Temporal Evolution of a Vortex Sheet in the Richtmyer-Meshkov Instability

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We study temporal evolution of an interface in the Richtmyer-Meshkov instability numerically. The interface is treated as a vortex sheet and the Birkhoff-Rott equation is used in order to describe motion of a vortex sheet. We show that redistribution of grid points to equal arclength and the application of the Fourier series expansion for numerical differentiations and integrations make it possible to perform long-time calculations. Successive profiles of a vortex sheet and the temporal evolution of the sheet strength are presented, and especially the evolution of the sheet strength of a vortex core, defined as a point at which the absolute values of the curvature and strength of a vortex sheet become maximum, is discussed. It is found that the sheet strength of a vortex core takes a maximal value at a finite time and turns to gradually decreasing when the Atwood number in the system is non-zero.

When an inhomogeneous vorticity initially distributes on an interface between two fluids with different densities and it is driven by some external force such as a shock wave, a corrugated interface eventually rolls up to a mushroom-like structure. This phenomenon is known as the Richtmyer-Meshkov (RM) instability\(^1\), and the interface perturbation grows linearly with respect to time \( t \) in the linear regime. The RM instability is of direct importance in inertial confinement fusion when a shock crosses layers of materials with different densities. The RM instability can be regarded as a special case with the gravitational acceleration term is zero in the Rayleigh-Taylor (RT) instability, although the initial vorticity on the interface is not zero. The absence of the gravity indicates that the system does not have the linear regime, therefore, it is difficult to investigate the temporal evolution of the system with the linear stability analysis or weakly nonlinear analysis. In this article, we regard the interface in the RM instability as a vortex sheet and examine the motion numerically using the Birkhoff-Rott equation\(^2\) coupled with an evolution equation for the sheet strength which is derived from the Bernoulli equation.

Bernoulli equation, the pressure continuous condition at the interface in this system, is given by\(^3\)

\[
\frac{d\Gamma}{dt} = -2A \frac{d\Phi}{dt} + Aq \cdot q - \frac{A - 2\alpha}{4} \kappa \cdot \kappa + \alpha A \kappa \cdot q, \tag{1}
\]
where \( \mathbf{q} = (\mathbf{u}_1 + \mathbf{u}_2)/2 \) is an average of the velocities on two sides with the velocity \( \mathbf{u}_i \) \((i = 1, 2)\) in each side, \( \boldsymbol{\kappa} = \mathbf{u}_2 - \mathbf{u}_1 \) and the Atwood number is defined by two fluid densities \( \rho_1 \) and \( \rho_2 \) as \( A = (\rho_2 - \rho_1)/(\rho_1 + \rho_2) \). The circulation \( \Gamma \) related to \( \boldsymbol{\kappa} = \nabla \Gamma \) is defined by using the velocity potential \( \phi_i (i = 1, 2) \) in each side as \( \Gamma = \phi_2 - \phi_1 \) and \( \Phi = (\phi_1 + \phi_2)/2 \) is the average velocity potential related with \( \mathbf{u}_i \). The system is assumed to be incompressible, therefore, the velocity potential satisfies the Laplace equation \( \Delta \phi_i = 0 \) \((i = 1, 2)\) in each region. The differentiation \( \frac{d}{dt} \) following a fluid particle on the interface with the velocity \( \mathbf{\bar{u}} \) is defined to be

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{\bar{u}} \cdot \nabla, \quad \mathbf{\bar{u}} \equiv \mathbf{q} + \frac{\alpha \boldsymbol{\kappa}}{2},
\]

where \( |\alpha| \leq 1 \) is a weighting factor such that \( \alpha \neq 0 \) when the Atwood number \( A \neq 0 \). Analytically, any value is allowed for the factor \( \alpha \), however, we must carefully take it when we intend to perform stable long-time numerical computations. We choose here \( \alpha = -|A|^2 \) for small Atwood numbers, while other value is taken for larger Atwood numbers \((|A| > 0.3)\).

We consider an interface in the RM instability as a vortex sheet and suppose that the vortex sheet is described by equations for \( x = X(\theta, t) \) and \( y = Y(\theta, t) \) with \( Z = X + iY \), where \( \theta \) is a Lagrangian parameter which parametrizes the interface. The strength of the vortex sheet \( \kappa(\theta, t) \) and the circulation \( \Gamma \) are related to \( \kappa = \partial \Gamma / \partial s = \Gamma_{\theta}/s_{\theta} \) where the subscript denotes the differentiation with respect to the variable and \( s \) is arclength of the sheet. The strength \( \kappa \) is also related with the velocity difference \( \boldsymbol{\kappa} \) as \( \kappa t = \boldsymbol{\kappa} \), where \( t = (X_{\theta}, Y_{\theta})/\sqrt{X_{\theta}^2 + Y_{\theta}^2} \) is a unit tangential vector on the interface. The Birkhoff-Rott equation, known as an integro-differential equation which describes motion of a vortex sheet in an inviscid and incompressible fluid is given here as

\[
\frac{\partial Z^*(\theta, t)}{\partial t} = \frac{1}{2\pi i} \text{PV} \int \frac{\kappa(\theta', t)s_{\theta}(\theta')d\theta'}{Z(\theta, t) - Z^*(\theta, t)} + \frac{\alpha \kappa(\theta, t)Z^*_\theta(\theta, t)}{2s_{\theta}(\theta)}, \tag{2}
\]

where \( Z^* \) is the complex conjugate of \( Z \). Differentiating Eq. (1) with respect to \( \theta \), we obtain the following Fredholm integral equation of the second kind:

\[
\kappa_t = -\frac{2A}{s_{\theta}}(X_{\theta}U_{\theta} + Y_{\theta}V_{\theta}) - \frac{(1 - \alpha A)\kappa}{s_{\theta}^2}(X_{\theta}U_{\theta} + Y_{\theta}V_{\theta}) - \frac{A - \alpha}{4s_{\theta}}(\kappa^2)_\theta. \tag{3}
\]

Equations (2) and (3) are equations which we should solve here.

If we add the term \(-2Ag_{\theta}/s_{\theta}\) to the right hand side of Eq. (3), Eqs. (2) and (3) can also describe the RT instability, where \( g \) is the gravitational acceleration which is assumed to be in the direction of the \( y \)-axis. For the RT instability, some numerical studies were made, in which the most stable scheme for long-time computations is presented by Kerr\(^4\). He calculated the roll-up of the interface in the RT instability for various Atwood numbers by using the semi-Lagrangian scheme with the second order Runge-Kutta method for the temporal integration. Sohn\(^5\) applied the Godunov method in order to compute the \( (\kappa^2)_\theta \) term in Eq. (3), which causes the strong numerical instability, and succeeded in long-time calculations for both RM and RT instabilities with Euler’s method for the temporal integration, although the accuracy of computations is lower than that by Kerr. Both Kerr and Sohn use the central differences for calculating derivative \( \partial / \partial \theta \) in the equations.

In the present work, we expand dependent variables \( Z \) and \( \kappa \) into discrete Fourier series and calculate all derivatives in Eqs. (2) and (3) using these series in order to avoid errors due to finite differences. Then we redistribute grid points so that they are arranged equally\(^6\) in order to smooth the effect by the term \( (\kappa^2)_\theta \) in Eq. (3). Fourier series expansion and the redistribution
Fig. 1: Interface profiles (a) - (c) and the sheet strength $\kappa$ (d) - (f) in the Richtmyer-Meshkov instability at (a) and (d) $t=3$, (b) and (e) $t=5$, (c) and (f) $t=9$, where $\theta$, the Lagrangian parameter is taken over $-\pi \leq \theta \leq \pi$. Grid number $N = 512$, Krasny's $\delta = 0.15$, time step $\Delta t = 0.002$, Atwood number $A = 0.01$ and $\alpha = -|A|^2$.

of grid points of the interface enable us to perform stable long-time computations. The fourth-order Runge-Kutta method is adopted for the integration with respect to time $t$ and Krasny's $\delta$, with which the Cauchy kernel $1/Z$ in Eq. (2) is regularized as $Z^*/(|Z|^2 + \delta^2)^{3/2}$, is used for the calculation of the singular integral in Eq.(2).

Initial configurations of the interface and the sheet strength $\kappa$ are set to be

$$Z(\theta, 0) = \theta + ia_0 \cos \theta,$$
$$\kappa(\theta, 0) = -2 \sin \theta / s_\theta(0),$$

where $a_0$ is a dimensionless initial amplitude of the interface. All physical quantities are normalized with the wave number and the linear growth rate in the system. For details of the normalization, refer to the reference 1). Throughout this paper, we take the initial amplitude $a_0 = 0.2 \ (0.508 \text{ cm in the real scale})$, Krasny's $\delta = 0.15$ and the number of grid points $N = 512$.

Fig.1 shows the temporal evolution of an interface and the strenght $\kappa$ for $A = 0.01$ and time step $\Delta t = 0.002$ over the normalized time $0 \leq t \leq 9$, which approximately corresponds to $0 \leq \tau \leq 940$ ms. for real time $\tau$. The contribution from the term $(\kappa^2)_{\theta}$ in Eq. (3) which causes strong numerical instability is small for small Atwood numbers, therefore, we can perform a long-time computation for this Atwood number without grid redistribution. Two peaks with opposite signs found in (d) - (f) in the figure correspond to spiral cores. We can see that the height of a bubble ($\theta = \pm \pi$) is almost same as that of a spike ($\theta = 0$) for $A = 0.01$.

Analogous profiles for $A = 0.5$ are shown in Fig.2, in which the time step is taken that $\Delta t = 0.001$ over $0 \leq t \leq 6$ and $\Delta t = 0.0002$ over $6 < t \leq 7.2$, where the grid redistribution
is performed every four time and five time steps over the intervals $0 \leq t \leq 6$ and $6 < t \leq 7.2$, respectively. We can see that the difference between the height of a spike and bubble is large and the roll-up to a spiral is weak compared to that for $A = 0.01$.

The strength $\kappa = \kappa_m$ of a vortex core which appears in $-\pi \leq \theta \leq 0$ is shown in Fig.3 for the two cases of $A = 0.01$ and $A = 0.5$. Here, a vortex core is defined as a point at which the absolute values of the curvature and strength of a vortex sheet become maximum (see also Fig.1 and Fig.2). We see that the curves take maximal values in $4 < t < 5$ for both cases of $A = 0.01$ and $A = 0.5$ and turn to decrease with oscillations. These oscillations are connected with motion of the vortex cores. The tendency that the core strength $\kappa_m$ takes a maximal value and turns to decrease afterwards is found for all Atwood numbers satisfied the condition $A < 0.8$. For $A \geq 0.8$, the roll-up of a vortex sheet itself does not occur and therefore, the concentration of the sheet strength $\kappa$ on cores is not so clear as found for $A = 0.01$ or $A = 0.5$. For $A = 0$, the decrease of the strength of a vortex core can not be found up to the end of computations, although the oscillation with motion of a core exists. Detailed computations for higher Atwood numbers and explanation of the grid redistribution method will be published elsewhere.
Fig. 3: Core strength $\kappa_m$ for $A = 0.01$, $0 \leq t \leq 9$ (left) and $A = 0.5$, $0 \leq t \leq 7.2$ (right)

REFERENCES


