis represents translation $b$ which corresponding to time. The vertical axis represents scale $a$. The color at each $x$--$y$ point represents the magnitude of the wavelet coefficient $W_f(a,b)$. Larger coefficient is reflected by brighter colors. These CWT coefficient plots are precisely the time-scale view of the original time history signals.

Fig. 5(a) shows the amplitude behaviors of chaotic time history given in Fig. 3(a) in the time-scale domain when Haar wavelet is selected as the mother wavelet. The amplitude behaviors of almost periodic time history in Fig. 3(b) is shown in Fig. 5(b). It is clear that, in Fig. 5(a) the change of pattern along the horizontal axis is indistinct. Furthermore, the difference between the pattern in Fig. 5(a) and Fig. 5(b) is not clear, which indicates that it is difficult to distinguish the chaotic and almost periodic motions by the results of the CWT when Haar wavelet, a skewsymmetry type, is used.

The CWT for the time history in Fig. 3 is shown in Fig. 6 when Mexican Hat wavelet is used. In Fig. 6(b), the pattern of straight contours paralleling the vertical axis indicate that there is not significant change in amplitude and frequency of the original time history. On the other hand, the changing pattern in Fig. 6(a) implies the changes in amplitude and frequency of the analyzed time history. Thus, the chaotic and almost periodic motions can be distinguished by comparison of the results of CWT. However, the patterns in high scale are not clearly visible.

The differences between chaotic and almost periodic time history can be seen more clearly by applying Meyer or Morlet mother wavelet. Both of Fig. 7(a) and Fig. 8(a) show that, for the chaotic time history, the patterns vary irregularly along the translation axis. Moreover, in the result of the CWT for almost periodic time history, there is not significant change of pattern. This can be seen from Fig. 7(b) and Fig. 8(b). The results of CWT by using the symmetric type of mother wavelet, Mexican Hat, Meyer and Morlet, are found to be in agreement.
Figure 3: Response of $m_3$: (a) time history of chaotic motion; (b) almost-periodic motion.

Figure 4: Power spectral density: (a) for chaotic motion; (b) for almost-periodic motion.

Figure 5: The CWT for the time history in Fig. 3 as Haar wavelet is used: (a) for chaotic response; (b) for almost-periodic response.

Figure 6: The CWT for the time history in Fig. 3 as Mexican Hat wavelet is used: (a) for chaotic response; (b) for almost-periodic response.
Figure 7: The CWT for the time history in Fig. 3 as Meyer wavelet is used: (a) for chaotic response; (b) for almost-periodic response.

Figure 8: The CWT for the time history in Fig. 3 as Morlet wavelet is used: (a) for chaotic response; (b) for almost-periodic response.

Figure 9: The CWT for the time history in Fig. 3 as Daubechies wavelet (db2) is used: (a) for chaotic response; (b) for almost-periodic response.

Figure 10: The CWT for the time history in Fig. 3 as Daubechies wavelet (db6) is used: (a) for chaotic response; (b) for almost-periodic response.
Figure 11: The CWT for the time history in Fig. 3 as Daubechies wavelet (db10) is used: (a) for chaotic response; (b) for almost-periodic response.

As the results of the CWT by using Daubechies wavelet, Figs. 9~11 show that the chaotic and almost periodic time history can be distinguished if the order of the mother wavelet is chosen properly. Fig. 9 shows the transform results as the order $N = 2$. It is difficult to see the difference between the patterns in Figs. 9(a) and 9(b). When the order $N = 6$, the variation of pattern in Fig. 10(a) can be observed clearly. On the other hand, the pattern for non-chaotic time history in Fig. 10(b) appears not change over time. When the order of Daubechies wavelet is increased to 10, by comparison of results of CWT which are shown in Figs. 11(a) and 11(b), the change of pattern for chaotic time history and no change for non-chaotic time history become more clearly.

It should be mentioned that, to detect the non-regular signal, such as a pulse, in a time history which is generated from a second order differential equation, the order of Daubechies wavelet needs to be greater than 7. However, for detecting chaos in the seven degrees of full-vehicle model, db6 gives a satisfactory result. The Daubechies mother wavelet is asymmetric type. From $N = 2$ and with the increase of the order $N$, the shape of Daubechies wavelet tends to symmetric.

4. CONCLUDING REMARKS

This limited investigation aims at highlighting the effectiveness of using continuous wavelet transform for distinguishing chaotic response and almost periodic response of a full nonlinear vehicle mode. The main conclusions deduced from the present investigation are as follows:

1. The chaotic and almost periodic response of the system can be detected and distinguished by observing the pattern changes in the results of CWT. Chaotic and almost periodic responses can also be distinguished. By applying CWT, the behavior of the almost periodic motion can be seen more clearly than by applying power spectra analysis. Since the responses of a nonlinear system often vary from almost periodic motion to chaotic one, CWT could be a useful tool in searching for the boundaries of chaotic and non-chaotic responses.

2. The mother wavelet needs to be selected. The results in Figs. 5~11 show that, the shape of mother wavelet will affect the results of the transform. It is known that, the mother wavelet should reflect the type of features present in the time series. For time series with sharp jumps or steps, one would choose a boxcar-like function such as the Haar, while for smoothly varying time series one would choose a smooth function such as a damped cosine. For the time history shown in Fig. 3, there not only exist sharp jumps, but also the relatively smooth variations. In this case, the using skewsymmetry type function, such as Haar mother wavelet will not give the ideal results for the identification. On the other hand, the three kinds of symmetry type functions, Mexican Hat, Meyer and Morlet mother wavelet can be used in the identification. However, the results of transform by using Meyer and Morlet mother
wavelet are better. It should be noticed that, the shape of Meyer and Morlet mother wavelet is more complex than that of Mexican Hat. Using Daubechies wavelet can also obtain the good result if \( N \geq 6 \) for this case study. Therefore, Daubechies, Meyer and Morlet mother wavelet could be a good candidate for the purpose of identification of chaos. The results of this case study show that, the symmetry and asymmetric type function which has complex shape would be more suitable for detecting chaos. The discussion on general rule of optimal selection of wavelet for identifying chaotic motion is left for further research.

3. The identification of chaotic motion with CWT is made by observation of pattern of the results. Thus, for a same result of CWT, different conclusions may be obtained by different observers. The quantitative measurement for evaluating change of pattern in the result of CWT is needed.

REFERENCES


11) S. Qian, Introduction to Time-frequency and Wavelet Transforms, (Prentice Hall PTR, 2002).


