A Design Method of Robust Stabilizing Modified PID Controllers

Kou YAMADA*, Takaaki HAGIWARA* and Yosuke SHIMIZU**

* Graduate School of Engineering, Gunma University, Kiryu, Gunma
** Department of Mechanical System Engineering, Gunma University, Kiryu, Gunma

In this paper, we examine a design method of robust stabilizing modified PID (Proportional-Integral-Derivative) controllers for single-input/single-output plants. PID controller structure is the most widely used one in industrial applications. The plants of which the PID controller is applicable are restricted. Yamada and Hagiwara proposed a design method of modified PID controllers such that modified PID controllers make the closed-loop system for any unstable plants stable. However, no method has been published to guarantee the robust stability of PID control system for any plants with uncertainty. In this paper, we propose a design method of robust stabilizing modified PID controllers for any plants with uncertainty.

1 Introduction

PID (Proportional-Integral-Derivative) controller is most widely used controller structure in industrial applications [1]. Its structural simplicity and sufficient ability of solving many practical control problems have contributed to this wide acceptance. Several papers on tuning methods for PID parameters have been considered [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. However the method in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] do not guarantee the stability of closed-loop system. The references in [13, 14, 15, 16, 17, 18] propose design methods of PID controllers to guarantee the stability of closed-loop system. However, plants to which the method in [13, 14, 15, 16, 17, 18] are restricted. Yamada and Hagiwara gave a design method of modified PID controllers to make the closed-loop system stable for any unstable plants [19]. However the method in [19] cannot apply for plants with uncertainty. The stability problem with uncertainty is known as the robust stability problem [20]. Since almost all practical plants include uncertainty, the problem to design robust stabilizing modified PID controllers for any plants with uncertainty is important. Several papers on design methods of robust stabilizing PID controllers have been considered [20, 21, 22, 23, 24, 25, 26, 27]. However, no design method of modified PID controllers has been published to guarantee the robust stability of PID control system for any plants with uncertainty.

In this paper, we propose a design method of robust stabilizing modified PID controllers such that modified PID controller makes the closed-loop system stable for any plants with uncertainty. The basic idea of robust stabilizing modified PID controller is very simple. If the modified PID control system is robustly stable for the plant with uncertainty, then the modified PID controller must satisfy the robust stability condition. This implies that if the modified PID control system is robustly stable, then the modified PID controller is included in the parametrization of all robust stabilizing controllers for the plant with uncertainty. The parametrization of all robust stabilizing controllers for the plant with uncertainty is obtained using $H_{\infty}$ control theory based on the Riccati equation [20, 29] and the Linear Matrix Inequality (LMI) [30, 31]. Robust stabilizing controllers for the plant with uncertainty include a free parameter, which is designed to achieve desirable control characteristics. When the free parameter of the parametrization of all robust stabilizing controllers is adequately chosen, then the controller works as a robust stabilizing modified PID controller. A numerical example is illustrated to show the effectiveness of the proposed method.

Notations

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\( R \) the set of real numbers.
\( R_+ \) \( R \cup \{ \infty \} \).
\( R(s) \) the set of real rational function with \( s \).
\( RH_{\infty} \) the set of stable proper real rational functions.
\( H_{\infty} \) the set of stable causal functions.
\( D^\perp \) orthogonal complement of \( D \), i.e., \( \begin{bmatrix} D & D^\perp \end{bmatrix} \) or \( \begin{bmatrix} D \\ D^\perp \end{bmatrix} \) is unitary.
\( A^T \) transpose of \( A \).
\( A^\dagger \) pseudo inverse of \( A \).
\( \rho(\{ \cdot \}) \) spectral radius of \( \{ \cdot \} \).
\( \sigma(\{ \cdot \}) \) largest singular value of \( \{ \cdot \} \).
\( \| \{ \cdot \} \|_\infty \) \( H_\infty \) norm of \( \{ \cdot \} \).
\( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) represents the state space description \( C(sI - A)^{-1}B + D \).

\section{Problem formulation}

Consider the closed-loop system written by

\[
\begin{aligned}
y &= G(s) u \\
u &= C(s)(r - y)
\end{aligned}
\tag{1}
\]

where \( G(s) \in R(s) \) is the plant, \( C(s) \in R(s) \) is the controller, \( r \in R \) is the reference input, \( u \in R \) is the control input and \( y \in R \) is the output. The nominal plant of \( G(s) \) is denoted by \( G_m(s) \in R(s) \). Both \( G(s) \) and \( G_m(s) \) are assumed to have no zero or pole on the imaginary axis. In addition, it is assumed that the number of poles of \( G(s) \) in the closed right half plane is equal to the number of poles of \( G_m(s) \) in the closed right half plane. The relation between the plant \( G(s) \) and the nominal plant \( G_m(s) \) is written as

\[
G(s) = G_m(s)(1 + \Delta(s)),
\tag{2}
\]

where \( \Delta(s) \in R(s) \) is the uncertainty. The set of \( \Delta(s) \) is all rational functions satisfying

\[
|\Delta(j\omega)| < |W_T(j\omega)| \quad (\forall \omega \in R_+),
\tag{3}
\]

where \( W_T(s) \) is an asymptotically stable rational function. Under these assumption, the robust stability condition for the plant \( G(s) \) with uncertainty \( \Delta(s) \) satisfying (3) is given by

\[
\|T(s)W_T(s)\|_\infty < 1,
\tag{4}
\]

where \( T(s) \) is the complementary sensitivity function given by

\[
T(s) = \frac{G_m(s)C(s)}{1 + G_m(s)C(s)}.
\tag{5}
\]

When the controller \( C(s) \) has the form written by

\[
C(s) = a_P + \frac{a_I}{s} + a_D s,
\tag{6}
\]

then the controller \( C(s) \) is called PID controller \(^1\), where \( a_P \in R \) is the P-parameter, \( a_I \in R \) is the I-parameter and \( a_D \in R \) is the D-parameter. \( a_P \), \( a_I \) and \( a_D \) are settled so that the closed-loop system in (1) has desirable control characteristics such as steady state characteristic and transient characteristic. For easy explanation, we call \( C(s) \) in (6) the conventional PID controller.

The purpose of this paper is to propose a design method of robust stabilizing modified PID controllers \( C(s) \) to make the closed-loop system in (1) stable for any plant \( G(s) \) in (2) with uncertainty \( \Delta(s) \) satisfying (3).

\section{The basic idea}

In this section, we describe the basic idea to design of robust stabilizing modified PID controllers \( C(s) \) to make the closed-loop system in (1) stable for the plant \( G(s) \) with uncertainty \( \Delta(s) \).
In order to design robust stabilizing modified PID controllers \( C(s) \) that can be applied to any plant \( G(s) \) with uncertainty \( \Delta(s) \), we must see that the robust stabilizing controllers hold (4). The problem of obtaining the controller \( C(s) \), which is not necessarily a PID controller, satisfying (4) is equivalent to the following \( H_\infty \) problem. In order to obtain the controller \( C(s) \) satisfying (4), we consider the control system shown in Fig. 1. \( P(s) \) is selected such that the transfer function from \( w \) to \( z \) in Fig. 1 is equal to \( T(s)W_T(s) \). The state space description of \( P(s) \) is, in general,

\[
\begin{align*}
    \dot{z}(t) &= Az(t) + B_1w(t) + B_2u(t) \\
    z(t) &= C_1z(t) + D_{12}u(t) \\
    y(t) &= C_2z(t) + D_{21}w(t)
\end{align*}
\]

where \( A \in \mathbb{R}^{n \times n}, B_1 \in \mathbb{R}^n, B_2 \in \mathbb{R}^n, C_1 \in \mathbb{R}^{1 \times n}, C_2 \in \mathbb{R}^{1 \times n}, D_{12} \in \mathbb{R}, D_{21} \in \mathbb{R} \). \( P(s) \) is called the generalized plant \( 28 \). \( P(s) \) is assumed to satisfy the following standard assumptions in \( 28, 29 \):

1) \( (A, B_2) \) is stabilizable and \( (C_2, A) \) is detectable;

2) \( D_{12} \) has full column rank and \( D_{21} \) has full row rank;

3) \[
\begin{bmatrix}
    A - j\omega I & B_2 \\
    C_1 & D_{12}
\end{bmatrix}
\]
has full column rank for all \( \omega \) and \[
\begin{bmatrix}
    A - j\omega I & B_1 \\
    C_2 & D_{21}
\end{bmatrix}
\]
has full row rank for all \( \omega \).

Under these assumptions, according to \( 28, 29 \), the parametrization of all robust stabilizing controllers \( C(s) \) is written by

\[
C(s) = C_{11}(s) + C_{12}(s)Q(s)(I - C_{22}(s)Q(s))^{-1}C_{21}(s),
\]

where

\[
\begin{bmatrix}
    C_{11}(s) & C_{12}(s) \\
    C_{21}(s) & C_{22}(s)
\end{bmatrix}
= \begin{bmatrix}
    A_c & B_{c1} & B_{c2} \\
    C_{c1} & D_{c11} & D_{c12} \\
    C_{c2} & D_{c21} & D_{c22}
\end{bmatrix}
\]

\[
A_c = A + B_1B_1^TX - B_2\left(D_{12}^TC_1 + E_{12}^{-1}B_1^TX\right) - (I - XY)^{-1}\left(B_1D_{21} + YC_1^TE_{21}^{-1}\right)(C_2 + D_{21}B_1^TX)
\]

\[
B_{c1} = (I - XY)^{-1}\left(B_1D_{21} + YC_1^TE_{21}^{-1}\right), \quad B_{c2} = (I - XY)^{-1}\left(B_2 + YC_1^TD_{12}\right)E_{12}^{-1/2}
\]

\[
C_{c1} = -D_{12}^TC_1 - E_{12}^{-1}B_1^TX, \quad C_{c2} = -E_{21}^{-1/2}(C_2 + D_{21}B_1^TX)
\]

\[
D_{c11} = 0, \quad D_{c12} = E_{12}^{-1/2}, \quad D_{c21} = E_{21}^{-1/2}, \quad D_{c22} = 0,
\]

\[
E_{12} = D_{12}^TD_{12}, E_{21} = D_{21}D_{21}^T,
\]

\( X \geq 0 \) and \( Y \geq 0 \) are solutions of

\[
X \left(A - B_2D_{12}^TC_1\right) + \left(A - B_2D_{12}^TC_1\right)^T + X \left(B_1B_1^T - B_2(D_{12}^TD_{12})^{-1}B_1^T\right)X + \left(D_{12}^TB_1^T\right)^TD_{12}^TC_1^TY = 0
\]
and

$$Y \left( A - B_1 D_{21}^1 C_2 \right)^T + \left( A - B_1 D_{21}^2 C_2 \right) Y + Y \left( C_1^T C_1 - C_2^T \left[ D_{21} D_{21}^T \right]^{-1} C_2 \right) Y + B_1 D_{21}^2 \left( B_1 D_{21}^1 \right)^T = 0$$

(11)
such that

$$\rho \left( XY \right) < 1$$

(12)
and both $A - B_2 D_{12}^1 C_1 + \left( B_1 B_1^T - B_2 \left( D_{12}^1 D_{12}^T \right)^{-1} B_2^T \right) X$ and $A - B_1 D_{21}^2 C_2 + Y \left( C_1^T C_1 - C_2 \left( D_{21} D_{21}^T \right)^{-1} C_2 \right)$ have no eigenvalue in the closed right half plane and the free parameter $Q(s) \in RH_{\infty}$ is any function satisfying $\|Q(s)\|_{\infty} < 1$.

On the parametrization of all robust stabilizing controllers $C(s)$ in (8) for $G(s)$, the controller $C(s)$ in (8) includes free-parameter $Q(s)$. Using free-parameter $Q(s)$ in (8), we propose a design method of robust stabilizing modified PID controllers $C(s)$ to make the closed-loop system in (1) stable. In order to design the robust stabilizing modified PID controllers $C(s)$, the free parameter $Q(s)$ in (8) is settled for $C(s)$ in (8) to have the same characteristics to conventional PID controller $C(s)$ in (6). Therefore, next, we describe the role of conventional PID controller $C(s)$ in (6) in order to clarify the condition that the modified PID controller $C(s)$ must be satisfied. From (6), using $C(s)$, the $P$-parameter $a_P$, the $I$-parameter $a_I$ and the $D$-parameter $a_D$ are decided by

$$a_P = \lim_{s \to \infty} \left\{ -s^2 \frac{d}{ds} \left( \frac{1}{s} C(s) \right) \right\},$$

(13)

$$a_I = \lim_{s \to 0} \left\{ sC(s) \right\}$$

(14)

and

$$a_D = \lim_{s \to \infty} \frac{d}{ds} \left\{ C(s) \right\},$$

(15)

respectively. Therefore, if the controller $C(s)$ holds (13), (14) and (15), the role of controller $C(s)$ is equivalent to the conventional PID controller $C(s)$ in (8). That is, we can design robust stabilizing modified PID controllers such that the role of controller $C(s)$ (8) is equivalent to the conventional PID controller $C(s)$ in (6).

In the next section, using the idea described in this section, we propose a design method of robust stabilizing modified PID controllers that satisfies (13), (14) and (15).

4 Robust Stabilizing Modified PID controller

In this section, we propose a design method of robust stabilizing modified PID controllers.

4.1 Robust Stabilizing Modified PID controller

The robust stabilizing modified PID controller $C(s)$ satisfying (13) is written by (8), where

$$Q(s) = \lim_{s \to \infty} \left( C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s) \right).$$

(16)

If $a_P$ satisfies

$$-\left| \lim_{s \to \infty} \left( C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s) \right) \right| < a_P < \left| \lim_{s \to \infty} \left( C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s) \right) \right|,$$

(17)

$Q(s)$ in (16) satisfies $|Q(s)|_{\infty} < 1$. This implies that when (17) holds true, the controller $C(s)$ in (8) with (16) makes the closed-loop system in (1) stable for the plant $G(s)$ with uncertainty $\Delta(s)$.

4.2 Robust Stabilizing Modified I controller

The robust stabilizing modified I controller $C(s)$ satisfying (14) is written by (8), where

$$Q(s) = \frac{q_0 + q_1 s}{\tau_0 + \tau_1 s}$$

(18)
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\[ q_0 = \frac{\tau_0}{C_{22}(0)}, \]  

\[ q_1 = \frac{\tau_1}{C_{22}(0)} - \frac{\tau_0}{a_1 C_{22}^2(0)} \left\{ \frac{d}{ds} \left( C_{22}(s) \right) \bigg|_{s=0} a_I + C_{12}(0)C_{21}(0) \right\}, \]  

\[ \tau_i \in R > 0 \ (i = 0, 1). \]  

If \[ |C_{22}(0)| < 0 \]  

and

\[ -1 < \frac{1}{C_{22}(0)} - \frac{\tau_0}{\tau_1 a_1 C_{22}^2(0)} \left\{ \frac{d}{ds} \left( C_{22}(s) \right) \bigg|_{s=0} a_I + C_{12}(0)C_{21}(0) \right\} < 1 \]  

hold true, then \( Q(s) \) in (18) satisfies \( \|Q(s)\|_\infty < 1 \). This implies that when (21) and (22) hold true, the controller \( C(s) \) in (8) with (18) makes the closed-loop system in (1) stable for the plant \( G(s) \) with uncertainty \( \Delta(s) \).

### 4.3 Robust Stabilizing Modified D controller

The robust stabilizing modified D controller \( C(s) \) satisfying (15) is written by (8), where

\[ Q(s) = \frac{a_D}{\lim_{s \to \infty} \left[ (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) + a_D \lim_{s \to \infty} (sC_{22}(s)) \right] s}. \]  

Since \( Q(s) \) in (23) is improper, \( Q(s) \) in (23) is not included in \( RH_\infty \). In order for \( Q(s) \) to be included in \( RH_\infty \), (23) is modified as

\[ Q(s) = \frac{a_D}{\lim_{s \to \infty} \left[ (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) + a_D \lim_{s \to \infty} (sC_{22}(s)) \right] \frac{s}{1 + \tau_D s}}, \]  

where \( \tau_D \in R > 0 \). From \( \tau_D > 0 \) in (24), \( Q(s) \) in (24) is included in \( RH_\infty \). If

\[ -1 < \frac{a_D}{\tau_D \left\{ \lim_{s \to \infty} \left[ (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) + a_D \lim_{s \to \infty} (sC_{22}(s)) \right] \right\} < 1 \]  

is satisfied, then \( Q(s) \) in (24) satisfies \( \|Q(s)\|_\infty < 1 \). This implies that when (25) is satisfied, the controller \( C(s) \) in (8) with (24) makes the closed-loop system in (1) stable for the plant \( G(s) \) with uncertainty \( \Delta(s) \).

### 4.4 Robust Stabilizing Modified PI controller

The robust stabilizing modified PI controller \( C(s) \) satisfying (13) and (14) is written by (8), where

\[ Q(s) = \frac{q_0 + q_1 s + q_2 s^2}{\tau_0 + \tau_1 s + \tau_2 s^2}, \]  

\[ q_0 = \frac{\tau_0}{C_{22}(0)}, \]  

\[ q_1 = \frac{\tau_1}{C_{22}(0)} - \frac{\tau_0}{a_1 C_{22}^2(0)} \left\{ \frac{d}{ds} \left( C_{22}(s) \right) \bigg|_{s=0} a_I + C_{12}(0)C_{21}(0) \right\}, \]  

\[ q_2 = \lim_{s \to \infty} \left[ (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) \right] \]  

and \( \tau_i \in R > 0 \ (i = 0, 1, 2) \). From \( \tau_i > 0 \ (i = 0, 1, 2) \), \( Q(s) \) in (26) is included in \( RH_\infty \). If \( a_P \) and \( a_I \) are settled to make \( Q(s) \) in (26) satisfy \( \|Q(s)\|_\infty < 1 \), then the controller \( C(s) \) in (8) with (26) makes the closed-loop system in (1) stable for the plant \( G(s) \) with uncertainty \( \Delta(s) \).
4.5 Robust Stabilizing Modified PD controller

The robust stabilizing modified PD controller $C(s)$ satisfying (13) and (15) is written by (8), where

$$Q(s) = q_0 + q_1 s,$$

(30)

$$q_1 = \frac{a_D}{\lim_{s \to -\infty} (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) + a_D \lim_{s \to -\infty} \{ sC_{22}(s) \}}$$

(31)

and

$$q_0 = \left\{ 1 - \lim_{s \to -\infty} (sC_{22}(s)) q_1 \right\}^2 a_P \frac{1}{\lim_{s \to -\infty} (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) \left\{ 1 - \lim_{s \to -\infty} (sC_{22}(s)) q_1 - \lim_{s \to -\infty} \left( \frac{d}{ds} (C_{22}(s)) q_1 \right) \right\}}$$

$$+ \lim_{s \to -\infty} \left( \frac{d}{ds} (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) \right) \left\{ q_1 - \lim_{s \to -\infty} (sC_{22}(s)) q_1^2 \right\}$$

$$+ \lim_{s \to -\infty} \left( C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s) \right) \left\{ \lim_{s \to -\infty} \left( \frac{d}{ds} (C_{22}(s)) \right) + \lim_{s \to -\infty} (s^2C_{22}(s)) \right\} q_1^2. \right.$$

(32)

Since $Q(s)$ in (30) is improper, $Q(s)$ in (30) is not included in $RH_{\infty}$. In order for $Q(s)$ to be included in $RH_{\infty}$, (30) is modified as

$$Q(s) = q_0 + \frac{q_1 s}{1 + \tau_D s},$$

(33)

where $\tau_D \in R > 0$. From $\tau_D > 0$ in (33), $Q(s)$ in (33) is included in $RH_{\infty}$. If

$$\left\{ 1 - \lim_{s \to -\infty} (sC_{22}(s)) q_1 \right\}^2 a_P \frac{1}{\lim_{s \to -\infty} (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) \left\{ 1 - \lim_{s \to -\infty} (sC_{22}(s)) q_1 - \lim_{s \to -\infty} \left( \frac{d}{ds} (C_{22}(s)) q_1 \right) \right\}}$$

$$+ \lim_{s \to -\infty} \left( C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s) \right) \left\{ q_1 - \lim_{s \to -\infty} (sC_{22}(s)) q_1^2 \right\}$$

$$+ \lim_{s \to -\infty} \left( C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s) \right) \left\{ \lim_{s \to -\infty} \left( \frac{d}{ds} (C_{22}(s)) \right) + \lim_{s \to -\infty} (s^2C_{22}(s)) \right\} q_1^2 \right\} < 1 \right.$$

(34)

and

$$\left\{ 1 - \lim_{s \to -\infty} (sC_{22}(s)) q_1 \right\}^2 a_P \frac{1}{\lim_{s \to -\infty} (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) \left\{ 1 - \lim_{s \to -\infty} (sC_{22}(s)) q_1 - \lim_{s \to -\infty} \left( \frac{d}{ds} (C_{22}(s)) q_1 \right) \right\}}$$

$$+ \lim_{s \to -\infty} \left( C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s) \right) \left\{ q_1 - \lim_{s \to -\infty} (sC_{22}(s)) q_1^2 \right\}$$

$$+ \lim_{s \to -\infty} \left( C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s) \right) \left\{ \lim_{s \to -\infty} \left( \frac{d}{ds} (C_{22}(s)) \right) + \lim_{s \to -\infty} (s^2C_{22}(s)) \right\} q_1^2 \right\} \left\{ \left( C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s) \right) + a_D \lim_{s \to -\infty} \{ sC_{22}(s) \} \right\} < 1 \right.$$

(35)

hold true, then $Q(s)$ in (33) satisfy $\|Q(s)\|_{\infty} < 1$. This implies that if (34) and (35) hold true, then the controller $C(s)$ in (8) with (33) makes the closed-loop system in (1) stable for the plant $G(s)$ with uncertainty $\Delta(s)$. 

4.6 Robust Stabilizing Modified PID controller

The robust stabilizing modified PID controller \( C(s) \) satisfying (13), (14) and (15) is written by (8), where

\[
Q(s) = \frac{q_0 + q_1 s + q_2 s^2}{\tau_0 + \tau_1 s + \tau_2 s^2} + q_3 s, \tag{36}
\]

\[
q_0 = \frac{\tau_0}{C_{22}(0)}, \tag{37}
\]

\[
q_1 = \frac{\tau_1}{C_{22}(0)} - q_0 \tau_0 - \frac{\tau_0}{a_I C_{22}(0)} \left( \frac{d}{ds} (C_{22}(s)) \right)_{s=0} \left( a_I + C_{12}(0)C_{21}(0) \right), \tag{38}
\]

\[
q_2 = \frac{1 - \lim_{s \to \infty} (sC_{22}(s)) q_3}{\lim_{s \to \infty} (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) \left( 1 - \lim_{s \to \infty} (sC_{22}(s)) q_3 - \lim_{s \to \infty} \left( s^2 \frac{d}{ds} (C_{22}(s)) \right) q_3 \right)} \left( q_0 - \lim_{s \to \infty} (sC_{22}(s)) q_3 \right) \left( q_3 - \lim_{s \to \infty} (sC_{22}(s)) q_3 \right) + \lim_{s \to \infty} \left( s^2 \frac{d}{ds} (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) \left( q_3 - \lim_{s \to \infty} (sC_{22}(s)) q_3 \right) \right) + \lim_{s \to \infty} \left( s^3 \frac{d}{ds} (C_{22}(s)) \right) + \lim_{s \to \infty} (s^2 C_{22}(s)) \left( q_3 \right) \right), \tag{39}
\]

\[
q_3 = \frac{a_D}{\lim_{s \to \infty} (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) + a_D \lim_{s \to \infty} (sC_{22}(s))} \tag{40}
\]

and \( \tau_i \in R > 0 \) \( (i = 0, 1, 2) \). Since \( Q(s) \) in (36) is improper, \( Q(s) \) in (36) is not included in \( RH_\infty \). In order for \( Q(s) \) to be included in \( RH_\infty \), (36) is modified as

\[
Q(s) = \frac{q_0 + q_1 s + q_2 s^2}{\tau_0 + \tau_1 s + \tau_2 s^2} + \frac{q_3 s}{1 + \tau_D s}, \tag{41}
\]

where \( \tau_D \in R > 0 \). From \( \tau_D > 0 \) and \( \tau_i > 0 \) \( (i = 0, 1, 2) \) in (41), \( Q(s) \) in (41) is included in \( RH_\infty \). If \( a_P, a_I \) and \( a_D \) are settled to make \( Q(s) \) in (41) satisfy \( \|Q(s)\|_\infty < 1 \), then the controller \( C(s) \) in (8) with (41) makes the closed-loop system in (1) stable for the plant \( G(s) \) with uncertainty \( \Delta(s) \).

4.7 Controller structure

In this subsection, we explain the structure of modified PID controller \( C(s) \) in (8) with (36).

The structure of modified PID controller \( C(s) \) in (8) with (36) is shown in Fig. 2. Figure 2 shows that in order for the controller in (8) with (41) to specify (4) and to stabilize any plant \( G(s) \), Fig. 2 is complex than the structure of the conventional PID controller \( C(s) \) in (6). That is, the order of the conventional PID controller is 2, but the order of the modified PID controller is 3n + 6, which is greater than that of the conventional PID controller.

5 Numerical example

In this section, we illustrate a numerical example to show the effectiveness of the proposed method.

Consider the problem to design a robust stabilizing modified PID controller \( C(s) \) for the plant \( G(s) \) in (2) with uncertainty \( \Delta(s) \), where the nominal plant \( G_m(s) \) and the upper bound \( W_\gamma(s) \) of the set of \( \Delta(s) \) are given by

\[
G_m(s) = \frac{11}{s^3 - s^2 - 3s - 5} \tag{42}
\]
and

$$W_T(s) = \frac{(s + 2)(s + 10)(s + 50)}{2 \times 10^4},$$

respectively. Note that there exists no stabilizing conventional PID controllers for the nominal plant $G_m(s)$ in (42). Therefore, methods in [20, 21, 22, 23, 24, 25, 26, 27] cannot make the stabilizing PID controller.

Using the method in 3, the parametrization of all robust stabilizing controllers $C(s)$ in (8) is obtained. $Q(s)$ in (8) is designed as (36), where

$$
\begin{align*}
\begin{cases}
  a_P &= 10 \\
  a_I &= 100 \\
  a_D &= 1
\end{cases} \\
\begin{cases}
  \tau_0 &= 50.41 \\
  \tau_1 &= 14.2 \\
  \tau_2 &= 1
\end{cases}
\end{align*}
$$

and $\tau_D$ is selected by $\tau_D = 0.1$.

From the discussion in 4.6, designed $Q(s)$ in (8) must hold $||Q(s)||_\infty < 1$. Next, we confirm that designed $Q(s)$ satisfies $||Q(s)||_\infty < 1$. The gain plot of designed $Q(s)$ is shown in Fig. 3. Figure 3 shows that designed $Q(s)$ satisfies $||Q(s)||_\infty < 1$.

When $\Delta(s)$ is given by

$$\Delta(s) = \frac{s + 2}{500}.$$

the response of the output $y$ of the closed-loop system in (1) for the step reference input $r$ using the robust stabilizing modified PID controller $C(s)$ is shown in Fig. 4. Figure 4 shows that the robust stabilizing modified PID controller $C(s)$ makes the closed-loop system stable.

On the other hand, using conventional PID controller in (6) with (44), response of the output $y$ of the closed-loop system in (1) for the step reference input $r$ is shown in Fig. 5. Figure 5 shows that the conventional PID control system is unstable.

Next, when $a_P$, $a_I$ and $a_D$ in the robust stabilizing modified PID controller are varied, the comparison of step responses is examined. First, the comparison of step responses for various $a_P$ as $a_P = 1$, $a_P = 50$ and $a_P = 100$ is shown in Fig. 6. Here, the solid line, the dotted line and the broken line show the step response of the robust stabilizing modified control system using $a_P = 1$, $a_P = 50$ and $a_P = 100$, respectively. Figure 6 shows that as the value of $a_P$ increased, the overshoot is larger and the rise time is shorter. Since this characteristic is equivalent to the conventional PID controller, the role of P-parameter $a_P$ in the robust stabilizing modified PID controller is...
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Figure 3: Gain plot of the free parameter $Q(s)$

Figure 4: Step response of the closed-loop system using the robust stabilizing modified PID controller

equivalent to that of the conventional PID controller. Secondly, the comparison of step responses for various $a_I$ as $a_I = 0.2$, $a_I = 0.3$ and $a_I = 1$ is shown in Fig. 7. Here the solid line, the dotted line and the broken line show the step response of the robust stabilizing modified PID control system using $a_I = 0.2$, $a_I = 0.3$ and $a_I = 1$, respectively. Figure 7 shows that as the value of $a_I$ increased, the overshoot is smaller and the convergence speed is faster. Since this characteristic is equivalent to the conventional PID controller, the role of I-parameter $a_I$ in the robust stabilizing modified PID controller is equivalent to that of the conventional PID controller. Thirdly, the comparison of step responses for various $a_D$ as $a_D = 10$, $a_D = 50$ and $a_D = 100$ is shown in Fig. 8. Here, the solid line, the dotted line and the broken line show the step response of the robust stabilizing modified PID control system using $a_D = 10$, $a_D = 50$ and $a_D = 100$, respectively. Figure 8 shows that as the value of $a_D$ increased, the response is smoothly. Since this characteristic is equivalent to the conventional PID controller, the role of D-parameter $a_D$ in the robust stabilizing modified PID controller is equivalent to that of the conventional PID controller. Since these characteristics are equivalent to the conventional PID controller, the role of P-parameter $a_P$, I-parameter $a_I$ and D-parameter $a_D$ in the robust stabilizing modified PID controller is equivalent to that of the conventional PID controller.

In this way, it is shown that we can easily design a robust stabilizing modified PID controller for the plant $G(s)$ in (2) with uncertainty $\Delta f(s)$, which has same characteristic to conventional PID controller, and guarantee the stability of the closed-loop system.

6 Conclusions

In this paper, we proposed a design method of robust stabilizing modified PID controllers such that modified PID controller makes the closed-loop system stable for any plants with uncertainly. Proposed modified PID controllers lose the advantage of the conventional PID controllers such as
Figure 5: Step response of the closed-loop system using conventional PID controller

Figure 6: Step response using the robust stabilizing modified P controller with $a_p = 1, 50, 100$

1. The control structure is simple
2. The order of the controller is 1

but have following advantages:

1. The modified PID controller makes the control system stable for any plant $G(s)$ with uncertainty.
2. The roles of P-parameter $a_p$, I-parameter $a_I$ and D-parameter $a_D$ in the robust stabilizing modified PID controller are equivalent to that of the conventional PID controller. That is, P-parameter $a_p$, I-parameter $a_I$ and D-parameter $a_D$ in the robust stabilizing modified PID controller can be tuned using previously proposed methods in 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

A numerical example was shown to illustrate the effectiveness of the proposed method.

References


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Figure 7: Step response using the robust stabilizing modified I controller with $a_I = 0.2, 0.3, 1$

Figure 8: Step response using the robust stabilizing modified D controller with $a_D = 10, 50, 100$


