Option Hedging Strategy with Transaction Costs

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Option replication is discussed in a discrete-time framework with transaction costs. The model represents an extension of the \( m(\geq 3) \)-nomial option pricing model to cover the case of proportional transaction costs. The method is based on an interval by a locally risk-minimizing strategy.

1. INTRODUCTION

The Black-Scholes\(^1\) option pricing methodology relies on perfect replication of the option payoff by a continuously rebalanced hedging portfolio involving the underlying stock. It is therefore inapplicable in markets with transaction costs, as the hedging costs would be ruinously expensive. Furthermore, in practice, the portfolio is rebalanced in discrete time, where the Black-Scholes methodology is also inapplicable. Here we propose a approach to the pricing and hedging of European option under transaction costs on trading the underlying stock in a general incomplete market in discrete time.

Attempts to circumvent these problems include seminal works of Leland\(^7\), Merton\(^9\) and Boyle and Vorst\(^9\), who used a fixed hedging time scale. This approach utilizes self-financing strategies which exactly replicate the option payoff at maturity or non-self-financing strategies which minimize hedging error. Our approach is similar to those. Bensaid et al\(^5\) replaced the replication strategy with a super-replicating strategy, in which the hedging portfolio is only required to dominate, rather than replicate, the option payoff at maturity. The third approach introduces preferences, usually in the form of utility functions, to obtain a valuation formula; proponents of this approach are for instance Hodges and Neuberger\(^6\), Davis\(^4\) and Monoyios\(^10\).

Merton\(^9\) and Boyle and Vorst\(^9\) discuss the option replication with transaction costs in the binomial option pricing model and show the exact replicating procedure. In practice, however, most contingent claims in an incomplete market will not permit the construction of such a strategy. In this paper, compared to these works, we discuss the European option replication with transaction costs in the \( m(\geq 3) \)-nomial option pricing model which turns out to be rather practical. Intuitively, the strategy, called locally risk-minimizing strategy, minimizes the hedging error locally.

On the other hand, Monoyios\(^10\) points out that the pricing bounds become wider as the hedging
error is reduced. It is correct. In a sense that any strategies, the pricing bounds become wider as the number of rebalancing the dynamic portfolio increases. Therefore in this paper we also propose an approach to decide the optimal number of rebalance in this paper.

The rest of the paper is as follows. Section 2 introduces our model, defines locally risk-minimizing strategy. Section 3 proposes the procedure for finding the optimal number of rebalancing the dynamic portfolio. We explore the numerical properties of prices and hedge error in Section 4. Section 5 concludes the paper.

2. OPTION REPLICATION WITH TRANSACTION COSTS

We employ risk-minimizing arguments to establish the cost of creating a long European call option by dynamic hedging when there are transaction costs in the \( m \geq 3 \)-nomial option pricing model. Let \( (\Omega, \mathcal{F}, P) \) be a probability space with a filtration \( \mathcal{F} = (\mathcal{F}_t)_{t=0,1,\cdots,n} \) for some fixed time horizon \( n \in \mathbb{N} \). The stochastic process \( S = (S_t)_{t=0,1,\cdots,n} \) describes some risky asset, here called stock, and we assume that \( S \) is adapted to \( \mathcal{F} \) and nonnegative. There is also a riskless asset called bond. \( K \) denotes strike price and \( n \) denotes the maturity time of the European call option. \( R \) is equal to one plus the one-period riskless rate.

For any stochastic process \( Y = (Y_t)_{t=0,1,\cdots,n} \), we denote by \( \Theta(Y) \) the space of all predictable processes \( \phi = (\phi_t)_{t=0,1,\cdots,n} \) such that \( \phi_{t-1} \Delta Y_t \in L^2(P) \) for \( i = 1, 2, \cdots, n \), where \( \Delta Y_t := Y_t - Y_{t-1} \).

A trading strategy \( \gamma \) is a pair of processes, \( \phi, \psi \) such that \( \phi \in \Theta(S) \), \( \psi = (\psi_t)_{t=0,1,\cdots,n} \) is adapted, and \( V_i(\gamma) := \phi_i S_t + \psi_i \in L^2(P) \) for \( i = 0, 1, \cdots, n \). The symbol \( \phi_i \) denotes the number of shares and \( \psi_i \) denotes the value of the share of the bond \( i = 0, 1, \cdots, n \). As we consider a long European call option, at the maturity, we can assume that if the stock price is in the money then \( \phi_n = 1 \), \( \psi_n = -K \) and if the stock price is out of the money then \( \phi_n = 0 \), \( \psi_n = 0 \) for calculation.

To introduce transaction costs, assume that proportional transaction costs are incurred when shares of the risky asset (stock) are traded. Let \( k \) be the transaction costs rate measured as a fraction of the amount traded. We assume that the transaction costs are not occurred at time \( 0 \).

Let us consider the replicating portfolio \( (\phi, \psi) \) at time \( i = 0, 1, \cdots, n-1 \). Since we consider the replicating strategy backward through time, we can assume that the portfolio \( (\phi_t, \psi_t)_{t=i+1,i+2,\cdots,n} \) is already decided.

Just before the trade at time \( i + 1 \) the portfolio amounts to

\[
\phi_i(S_i) S_{i+1} + \psi_i(S_i) R.
\]

(1)

When the trade is occurred we need

\[
\phi_{i+1}(S_{i+1}) S_{i+1} + \psi_{i+1}(S_{i+1}) + k | \phi_i(S_i) - \phi_{i+1}(S_{i+1}) | S_{i+1}.
\]

(2)

If possible, we want to pay the cost of (2) by (1). But, since we utilize the \( m \geq 3 \)-nomial option pricing model which is the incomplete market model, (1) and (2) can not be equal (if we use binomial model they can). In other words, we can not construct a self-financing strategy. Therefore we want to make the difference between (1) and (2) (hedging error) minimum.

Now we denote by \( Z_i(S_{i+1}) \) the difference between (1) and (2) as

\[
Z_i(S_{i+1}) := [\phi_i(S_i) S_{i+1} + \psi_i(S_i) R]
\]

\[
-|\phi_{i+1}(S_{i+1}) S_{i+1} + \psi_{i+1} + k | \phi_i(S_i) - \phi_{i+1}(S_{i+1}) | S_{i+1}].
\]

(3)

**Definition 1**

The locally risk process of a strategy \( \gamma \) is

\[
M_i(\gamma, S_i) := E[Z(S_{i+1})^2 | S_i]
\]

for \( i = 0, 1, \cdots, n-1 \).

(4)

An \( L^2 \)-admissible strategy \( \gamma \) is called a locally risk-minimizing strategy if

\[
M_i(\gamma, S_i) \leq M_i(\gamma, S_i) \text{ P-a.s.}
\]

(5)
for all $i$ and each $L^2$-admissible strategy $\gamma$. □

According to the above definition, if $M_t$ is minimized then the hedging error at time $i$ is minimized on average.

We assume that if $S_i R \leq S_i+1$ then $\phi_i \leq \phi_i+1$ and if $S_i R > S_i+1$ then $\phi_i > \phi_i+1$. Hence we can rewrite (3) as

$$Z(S_{i+1}) = \begin{cases} 
\phi_i S_{i+1}(1 + k) + \psi_i R - (\phi_{i+1} S_{i+1}(1 + k) + \psi_{i+1}) & \text{if } S_i R \leq S_i+1 \\
\phi_i S_{i+1}(1 - k) + \psi_i R - (\phi_{i+1} S_{i+1}(1 - k) + \psi_{i+1}) & \text{if } S_i R > S_i+1 
\end{cases}$$

where

$$F[S_{i+1}] = \begin{cases} 
S_{i+1}(1 + k) & \text{if } S_i R \leq S_i+1 \\
S_{i+1}(1 - k) & \text{if } S_i R > S_i+1 
\end{cases}$$

$$L[S_{i+1}] = \begin{cases} 
\phi_{i+1} S_{i+1}(1 + k) + \psi_{i+1} & \text{if } S_i R \leq S_i+1 \\
\phi_{i+1} S_{i+1}(1 - k) + \psi_{i+1} & \text{if } S_i R > S_i+1 
\end{cases}$$

Because of (4) and (6), if $\gamma_i = (\phi_i, \psi_i)$ satisfies (5) then $(\phi_i, \psi_i)$ is the solution of $\partial M_t[\phi_i, \psi_i]/\partial \phi_i = 0$, $\partial M_t[\phi_i, \psi_i]/\partial \psi_i = 0$. And the solution is

$$\phi_i(S_i) = \text{Cov}(F[S_{i+1}], L[S_{i+1}] | S_i)$$

$$\psi_i(S_i) = \frac{E(L[S_{i+1}] | S_i) - \phi_i(S_i) E(F[S_{i+1}] | S_i)}{R}$$

After all, the lowest initial value $C(0, S_0)$ which satisfies the locally risk-minimizing strategy is

$$C(0, S_0) = \phi_0 S_0 + \psi_0.$$ 

We remark that even if we consider the binomial model, the locally risk-minimizing strategy is satisfied. In other words, Merton's or Boyle and Vorst's strategies are the special cases of the locally risk-minimizing strategy developed here.

3. OPTIMAL NUMBER OF REBALANCING THE DYNAMIC PORTFOLIO

In the market with transaction costs, as $n$ (number of rebalancing the portfolio) is increased the value $C(0, S_0)$ is increased, but the total hedge error is decreased. We construct the optimal number of rebalance using this property. Under the given $k$ (transaction cost rate), we denote by $H(i, k) = Z(S_i)/R^i$ the discounted hedge error at time $i$. The total discounted hedge error is denoted by $\sum_{i=1}^{n} H(i, k)$ which is provided by simulations and its standard deviation is denoted by $\sqrt{E[(\sum_{i=1}^{n} H(i, k))^2]}$ under the assumption of zero mean. The standard deviation is decreased as $n$ is increased. If a test value is denoted by

$$C(0, S_0) + \sqrt{E\left[\left(\sum_{i=1}^{n} H(i, k)\right)^2\right]},$$

then we can decide the optimal number $n$ which minimize the test value (7). This way of consideration is somewhat similar to the value at risk (VaR).
4. IMPLEMENTATION

In this section we compute long call option prices and hedge errors for a range of parameter values. For Table 1, stock prices are leaded by the 4-nomial tree model. The 4-nomial tree model is constructed by the binomial tree model. In other words, when we write down the binomial tree model, at the third term, we find that there are four nodes. We regard the third term of the binomial tree model as the first term of the 4-nomial model. So do the later terms. We apply the binomial model of Jarrow and Rudd.

We take the current price of stock to be 100, the strike price of the European call option to be 100, the time to option expiry to be one year, and the interest rate to be 1%. For our base case assumptions, the drift and the standard deviation of the return on the stock price are 5%, 15% respectively, which are chosen suitably. We assume that the transaction cost is not occurred at the maturity time.

Table 1 provides the long call prices and total hedge error for a range of number of rebalancing the dynamic portfolio and transaction cost rate. [Test] denotes (7). From Table 1 we can find that as the number of rebalance is increased the price is increased and the total discounted hedge error is decreased. According to the procedure of Section 3 which finds the optimal number of rebalance, when the transaction cost rate is 0%, the optimal number of rebalance is 500 among the choices of 50, 100, 250 and 500 because [Test] number of 500 is minimum. As checking in the same way, when the transaction cost rate is 0.1% the optimal number is 100. When the transaction cost rate is 0.5% or 1.0% or 2.0%, the optimal number is 50. From these results, we may conclude that if the transaction cost rate is large then we should not rebalance the dynamic portfolio too much.

The Black-Scholes price of the previous parameters is 6.459. It is near the value of Table 1 when transaction cost rate is 0%. From this result, the locally risk-minimizing strategy may have strong connection with the no-arbitrage theory, which is discussed in part in Föllmer and Schied.

5. CONCLUSION

This paper derived a procedure for computing option prices under transaction costs on trading the underlying stock in a general incomplete market in discrete time, and proposed a procedure for finding the optimal number of rebalancing the dynamic portfolio. Since these procedures provide plausible parameter values and are easy to implement, we believe that our methods are ready to be in use.

Table 1: Stock price = Strike price = 100, Drift = 5%, Volatility = 15%, time to expiry = 1 year, interest rate = 1%. The number of simulations = 10,000. [Mean] is the mean of the total discounted hedge error and [SD] is its standard deviation. [Test] is defined by (7). We use the 4-nomial tree model.

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<tr>
<th>Hedging number</th>
<th>Cost</th>
<th>Mean</th>
<th>SD</th>
<th>Price</th>
<th>Test</th>
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ACKNOWLEDGMENTS

I would like to thank Naoyuki Ishimura, Takehiko Fujita and Koichiro Takaoka who are my teachers in Hitotsubashi University for their advice. In particular Ishimura gave me a chance of writing of this paper.

I am especially grateful to Noriko Yamamoto, who inspired, encouraged and motivated me.

REMARK

The view expressed here are those of the author and do not reflect those of Mizuho Securities Co., Ltd.

REFERENCES


