Relaxation Oscillations of Point Vortices in a Plane

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The motion of point vortices in a plane is considered. The aim of this paper is to discuss the vortices which exhibit the relaxation oscillations. Five vortices as well as three ones are treated. The numerical simulations and mathematical result are shown.

1. INTRODUCTION

In this paper, we consider the motion of assembly of vortices in the two-dimensional Euler flow. If we use the point vortex formulation, the problem is described by the ordinal differential equation

\[
\frac{d}{dt} z_j(t) = \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{\Gamma_k}{z_j(t) - z_k(t)} \quad (j = 1, 2, \ldots, n),
\]

(1)

where \( z_j(t) \) and \( \Gamma_j \) are the complex coordinate and the circulation of \( j \)th point vortices, respectively, \( n \) is the number of vortices in the fluid, \( \overline{z} \) is the complex conjugate of \( z \) in \( C \), and \( i = \sqrt{-1} \). Since the circulation is independent of time by Kelvin’s circulation theorem\(^9\), we treat \( \Gamma_j \) as a given constant \((j = 1, 2, \ldots, n)\). The equation (1) is considered with the initial condition

\[
z_j(0) = \xi_j \quad (j = 1, 2, \ldots, n),
\]

(2)

where \( \xi_j \in C \) is a given number \((j = 1, 2, \ldots, n)\).

Let us briefly mention some known results on (1). When \( n = 2 \), the distance between two vortices \(|z_j(t) - z_k(t)|\) is constant and it is easy to analyze the behavior of vortices. Aref\(^1\) shows the qualitative analysis of three point vortices. Novikov\(^6\) and Selivanova\(^11\) also consider the case of \( n = 3 \). For \( n \geq 4 \), the equation (1) is not solved yet in general case, however, some special cases are treated. For examples, Aref and Pomphrey\(^2\) show the chaotic behavior of four point vortices. Newton\(^9\) also treats the case of \( n = 4 \) as well as that of \( n = 3 \). Nakaki\(^8\) analyzes the stability of the relative equilibria for \( n = 5 \). Morikawa and Swenson\(^5\) consider the case where \( \Gamma_j = 1 \) and \( \xi_j = \exp(2\pi i (j - 1)/n) \) \((j = 1, 2, \ldots, n)\) (see also Cabral and Schmidt\(^9\)). The
relaxation oscillation is treated by Nakaki\(^{5}\) for five point vortices which satisfy

\[
\Gamma_j = \begin{cases} 
1 & \text{if } 1 \leq j \leq 4, \\
-1.5 & \text{if } j = 5,
\end{cases} \quad \xi_j = \begin{cases} 
\exp \{ i \pi (j - 1) / 2 \} & \text{if } 1 \leq j \leq 4, \\
0 & \text{if } j = 5.
\end{cases}
\tag{3}
\]

The aim of this paper is to study another configurations of vortices which exhibit relaxation oscillations. In the following section, we describe the oscillations of five point vortices in a diamond-shaped configuration\(^{7}\). Some numerical simulations are demonstrated. The oscillation of three vortices is treated in Section 3, and a mathematical result is shown.

2. RELAXATION OSCILLATION OF FIVE POINT VORTEXES

In this section, we deal with the motion of five point vortices, which satisfy

\[
\left\{ \begin{array}{l}
\Gamma_1 = \Gamma_3 = 1, \\
\Gamma_2 = \Gamma_4 = \Gamma_v, \\
\xi_1 = -\xi_3 = 1, \\
\xi_2 = -\xi_4 = bi, \\
\xi_5 = 0,
\end{array} \right.
\tag{4}
\]

where \( b \in (0, 1) \) and \( \Gamma_v \in \mathbf{R} \) are parameters, and \( \Gamma_v \in \mathbf{R} \) is the constant satisfying

\[
\exp \{-i \Omega t \} z_j(t) = z_j(0) \quad (j = 1, 2, \ldots, 5, \ t \geq 0)
\tag{5}
\]

for some \( \Omega \in \mathbf{R} \). In other words, \( \{z_j(t)\} \) is a relative equilibrium to (1). Such a value of \( \Gamma_v \) exists. In fact, it follows that

\[
\Gamma_v = \frac{-b^4 - 3b^2(\Gamma_v - 1) + \Gamma_v}{2(b^4 - 1)}, \quad \Omega = \frac{-1 + 3\Gamma_v - b^2(\Gamma_v + 3)}{4\pi(b^4 - 1)}.
\tag{6}
\]

To fix the rotation with angular velocity \( \Omega \), let us define \( w_j(t) = z_j(t) \exp \{-i \Omega t \} \). Then (1) is rewritten as

\[
\frac{d}{dt} w_j(t) = \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{\Gamma_k}{w_j(t) - w_k(t)} + i\Omega w_j(t) \quad (j = 1, 2, \ldots, n).
\tag{7}
\]

For this problem, Nakaki\(^{5}\) shows by numerical simulations such that, if the initial value is close to the equilibrium, the solution to (7) exhibits the relaxation oscillation (see Fig. 1). This fact can be explained by the existence of the chain of heteroclinic orbits of (1): By the assumption on the initial value, the solution to (1) travels along the heteroclinic orbit and approaches to the another equilibrium. After that the solution moves toward the third equilibrium along another heteroclinic orbit. The motion is slow if it is near the equilibria and fast if the solution is far from the equilibria. As a result, the solution exhibits the relaxation oscillation.

For the simple case where \( \Gamma_v = 0 \), we obtain the existence of heteroclinic orbit.

**Theorem 2.1** (Nakaki\(^{5}\)). Let \( \Gamma_v = 0 \) and \( b \in (0, 1) \). Then there exists a solution to (7) satisfying

\[
w_j(-\infty) = \xi_j, \quad w_j(+\infty) = \xi_j \quad (j = 1, 2, \ldots, 5).
\tag{8}
\]

This theorem implies the existence of a heteroclinic orbit. Similarly we can construct the heteroclinic orbit which starts from \( \{\xi_j\} \) at \( t = -\infty \) and goes to another equilibrium at \( t = +\infty \). Hence we find the chain of heteroclinic orbits.

For \( \Gamma_v \neq 0 \), the existence of heteroclinic orbit is unknown. However, our numerical simulations suggest that the heteroclinic orbits exist for \( 0 < b < 1 \) and \( -b^{-2} < \Gamma_v < 1 \).
The aim of this section is to study the relaxation oscillations from a numerical point of view. As a typical case, we set $\beta = 0.6$. The range of $\Gamma_\nu$ is $-\beta^{-2} = -2.77 \cdots < \Gamma_\nu < 1$. The initial condition is

$$w_1(0) = -w_3(0) = 1, \quad w_2(0) = -w_4(0) = (b - 10^{-7})i, \quad w_5(0) = 0.$$  \hspace{1cm} (9)

Fig. 2 shows the velocity of vortices defined by $v(t) = \frac{1}{\beta} \sum_{j=1}^{n} \frac{dz_j(t)}{dt}$ for $\Gamma_\nu = -2, -1, 0, 0.5$. One can clearly observe the relaxation oscillation. We define the "period" of oscillation by the time difference between two peaks in Fig. 2. As shown in the figure, the "period" varies with respect to $\Gamma_\nu$. So does the maximum velocity of vortices. By our numerical simulation (see Fig. 3), we find that

- The "period" becomes longer as $\Gamma_\nu \uparrow 1$ or $\Gamma_\nu \downarrow -2.77 \cdots$;
- The maximum velocity decreases with respect to $\Gamma_\nu$.

From this simulation, one may expect that, as $\Gamma_\nu \uparrow 1$, the oscillation becomes slower and the "period" becomes longer. On the other hand, when $\Gamma_\nu \approx -2.77 \cdots$, we find the following by numerical simulations shown in Fig. 4 where the velocity $v(t)$, the argument of vortex $z_1$ defined by $\theta(t) = \text{arg} z_1(t) = \arctan \frac{2z_1(t)}{z_{11}(t)}$, and the diameter of vortices defined by $d(t) = \max_{j,k} |z_j(t) - z_k(t)|$ are displayed: The vortices gather around the origin, rotate with high angular velocity, and go back to the diamond-shaped configuration.

3. RELAXATION OSCILLATION OF THREE POINT VORTICES

We show in this section that some three point vortices exhibit the relaxation oscillation. Let us consider the ordinary differential equation (1) with $n = 3$ and

$$\Gamma_1 = \Gamma_2 = 2, \quad \Gamma_3 = -1$$  \hspace{1cm} (10)

subject to the initial condition (2), where

$$\xi_1 = -\xi_2 = 1, \quad \xi_3 = 0.$$  \hspace{1cm} (11)
Fig. 2: Time evolution of velocity of vortices for $\Gamma_v = -2, -1, 0, 0.5$.

Fig. 3: Left: "Period" of oscillation versus $\Gamma_v$. Right: Maximum velocity of vortices versus $\Gamma_v$. 
It is easy to verify that $(\xi_1, \xi_2, \xi_3)$ is an unstable equilibrium to (1). Hence, by a small perturbation, the solution goes away from the equilibrium. The numerical simulation, under the initial condition
\[ z_j(0) = \xi_j + \epsilon(-1+i)/4 \quad (j = 1, 2), \quad z_3(0) = \xi_3 + \epsilon(-1+i), \quad \epsilon = 10^{-17}, \]
\[ (12) \]
is shown in Fig. 5. One can observe that the relaxation oscillation occurs. In this case, we can show the existence of heteroclinic orbit as follows.

**Theorem 3.1** Let $n = 3$ and assume (10). Then there exists a solution to (1) satisfying
\[ z_j(-\infty) = \xi_j, \quad z_j(+\infty) = \eta_j \quad (j = 1, 2, 3) \]
\[ (13) \]
for some $\eta_j \in C$ $(j = 1, 2, 3)$.

The configurations of $(\xi_1, \xi_2, \xi_3)$ and $(\eta_1, \eta_2, \eta_3)$ are shown in $E_1$ and $E_3$ of Fig. 6, respectively.

Proof of Theorem 3.1. The proof shall be done by using idea of Aref\textsuperscript{1}). He introduces the phase plane with trilinear axes $b_1$, $b_2$ and $b_3$ defined by
\[ b_1 = \frac{\Gamma_3}{\Gamma_1 C}, \quad b_2 = \frac{\Gamma_1}{\Gamma_2 C}, \quad b_3 = \frac{\Gamma_2}{\Gamma_3 C}, \]
\[ (14) \]
Fig. 6: Three collinear configurations of vortices. $E_1$ and $E_3$ are equilibria.

where $l_{jk} = |z_j(t) - z_k(t)|$ and $C \in \mathbb{R}$. Since $\sum_{j \neq k} \Gamma_j \Gamma_k l_{jk}^2$ is constant in time $t$ (see Aref\(^1\), for example), the number $C$ becomes constant, which is determined to satisfy

$$b_1 + b_2 + b_3 = 3. \quad (15)$$

On the other hand, (1) has the Hamiltonian $H = -\frac{1}{4\pi} \sum_{j \neq k} \Gamma_j \Gamma_k \log l_{jk}$, which yields that

$$|b_1|^{1/T_1}|b_2|^{1/T_2}|b_3|^{1/T_3} \text{ is constant in } t. \quad (16)$$

It is also known that

$$\frac{d}{dt} l_{12}^2 = \frac{2}{\pi} \Gamma_3 \sigma A \left( \frac{1}{l_{23}^2} - \frac{1}{l_{13}^2} \right), \quad \frac{d}{dt} l_{23}^2 = \frac{2}{\pi} \Gamma_1 \sigma A \left( \frac{1}{l_{12}^2} - \frac{1}{l_{13}^2} \right), \quad \frac{d}{dt} l_{13}^2 = \frac{2}{\pi} \Gamma_2 \sigma A \left( \frac{1}{l_{12}^2} - \frac{1}{l_{23}^2} \right), \quad (17)$$

where $A$ is the area of triangle spanned by three point vortices $z_1$, $z_2$, and $z_3$, and

$$\sigma = \begin{cases} 1, & \text{if } z_1, z_2, z_3 \text{ appear in counterclockwise order,} \\
-1, & \text{if } z_1, z_2, z_3 \text{ appear in clockwise order.} \end{cases} \quad (18)$$

The solution to (17) determines the relative configuration of three point vortices $z_1$, $z_2$, $z_3$, and what we should do is to solve (17).

Our proof is done by constructing a homoclinic orbit in the phase plane on $b_1, b_2, b_3$ (we note that the two collinear equilibria $E_1$ and $E_3$ correspond to the same point $(b_1, b_2, b_3) = (\frac{-1}{2}, \frac{-1}{2}, 4)$). See Fig. 7. To this end, by using two invariants (15) and (16), let us lead an ordinary differential equation on $b_1$.

By assumptions (10), (11) and (13), we have $C = -1$ and

$$b_1 = -\frac{l_{23}^2}{2}, \quad b_2 = -\frac{l_{31}^2}{2}, \quad b_3 = l_{12}^2. \quad (19)$$

The invariant (16) is rewritten as

$$64b_1b_2 = b_3^2. \quad (20)$$

Combining this and (15) gives $b_2 = 3 + 31b_1 \pm 8\sqrt{3b_1(1 + 5b_1)}$. Our homoclinic orbit is constructed so as to satisfy

$$b_2 = \phi(b_1) \equiv 3 + 31b_1 + 8\sqrt{3b_1(1 + 5b_1)}. \quad (21)$$
In this case, from (15) and (17), we have

\[
\frac{d}{dt}b_1 = \frac{2}{\pi} \sigma A f,
\]

where \( f = f(b_1) = \frac{1}{2\pi b_1} + \frac{1}{3 - b_1 - \sqrt{b_1}} \). From (15), (19) and (21), it follows that, if a function \( b_1(t) \) satisfies (22), three ordinary differential equations in (17) hold. Hence, it remains only to prove the existence of homoclinic orbit to (22).

We shall construct the solution to (22) connecting two collinear configurations \( E_1 \) and \( E_2 \) shown in Fig. 6. A straightforward calculation gives

\[
f = 0 \quad \text{if and only if} \quad b_1 = \frac{1}{5},
\]

\[
A = 0 \quad \text{if and only if} \quad b_1 = \frac{1}{2} - \frac{3(3 + 2\sqrt{2})}{2}.
\]

We note that \( b_1 = \frac{3(3 + 2\sqrt{2})}{2} \) corresponds to the collinear equilibrium \( E_2 \). Let us construct the solution \( b_1(t) \) which satisfies \( b_1(0) = \beta_1 \) and \( \sigma \left|_{t=0} = 1 \right. \), where \( \beta_1 \in \left( -\frac{3(3 + 2\sqrt{2})}{2}, \frac{1}{2} \right) \) is a given constant. Since

\[
\frac{d}{dt} b_1(0) = \frac{2}{\pi} \sigma A f \big|_{t=0} < 0
\]

and the right hand side of (22) is bounded and Lipschitz continuous on \( \left( -\frac{3(3 + 2\sqrt{2})}{2}, \frac{1}{2} \right) \), there exists a decreasing function \( b_1(t) \) defined on \( (-\infty, t^*) \) satisfying (22), \( \sigma = 1 \) and

\[
\begin{align*}
 b_1(-\infty) &= -\frac{1}{2}, \quad b_1(t^*) = -\frac{3(3 + 2\sqrt{2})}{2} \\
 b_1(-\infty) &= -\frac{1}{2}, \quad b_1(t^*) = -\frac{3(3 + 2\sqrt{2})}{2}
\end{align*}
\]

for some \( t^* > 0 \). Here we use the facts that \( f(b_1) < 0 \) if \( b_1 \in \left( -\frac{3(3 + 2\sqrt{2})}{2}, \frac{1}{2} \right) \) and that \( E_1 \) (that is, \( b_1 = -\frac{1}{2} \)) is an equilibrium while \( E_2 \) (that is, \( b_1 = \frac{3(3 + 2\sqrt{2})}{2} \)) is not. Hence the solution connecting \( E_1 \) and \( E_2 \) is constructed.

The existence of the solution between \( E_2 \) and \( E_3 \) in Fig. 6 can be similarly shown. These two solutions can connect at \( E_2 \) because \( \frac{d}{dt} b_1 = 0 \) holds at that point. Thus we have proved the theorem.

The phase diagram of solution in Theorem 3.1 is shown in Fig. 7. The homoclinic orbit starts at \( E_1 \) at \( t = -\infty \), goes toward \( E_2 \), and retraces its steps back to \( E_1 \).

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Fig. 7: Phase diagram of orbit in Theorem 3.1. The trajectory in the phase plane with trilinear axes $b_1, b_2, b_3$ is shown.


