Shape Optimization Using Adjoint Variable Method for Reducing the Surface Force of an Object under Unsteady Flow

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To obtain the optimal shape of a 3D object minimizing the fluid surface force, an adjoint variable method based on the variational principle is formulated and applied to the finite element method. This method is a type of sensitivity analysis (the sensitivity is the gradient of the Lagrange function with respect to spatial coordinates), and is based on the calculus of variations. The constrained optimization problem of the cost function is converted into an unconstrained optimization problem of the Lagrange function by introducing Lagrange multipliers called adjoint variables. The optimality condition of the adjoint variable method consists of the state equations, the adjoint equations, and the sensitivity equations. The equations for reducing the fluid surface force under a constant volume condition are formulated. By using the 3D shape optimization system, the surface force of an object located in Reynolds number 1000 can be reduced by about 57.6%.

1. INTRODUCTION

Many optimization problems in industry such as obtaining minimum drag for a given lift or obtaining minimum drag for a given surface area have been approached by using optimization techniques[13].

Development of shape optimization algorithms to minimize the drag on an object in viscous flow under a constant volume has started since 1973[4]. The minimum drag problem of an object placed in flow has been mainly solved for small-to-intermediate Reynolds numbers, in the range 1-250[9]. In high Reynolds number cases (between 1000 and 100000), Glowinski and Pironneau (1974) presented a numerical algorithm to compute the minimum-drag profile of a two-dimensional body, although with a boundary layer approximation[9]. Huan and Modi investigated two-dimensional minimum drag bodies for a range of Reynolds numbers varying from 20 to 100000[11,12]. At Reynolds number 500, Ghattas et al. obtained the optimal shape in viscous energy dissipation[13].

Studies done in high Reynolds number have to deal with numerical instabilities in the flow analysis. Moreover, complex flows cannot be simulated if low resolution meshes are used in the flow analysis. Parallel techniques to distribute the computation are naturally needed for dealing with large scale meshes. In conclusion, stabilization techniques and parallelization need to be used in conjunction with adjoint variable method.

To implement the above-mentioned techniques into the sensitivity analysis based on the adjoint variable method, both the calculation procedure and the theoretical derivation (to obtain the state, the adjoint and the sensitivity equations) of this method should be specified. Fusion methods to combine these techniques with the adjoint variable method are demanded. In this study, the proposed techniques were developed to optimize an object placed under unsteady flow. Lagrange function is transformed to the weak formulation by using the Gauss Green theorem. The state, the adjoint and the sensitivity equations including the boundary conditions are naturally derived from the stationary condition of the weak formulation. Three stationary conditions (which consist of state equations, adjoint equations and sensitivity equations), boundary conditions and time interval conditions were derived from the Lagrange function. A 3D adjoint variable method used to decrease the drag force of an object in an unsteady flow is formulated by using FEM. The particularity of this method resides in the fact that the start time and the end time in the optimization are determined by the stationary condition of the Lagrange function. The state variable is calculated from the start time to the end time in forward time and this data is stored, while the adjoint variable is calculated in backward time by using the stored data. By using this method, robust convergence of the cost function can be attained. This robustness makes the shape optimization possible even under an unsteady flow with Karman vortices. The 3D high resolution analysis was realized by using the parallel library, HEC-MW[14]. The cost function could robustly converge in Reynolds number 1000 and the reduction in surface force for the optimal shape was 57.6 percent.
2. ADJOINT VARIABLE METHOD

2.1 Definitions

In Fig. 1, the solid line and the dotted line schematically show the initial shape and the optimized shape in the fluid, respectively. We denote by $\Omega$ the computational domain. We denote by $\Gamma$, $\gamma$ and $\Psi$ the boundaries of the computational domain. We define $\Gamma$, $\gamma$ and $\Psi$ as follows:

$$\Gamma = \Gamma_E + \Gamma_W + \Gamma_S + \Gamma_N + \Gamma_U + \gamma = \Psi + \gamma$$  \hspace{1cm} (1)

where subscripts $E, W, S, N, U$ and $D$ indicate the boundary parts. We denote time and the three-dimensional spatial coordinate vector as follows:

$$X = (x_1, x_2, x_3)^T = (x, y, z)^T \in \mathbb{R}^3 \text{ in } \Omega$$  \hspace{1cm} (2)

A unit normal vector on the boundary is also defined as follows:

$$n(X) = (n_1(X), n_2(X), n_3(X))^T \in \mathbb{R}^3 \text{ on } \Gamma_E, \Gamma_S, \Gamma_N, \Gamma_U, \Gamma_D$$  \hspace{1cm} (3)

We define the velocity vector as follows:

$$u(t, X) = (u_1(t, X), u_2(t, X), u_3(t, X))^T \in \mathbb{R}^3 \text{ in } \Omega$$  \hspace{1cm} (4)

and the state variable vector as:

$$W(t, X) = (W_1(t, X), W_2(t, X), W_3(t, X), W_4(t, X))^T = (p(t, X), u(t, X), v(t, X), w(t, X))^T \in \mathbb{R}^4 \text{ in } \Omega$$  \hspace{1cm} (5)

where $p$ denotes the pressure. The adjoint variable vector, which depends on time and spatial coordinates, is defined as follows:

$$\lambda(t, X) = (\lambda_0(t, X), \lambda_1(t, X), \lambda_2(t, X), \lambda_3(t, X))^T \in \mathbb{R}^4 \text{ in } \Omega$$  \hspace{1cm} (6)

where $\lambda_0$ represents the adjoint pressure, and $\lambda_1, \lambda_2$ represent the adjoint velocity vectors. Tractions in the flow analysis and the adjoint analysis are defined as follows:

$$T(t, X) = (T_1(t, X), T_2(t, X), T_3(t, X))^T \in \mathbb{R}^3 \text{ in } \Omega$$  \hspace{1cm} (7)

$$M(t, X) = (M_1(t, X), M_2(t, X), M_3(t, X))^T \in \mathbb{R}^3 \text{ in } \Omega$$  \hspace{1cm} (8)

The superscript $(n)$ shows the $n$th time step. The subscript $(k)$ shows the $k$th shape step. The shape step represents the number of shape modifications from the initial step. For example, the $k$th velocity is defined as follows:

$$u_{(n)}^{(k)} = (u_{(n)}^{(k)}, u_{(n)}^{(k)}, u_{(n)}^{(k)}, v_{(n)}^{(k)}, w_{(n)}^{(k)})^T \text{ for } k = 0, 1, \ldots \in \mathbb{R}^4 \text{ in } \Omega$$  \hspace{1cm} (9)

In the case of a subscript being used more than one time in the same term, it should be interpreted according to the following summation convention:

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \text{ for } i = 1, 2, 3 \in \mathbb{R}^3$$  \hspace{1cm} (10)

2.2 The problem

To minimize the cost function under constraints, we formulated the Lagrange function by introducing adjoint variables. The adjoint variable method is based on the variational method. By introducing Lagrange multipliers called adjoint variables, the constrained optimization of the cost function is transformed to the unconstrained optimization of the Lagrange function. A circular cylinder is placed in the computational domain $\Omega$, as shown in Fig. 1. $\Gamma$ is the $N-S-E-W$ boundary at north, south, east, and west. We denote by $\gamma$ the surface of the object under optimization. A fluid flows in on the boundary $\Gamma_W$ and flows out on the boundary $\Gamma_E$. The origin of coordinates is at the center of the cylinder. Units are dimensionless. In this paper, as the cost function, the surface force on the surface $\gamma$ is defined as:
\[ J = -\int_{t_1}^{t_2} \left( -\frac{P}{\rho} + 2\nu \frac{\partial u}{\partial x} \right) p_1 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) p_2 + \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \right) n_3 \right) dt \in \mathbb{R}^1 \] (11)

where the start of test time and the end of test time in the optimization are denoted by \( t_1 \) and \( t_2 \). In the case that the flow is generated from the boundary \( \Gamma_N \) to the boundary \( \Gamma_D \), the direction of the drag on the object becomes inverse in comparison with flow from \( \Gamma_N \) to \( \Gamma_D \). Therefore, the cost function becomes negative. In the case of the inverse flow, the second power of the cost function may be defined. In this study, the flow from \( \Gamma_N \) to \( \Gamma_D \) is as shown in Fig. 1. Inverse flow is not considered in this study. We formulated the Lagrange function by introducing the adjoint variable as follows:

\[
L = -\int_{t_1}^{t_2} \left( \rho \frac{\partial u}{\partial x} \right) p_1 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) p_2 + \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \right) n_3 \right) \phi dx dt + \int_{t_1}^{t_2} \int_{\Omega} \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \phi dx dt
\] (12)

\[
+ \int_{t_1}^{t_2} \int_{\Omega} \rho \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \phi dx dt + \int_{t_1}^{t_2} \int_{\Omega} \rho \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial z} \right) \phi dx dt
\] (13)

\[
L = -\int_{t_1}^{t_2} \int_{\Omega} F_1 \phi dx dt + \int_{t_1}^{t_2} \int_{\Omega} F_2 \phi dx dt + \int_{t_1}^{t_2} \int_{\Omega} F_3 \phi dx dt \in \mathbb{R}^1
\] (14)

In order to derive both the adjoint and the sensitivity equations, the Lagrange function is transformed as follows: (See Appendix A for details regarding this transformation):

\[
L = J + \int_{t_1}^{t_2} \int_{\Omega} \rho \left( \frac{\partial u}{\partial x} \right) p_1 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) p_2 + \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \right) n_3 \right) \phi dx dt - \int_{t_1}^{t_2} \int_{\Omega} \rho \phi \left( \frac{\partial u}{\partial x} \right) + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \phi \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \right) \phi \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} \right) \right) \phi dx dt
\] (15)

In the above equation, the summation convention is used. Integrands \( F_i \sim F_3 \) of the above equation are defined as follows:

\[
L = J + \int_{t_1}^{t_2} \int_{\Omega} F_1 \phi dx dt - \int_{t_1}^{t_2} \int_{\Omega} F_2 \phi dx dt - \int_{t_1}^{t_2} \int_{\Omega} F_3 \phi dx dt + \int_{t_1}^{t_2} \int_{\Omega} F_1 \phi dx dt + \int_{t_1}^{t_2} \int_{\Omega} F_2 \phi dx dt + \int_{t_1}^{t_2} \int_{\Omega} F_3 \phi dx dt \in \mathbb{R}^1
\] (16)

Fig. 1 Computational domain and boundary conditions.

<table>
<thead>
<tr>
<th>Table 1 Boundary conditions</th>
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<tbody>
<tr>
<td><strong>Domains</strong></td>
</tr>
<tr>
<td>( \Gamma_N )</td>
</tr>
<tr>
<td>( \Gamma_T )</td>
</tr>
<tr>
<td>( \Gamma_D )</td>
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<tr>
<td>( \Gamma_s )</td>
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<td>( \gamma )</td>
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</table>
2.3 State equations

The state variables are calculated by solving the state equations, the stationary conditions which are obtained by taking the first variation of the Lagrange function with respect to the adjoint variable \( \lambda \) as follows:

\[
\delta L = \int_{t_1}^{t_2} \left( \sum_{i=1}^{n} \frac{\partial F_i}{\partial \lambda_i} \delta \lambda_i \right) dt + \int_{t_1}^{t_2} \left( \sum_{i=1}^{n} \frac{\partial F_i}{\partial \lambda_i} \delta \lambda_i \right) \delta \lambda_i \delta \lambda_i dt
\]

\[
+ \int_{t_1}^{t_2} \sum_{j=1}^{m} \left( \frac{\partial F_j}{\partial \lambda_j} \delta \lambda_j \right) \delta \lambda_j dt = 0 \quad i = 1, 2, 3 \quad \in \mathbb{R}^1
\]

By the fundamental lemma of the calculus of variations, Eq.(16) becomes as follows:

\[
\frac{\partial F_i}{\partial \lambda_i} = 0 \quad \in \mathbb{R}^1 \quad \text{on} \quad \gamma
\]

\[
\frac{\partial F_i}{\partial \lambda_i} = 0 \quad i = 1, 2, 3 \quad \in \mathbb{R}^1 \quad \text{on} \quad \gamma
\]

\[
\frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} = 0 \quad i = 1, 2, 3 \quad \in \mathbb{R}^1 \quad \text{in} \quad \Omega
\]

\[
\frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} = 0 \quad i = 1, 2, 3 \quad \in \mathbb{R}^1 \quad \text{in} \quad \Omega
\]

These equations shown above consist of the continuum and the Navier-Stokes equations. The Navier-Stokes consists of the time derivative term, the advection term, the pressure term and the diffusion term as follows:

\[
\frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} = 0 \quad \text{in} \quad \Omega
\]

\[
\frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} = 0 \quad \text{in} \quad \Omega
\]

\[
\frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} = 0 \quad \text{in} \quad \Omega
\]

In the state equations, the boundary condition is shown in Table 1.

2.4 Adjoint equations

The adjoint variable is calculated by solving the adjoint equation, the stationary condition which is obtained by taking the first variation of the Lagrange function with respect to the state variable \( W \) as follows:

\[
\delta L = \int_{t_1}^{t_2} \left( \sum_{i=1}^{n} \frac{\partial F_i}{\partial \lambda_i} \delta \lambda_i \right) dt + \int_{t_1}^{t_2} \left( \sum_{i=1}^{n} \frac{\partial F_i}{\partial \lambda_i} \delta \lambda_i \right) \delta \lambda_i \delta \lambda_i dt
\]

\[
+ \int_{t_1}^{t_2} \sum_{j=1}^{m} \left( \frac{\partial F_j}{\partial \lambda_j} \delta \lambda_j \right) \delta \lambda_j dt = 0 \quad j = 1, 2, 3 \quad \in \mathbb{R}^1
\]

By the fundamental lemma of the calculus of variations, Eq.(25) becomes:

\[
\frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} + \frac{\partial F_i}{\partial \lambda_i} = 0 \quad i = 1, 2, 3, 4 \quad \text{on} \quad \gamma \quad \in \mathbb{R}^1
\]
\[
\frac{\partial F_8}{\partial W_j} - \frac{\partial F_7}{\partial W_j} - \frac{\partial F_8}{\partial W_j} = 0 \quad j = 1, 2, 3, 4 \quad \text{on} \quad \psi \in \mathbb{R}^1
\]  
(27)

\[
\frac{\partial F_8}{\partial W_j} + \frac{\partial F_{10}}{\partial W_j} + \frac{\partial F_{11}}{\partial W_j} = 0 \quad j = 1, 2, 3, 4 \quad \text{in} \quad \Omega \in \mathbb{R}^1
\]  
(28)

\[
-\frac{\partial F_{12}}{\partial W_j} = 0 \quad j = 1, 2, 3, 4 \quad \text{in} \quad \Omega \in \mathbb{R}^1
\]  
(29)

The \( \gamma \) term of Eq. (26) (j=1) is as follows:

\[
-\frac{\partial F_1}{\partial p} + \frac{\partial F_6}{\partial p} - \frac{\partial F_7}{\partial p} - \frac{\partial F_8}{\partial p} = \frac{1}{\rho} (\eta_1 - \lambda_1 \eta_1 - \lambda_2 \eta_2 - \lambda_3 \eta_3) = 0 \quad \gamma \in \mathbb{R}^1
\]  
(30)

The above equation should be satisfied at an arbitrary normal vector as follows:

\[
(\lambda_1, \lambda_2, \lambda_3) = (1, 0, 0) \quad \text{on} \quad \gamma
\]  
(31)

The \( \Psi \) term of Eq. (27) is as follows:

\[
-\frac{\partial F_8}{\partial p} + \frac{\partial F_7}{\partial p} - \frac{\partial F_8}{\partial p} = \frac{1}{\rho} (\lambda_1 \eta_1 - \lambda_2 \eta_2 - \lambda_3 \eta_3) = 0 \quad j = 1, 2, 3 \quad \text{on} \quad \psi \in \mathbb{R}^1
\]  
(32)

The above equation should also be satisfied at arbitrary normal vector as follows:

\[
(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0) \quad \text{on} \quad \psi
\]  
(33)

By using Eq. (27) (j=1) and Table 1, the \( \Psi \) term becomes:

\[
-\frac{\partial F_8}{\partial \mathbf{u}} + \frac{\partial F_7}{\partial \mathbf{u}} - \frac{\partial F_8}{\partial \mathbf{u}} = -\lambda_1 \eta_1 + \mu \left( \frac{\partial \gamma_1}{\partial x} + \frac{\partial \gamma_1}{\partial y} \right) \eta_1 + \mu \left( \frac{\partial \gamma_2}{\partial x} + \frac{\partial \gamma_2}{\partial y} \right) \eta_2 + \mu \left( \frac{\partial \gamma_3}{\partial x} + \frac{\partial \gamma_3}{\partial y} \right) \eta_3 = 0 \quad \text{on} \quad \psi \in \mathbb{R}^1
\]  
(34)

\[
-\frac{\partial F_8}{\partial \mathbf{v}} + \frac{\partial F_7}{\partial \mathbf{v}} - \frac{\partial F_8}{\partial \mathbf{v}} = -\lambda_2 \eta_2 + \mu \left( \frac{\partial \gamma_1}{\partial x} + \frac{\partial \gamma_1}{\partial y} \right) \eta_2 + \mu \left( \frac{\partial \gamma_2}{\partial x} + \frac{\partial \gamma_2}{\partial y} \right) \eta_2 + \mu \left( \frac{\partial \gamma_3}{\partial x} + \frac{\partial \gamma_3}{\partial y} \right) \eta_3 = 0 \quad \text{on} \quad \psi \in \mathbb{R}^1
\]  
(35)

\[
-\frac{\partial F_8}{\partial \omega} + \frac{\partial F_7}{\partial \omega} - \frac{\partial F_8}{\partial \omega} = -\lambda_3 \eta_3 + \mu \left( \frac{\partial \gamma_1}{\partial x} + \frac{\partial \gamma_1}{\partial y} \right) \eta_3 + \mu \left( \frac{\partial \gamma_2}{\partial x} + \frac{\partial \gamma_2}{\partial y} \right) \eta_3 + \mu \left( \frac{\partial \gamma_3}{\partial x} + \frac{\partial \gamma_3}{\partial y} \right) \eta_3 = 0 \quad \text{on} \quad \psi \in \mathbb{R}^1
\]  
(36)

From Eq. (28), we derive the following:

\[
\frac{\partial F_8}{\partial \mathbf{p}} + \frac{\partial F_{10}}{\partial \mathbf{p}} + \frac{\partial F_{11}}{\partial \mathbf{p}} = \frac{\partial \lambda_1}{\partial x} + \frac{\partial \lambda_2}{\partial x} + \frac{\partial \lambda_3}{\partial x} = 0 \quad \text{in} \quad \Omega \in \mathbb{R}^1
\]  
(37)

The subscript \( j=2 \) in Eq. (28) is as follows:

\[
\frac{\partial F_9}{\partial \mathbf{u}} + \frac{\partial F_{10}}{\partial \mathbf{u}} + \frac{\partial F_{11}}{\partial \mathbf{u}} = \lambda_1 - \frac{1}{\rho} \frac{\partial \lambda}{\partial \mathbf{u}} + \lambda_2 \left( \frac{\partial \lambda}{\partial \mathbf{u}} + \lambda_3 \frac{\partial \lambda}{\partial \mathbf{u}} \right) + \lambda_3 \left( \frac{\partial \lambda}{\partial \mathbf{u}} + \lambda_2 \frac{\partial \lambda}{\partial \mathbf{u}} \right) + \lambda_2 \left( \frac{\partial \lambda}{\partial \mathbf{u}} + \lambda_3 \frac{\partial \lambda}{\partial \mathbf{u}} \right) = 0 \quad \text{in} \quad \Omega \in \mathbb{R}^1
\]  
(38)

Eq. (38) causes inverse diffusion problems because the viscosity term in Eq. (38) has the same positive sign as the time derivative term\(^{15,16}\). Inverse diffusion problems cause numerical oscillations and can not converge. For stability reasons\(^ {15}\), the backward time is defined as follows:

\[
t = -t \quad \in \mathbb{R}^1 \quad \text{in} \quad \Omega
\]  
(39)

By using Eq. (39), we obtain:
\[ \frac{\partial \lambda}{\partial t} + \frac{1}{\rho} \frac{\partial \lambda}{\partial x} + \frac{w}{\partial y} \frac{\partial \lambda}{\partial x} + \frac{w}{\partial y} \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial x} + \frac{2}{\partial y} \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial y} = 0 \quad \text{in} \quad \Omega \]  

Similarly, the equations derived from Eq.(28) \((j=2, 3)\) are the following:

\[ \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial y} = \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial y} = 0 \quad \text{in} \quad \Omega \]  

\[ \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial y} = \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial x} + \frac{\partial \lambda}{\partial y} = 0 \quad \text{in} \quad \Omega \]  

Eq.(29) becomes:

\[ \frac{\partial F_{12}}{\partial y} = -\lambda_1 \left| \frac{\partial x}{\partial y} \right| - \lambda_2 \left| \frac{\partial x}{\partial y} \right| - \lambda_3 \left| \frac{\partial x}{\partial y} \right| = 0 \quad \text{in} \quad \Omega \in R^3 \]  

The remaining conditions are determined in a similar way.

\[ \lambda_2 \left| \frac{\partial x}{\partial y} \right| - \lambda_2 \left| \frac{\partial x}{\partial y} \right| = 0 \quad \text{in} \quad \Omega \]  

\[ \lambda_3 \left| \frac{\partial x}{\partial y} \right| - \lambda_3 \left| \frac{\partial x}{\partial y} \right| = 0 \quad \text{in} \quad \Omega \]  

2.5 Sensitivity equations

The first variation of the Lagrange function with respect to \(X \in R^3\) represents the sensitivity equation. The first variation as follows:

\[ \frac{\partial L}{\partial \frac{\partial \Gamma}{\partial y}} - \frac{\partial L}{\partial \frac{\partial \Gamma}{\partial y}} + \frac{\partial L}{\partial \frac{\partial \Gamma}{\partial y}} - \frac{\partial L}{\partial \frac{\partial \Gamma}{\partial y}} = -\frac{\partial L}{\partial \frac{\partial \Gamma}{\partial y}} + \frac{\partial L}{\partial \frac{\partial \Gamma}{\partial y}} = 0 \quad \text{in} \quad \Omega \]  

The sensitivity equation derived from Eq.(46) is as follows:

\[ \frac{\partial \lambda}{\partial y} = \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} = 0 \quad \text{on} \quad \gamma \]  

\[ \frac{\partial \lambda}{\partial y} = \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} = 0 \quad \text{on} \quad \gamma \]  

\[ \frac{\partial \lambda}{\partial y} = \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} + \frac{\partial \lambda}{\partial y} = 0 \quad \text{on} \quad \gamma \]
where $\mu$ are the viscosity coefficient. (See appendix B for details regarding this transformation.)

Sensitivities $G_1 \sim G_3$ in Eqs.(47)-(49) do not include the constant volume constraint. These sensitivities may be different from zero. By introducing the constant volume as described below (2.7 constant volume constraint), the optimal solution is decided by maintaining an equilibrium between the sensitivities Eqs.(47)-(49) and Eq.(52).

2.6 Shape modification

The variables $G_i \sim G_j$ of Eqs.(47)-(49) represent the sensitivity. When $G$ takes extremal values, this sensitivity equals to zero. By using the steepest descent method, the shape is modified so that the sensitivity becomes zero.

$$X_{(k+1)} = X_{(k)} + \beta \int_0^1 G_i(y) \gamma \, dy \quad k = 0, 1, \ldots \quad \in \mathbb{R}^3 \quad \text{on} \quad \gamma$$

The value of the coefficient $\beta$ should be small enough in order to robustly converge to the optimal coordinates and to avoid collapse of the mesh topology. The value is decided based on a heuristic search method$^{18}$.

2.7 Constant volume constraint

By using only Eqs.(47)-(49), due to the volume becoming negative, the object may cause an unrealistic deformation. This problem can be overcome by considering constraints. In this research, a constant volume constraint is implemented. Two approaches are considered.

Process A: By introducing a new adjoint variable, the constant volume constraint is added to the Lagrange function. The sensitivity is modified to satisfy the constant volume constraint.

Process B: After constructing the optimal shape with respect to the cost function, this shape is iteratively modified to satisfy the constant volume constraint.

In the former approach, the sensitivity at every shape step includes the constant volume constraint. The constant volume condition could not be sufficiently satisfied by this approach$^9$. In this study, the latter approach is implemented. Line search method is applied to the constant volume constraint. The function $h(X)$ is defined as follows:

$$h(X_{(k)}) = V(X_{(k)}) - V(X_{(0)}) \quad k = 0, 1, \ldots \quad \in \mathbb{R}^I \quad \text{on} \quad \gamma$$

Minimizing the function $h(X)$ means satisfying the constant volume constraint. To minimize $h(X)$ while maintaining the surface shape, the surface shape is deformed along an outward normal vector of the object as follows:

$$X_{(k), (l+1)} = X_{(k), (l)} + \alpha h(X_{(k), (l)}) n(X_{(k)}) \quad l = 0, 1, \ldots \quad \in \mathbb{R}^3 \quad \text{on} \quad \gamma$$

The lower subscript $(l)$ is the iteration number of the mesh deformation. As this $(l)$ is increased, the volume of the deformed shape gets closer to the volume of the initial shape. $n(X_{(k)})$ remains constant while the subscript $(l)$ is increased. The deformation amount is set to be small by multiplying the second term with a coefficient $\alpha$. In the beginning, the shape is deformed using sensitivity analysis based on the adjoint variable method. In case the deformed volume is smaller than the initial volume, the volume is slowly increased along an outward normal vector of the object surface by expanding the shape. In case the deformed volume is larger than the initial volume, the deformed volume is slowly decreased along an inward normal vector of the object surface by suppressing the shape. In other words, this algorithm is repeated until the deformed shape is in good agreement with the initial volume.

3. SHAPE OPTIMIZATION ALGORITHM

The algorithm of the proposed shape optimization method is shown in Fig.2. The fluid velocities are zero at the initial time ($t=0$). Therefore, the drag on the cylinder is zero. After that, the drag is gradually increasing while fluid inflows in the boundary $I_{w}$. The drag finally reaches a steady state within a certain range. Time is sufficiently advanced and the start of test time is set.

In the first phase of the algorithm, the state variables $(W_{(0)}^0)$ are calculated by using the state equations (Eqs (21)-(24)). The state equations are solved from the start of test time to the end of test time. All the nodal values of the state variables $(W_{(0)}^0)$ are stored at every time step. The incompressible unsteady Navier-Stokes equations are solved
by using the fractional step method. This streamline-upwind (SU) stabilization technique is applied to this method in order to prevent numerical oscillations in solving problems. In this feature of the SU method, the streamline-upwind term is only applied to the convective term in Navier Stokes equations. In the second phase of the algorithm, the adjoint variables \( \lambda_{a}^{(0)} \) are calculated by Eqs. (37), (40) - (42) from the start of test time to the end of test time. The equations \( \lambda_{a}^{(0)} = 0 \) are set up at the end time \( t_{e} \). The adjoint equations are also solved until the adjoint flow field reaches steady state. All the nodal values of the adjoint variables \( \lambda_{a}^{(0)} \) are saved at every time step. In this study, Eqs.(43)-(45) are not considered in solving the adjoint equations. This data is stored as files. In the third phase, the sensitivity at every time step is calculated by using the saved files containing the adjoint and state variables. The sensitivity represents the displacement of the nodes on the surface of the object. The sensitivity must have a small value in order to robustly converge to the optimal coordinates and to avoid collapse of the mesh topology. In the fourth phase, the shape is modified by using the time integral sensitivity. The optimization method is the steepest descent method. After that, the nodes of the mesh are relocated according to the time integral sensitivity. In the fifth phase, the shape is modified in order to satisfy the constraint of constant volume. In the case the shape converges to the optimum, the result is outputted. In the opposite case, the algorithm returns to the first phase.

Fig.2 Optimization algorithm by the adjoint variable method.

4. SHAPE OPTIMIZATION OBJECTS IN FLOW

4.1 Calculation model and conditions

The calculation conditions are shown in Table 2. The mesh is shown in Fig. 3. The mesh resolution is 121,240 nodes and 701,771 elements. The element type is 4-node tetrahedron element. The P1-P1 element with linear shape functions for velocity and pressure is used. Therefore, the tractions on the boundary \( T_{e} \) are treated as \( \rho = 0 \) and the adjoint tractions on the boundary \( \Psi \) are treated as \( \lambda_{p} = 0 \) while computing the shape optimization. The cylinder surface is shown in Fig. 4. The cylinder is divided into 72 sections along circumferential direction and 120 sections along the longitudinal direction.
4.2 Calculation results

The streamline and flow distribution of the initial shape are shown in Fig. 6. Here, views I and II are associated with the ones in Fig. 1. The fluid velocity which flows in the boundary $F_u$ decreases around the cylinder. The flow velocity on the cylinder surface is zero by the non-slip condition. The inflow velocity in the boundary $F_e$ is 10.0 and the Reynolds number is 1000. In the Reynolds number 10000, the fluid analysis in the SU method can be computed by increasing the effectiveness of the streamline upwind term, but the time history of the drag coefficient found in literature can not be obtained. Therefore the optimal shape is discussed in the Re=1000. The drag coefficient for the initial shape is shown in Fig. 5. This coefficient is almost in agreement with the forces found in literature. Lines show time spans (0.1(s), 2.5(s), 5.0(s)). In this study, three cases are calculated to confirm the optimum solution dependency with respect to time spans. The streamline of the initial shape is shown in Fig. 6. Karman vortices are not generated in the computational domain and the 3D flow is irregular. The streamline and flow distribution of the optimal shape are shown in Fig. 7. 3D streamlines generate irregularities. Intermediate shapes with respect to the shape step are shown in Fig. 8. Optimal shapes depend on the length of the time span and for different time spans, different optimal shape are obtained. The cylinder deforms like a bullet and the optimal shape is constructed as shown in Fig. 9 and Fig. 10. The normalized cost function is shown in Fig. 11. The horizontal axis represents the shape step and the vertical axis represents the normalized cost function with respect to the initial cost function. In all three cases, the cost functions are reduced. However, the reduction rates of three cost functions are different. Optimal shape depends on the time span and different optimal shapes by different time spans (the short 0.1(s), the intermediate span 2.5(s) and the long span 5.0(s)) are obtained. It is considered that these shapes are optimized at each time span. In practice, the time spans need to be determined by considering the time periodicity in flow. However, the periodicity is not obtained before the calculation. Therefore, it may be preferable to use long time spans. For all three cases, sensitivity distributions on the initial shape are shown in Fig. 12. The sensitivity distribution is symmetric with respect to the x axis in the case of long time span of 5.0(s) whereas it has asymmetry for the short time span of 0.1(s). A time span corresponding to the periodicity of the drag seems to lead to axis-symmetric objects.
5. CONCLUSION

By using the adjoint variable method, the proposed optimization technique succeeded in reducing the surface force on the object even under the difficult condition of Reynolds number 1000, which usually causes numerical instability, irregularity sensitivity distribution and the divergence of the cost function. Using smoothing, stabilization, robust mesh deformation, volume control by a volume constraint and parallelization with HEC-MW, the cost function was able to converge, making 3D shape optimization under unsteady flow possible. The reduction in the surface force for the optimal shape was 57.8% (time span 5.0(s)).

Fig.6 The streamline of the initial shape.

Fig.7 The streamline of the optimal shape.

Fig.8 Shape deformations (View II).

Fig.9 Optimal shape (View III) (The time span 5.0(s)).

Fig.10 Optimal shape (View I) (The time span 5.0(s)).

Fig.11 History of normalized cost function.
**ACKNOWLEDGMENTS**

This work was supported by the JAEA (Japan Atomic Energy Agency) and the IML (Intelligent Modeling Laboratory). The first author is grateful to JAEA for the financial support as a research student and to the IML for the financial support as a postdoctoral fellow. This work was also supported by the HEC-MW group (Frontier Simulation Software for Industrial Science (FSIS) project, which started in 2002 and has been driven for more than three years, has developed and released more than 60 pieces of most-advanced simulation software in the field of computational mechanics). This program, based on HEC-MW, was improved by useful advice from a large number of people, who also helped debug it.

**REFERENCES**


Fig.12 Sensitivity distributions (View II).
A. THE TRANSFORMATION OF THE LAGRANGE FUNCTION

The Lagrange function (Eq.(12)) is transformed to derive the adjoint equation and the sensitivity equation. By applying the Gauss-Green theorem, the second term of Eq.(12) becomes as follows:

\[
\frac{1}{\rho} \int_\Omega \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{u} \, d\Omega = \int_\Omega \lambda \rho \eta \eta \, d\Omega - \frac{1}{\rho} \int_\Omega \frac{\partial \lambda}{\partial x_i} u_i \, d\Omega i = 1, 2, 3
\]

(53)

The other term of Eq.(12) is as follows:

\[
\int_\Omega \lambda \rho \eta \eta \, d\Omega - \int_\Omega \lambda \rho \eta \eta \, d\Omega - \int_\Omega \lambda \rho \eta \eta \, d\Omega + \int_\Omega \lambda \rho \eta \eta \, d\Omega
\]

(54)

The p term, the u term, the v term and the w term in the integrand of \(\Omega\) term are rearranged by using the following equation:

\[
\int_\Omega \frac{\partial \lambda}{\partial x_i} u_i \, d\Omega = \int_\Omega \frac{\partial \lambda}{\partial x_i} u_i \, d\Omega
\]

(55)

\[
\int_\Omega \frac{\partial \lambda}{\partial x_i} u_i \, d\Omega = \int_\Omega \frac{\partial \lambda}{\partial x_i} u_i \, d\Omega
\]

(56)
B. THE TRANSFORMATION OF THE SENSITIVITY EQUATION

The sensitivity is calculated by solving the sensitivity equations, the stationary conditions which are obtained by taking the first variation of the Lagrange function with respect to the spatial coordinate \( x \). The first term in Eq.(46) (\( k=1 \)) as follows:

\[
\left[ \int_{t_s}^{t_e} \int_{\Omega} \frac{\partial F_k}{\partial x} \delta u \, dx \, dt \right] = - \int_{t_s}^{t_e} \int_{\Omega} \frac{1}{\rho} \frac{\partial p}{\partial x} n_j + \left( \frac{\partial u_j}{\partial x} + \frac{\partial u_j}{\partial x} \right) n_j \, dx \, dt
\]

\[
\left( \frac{\partial u_j}{\partial x} + \frac{\partial u_j}{\partial x} \right) n_j \, dx \, dt \quad j=1,2,3
\]  

(57)

The second term in Eq.(46) is as follows:

\[
\left[ \int_{t_s}^{t_e} \int_{\Omega} \frac{\partial F_k}{\partial x} \delta u \, dx \, dt \right] = \int_{t_s}^{t_e} \int_{\Omega} \frac{\partial \lambda_j}{\partial x} n_j + \lambda_j \frac{\partial u_j}{\partial x} + \lambda_j \frac{\partial u_j}{\partial x} n_j \, dx \, dt
\]

\[
+ \int_{t_s}^{t_e} \int_{\Omega} \frac{\partial u_j}{\partial x} + \lambda_j \frac{\partial u_j}{\partial x} \, dx \, dt
\]

(58)

The third term in Eq.(46) is as follows:

\[
\left[ \int_{t_s}^{t_e} \int_{\Omega} \frac{\partial F_k}{\partial x} \delta u \, dx \, dt \right] = - \int_{t_s}^{t_e} \int_{\Omega} \left( \frac{\partial \lambda_j}{\partial x} + \frac{\partial \lambda_j}{\partial x} \right) n_j \, dx \, dt
\]

(59)

By using Eq. (33) and Table 1, the fourth term in Eq.(46) is as follows:

\[
\left[ \int_{t_s}^{t_e} \int_{\Omega} \frac{\partial F_k}{\partial x} \delta u \, dx \, dt \right] = - \int_{t_s}^{t_e} \int_{\Omega} \left( u_j \lambda_j \frac{\partial u_j}{\partial x} + \lambda_j u_j \frac{\partial u_j}{\partial x} + \lambda_j u_j \frac{\partial u_j}{\partial x} + \lambda_j u_j \frac{\partial u_j}{\partial x} \right) \, dx \, dt = 0, \quad i,j=1,2,3
\]

(60)

The fifth term in Eq.(46) is as follows:

\[
\left[ \int_{t_s}^{t_e} \int_{\Omega} \frac{\partial \delta u}{\partial x} \delta x \, dt \right] = \int_{t_s}^{t_e} \int_{\Omega} \frac{p \partial \delta u}{\partial x} + \frac{\partial \lambda_j}{\partial x} \delta x \, dt
\]

\[
+ \int_{t_s}^{t_e} \int_{\Omega} \frac{p \partial \delta u}{\partial x} - \frac{\partial \lambda_j}{\partial x} \delta x \, dt
\]

(61)

The operation of the fourth term in Eq.(61) is as follows:

\[
\frac{\partial \delta u}{\partial x_i} = \delta \frac{\partial u}{\partial x_i} = \delta(0) = 0 \quad i \neq 1
\]

\[
\delta(1) = 0 \quad i = 1
\]

(62)

By Eq.(62), Eq.(61) is as follows:

\[
\left[ \int_{t_s}^{t_e} \int_{\Omega} \frac{\partial \delta u}{\partial x} \delta x \, dt \right] = \int_{t_s}^{t_e} \int_{\Omega} \frac{p \partial \delta u}{\partial x} + \frac{\partial \lambda_j}{\partial x} \delta x \, dt
\]

\[
+ \int_{t_s}^{t_e} \int_{\Omega} \frac{p \partial \delta u}{\partial x} - \frac{\partial \lambda_j}{\partial x} \delta x \, dt
\]

(63)

In the following equation, the operation is same. The sixth term in Eq.(46) is as follows:
\[ \int_{t_0}^{t_e} \int_{\Omega} \frac{\partial F_{\Omega}}{\partial x} \, d\mathbf{x} \, dt = \int_{t_0}^{t_e} \int_{\Omega} u \left( \frac{\partial \lambda}{\partial t} + \frac{\partial \rho}{\partial x} + u_j \frac{\partial \lambda}{\partial x_j} + v \left( \frac{\partial \lambda_j}{\partial x} + \frac{\partial \lambda}{\partial x_j} \right) \right) \, d\mathbf{x} \, dt \]

\[ + \int_{t_0}^{t_e} \int_{\Omega} \left[ \frac{\partial u}{\partial t} + \frac{\partial \rho}{\partial t} \right] \, d\mathbf{x} \, dt + \int_{t_0}^{t_e} \int_{\Omega} \frac{\partial \rho}{\partial t} \, d\mathbf{x} \, dt \]

\[ + \int_{t_0}^{t_e} \int_{\Omega} \left[ \frac{\partial \lambda_j}{\partial t} + \frac{\partial \lambda}{\partial x_j} \right] \, d\mathbf{x} \, dt + \int_{t_0}^{t_e} \int_{\Omega} \left[ \frac{\partial \lambda}{\partial t} + \frac{\partial \lambda_j}{\partial x_j} \right] \, d\mathbf{x} \, dt \]

\[ + \int_{t_0}^{t_e} \int_{\Omega} \frac{\partial u}{\partial t} \, d\mathbf{x} \, dt + \int_{t_0}^{t_e} \int_{\Omega} \frac{\partial \rho}{\partial t} \, d\mathbf{x} \, dt \]

\[ + \int_{t_0}^{t_e} \int_{\Omega} \frac{\partial \lambda_j}{\partial t} \, d\mathbf{x} \, dt + \int_{t_0}^{t_e} \int_{\Omega} \frac{\partial \lambda}{\partial t} \, d\mathbf{x} \, dt \]

By using Eq.(33) and Table 1, the seventh term in Eq.(46) is as follows:

\[ \int_{\Omega} \frac{\partial F_{\Omega}}{\partial x} \, d\mathbf{x} = \int_{t_0}^{t_e} \int_{\Omega} \frac{\partial u}{\partial t} \left( \frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x} \right) \, d\mathbf{x} \, dt = 0 \quad i, j = 1, 2, 3 \]

The eighth term in Eq.(46) is as follows:

\[ \int_{\Omega} \frac{\partial F_{\Omega}}{\partial x} \, d\mathbf{x} = -\int_{t_0}^{t_e} \int_{\Omega} \frac{\partial \lambda_j}{\partial x_j} \, d\mathbf{x} \, dt \quad i = 1, 2, 3 \]

By using Eq.(57)-(66), Eq.(46) is transformed as follows:

\[ \text{The last terms in the above equation as follows:} \]

\[ \int_{t_0}^{t_e} \int_{\Omega} \frac{\partial F_{\Omega}}{\partial x} \, d\mathbf{x} \, dt = 0 \quad i, j = 1, 2, 3 \quad \text{in} \quad \mathbb{R}^4 \]
\[ I_{is} \int \frac{\partial u_i}{\partial t} \left( \frac{\partial}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial x} \right) u_i + \frac{\partial \lambda}{\partial x} \left( \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial j} \right) \lambda_i \right) d\Omega d\alpha dt = 0 \in \mathbb{R}^1 \quad (68) \]

\[ I_{is} \int \frac{\partial u_j}{\partial t} \left( \frac{\partial}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial x} \right) u_j + \frac{\partial \lambda}{\partial x} \left( \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial j} \right) \lambda_j \right) d\Omega d\alpha dt = 0 \in \mathbb{R}^1 \quad (69) \]

The \( \Omega \) term in the Eq.(67) is as follows:

\[ \lambda_i(t_e, X) \frac{\partial u_i(t_e, X)}{\partial x} - \lambda_i(t_e, X) \frac{\partial u_i(t_e, X)}{\partial x} = 0 \in \mathbb{R}^1 \quad \text{in} \quad \Omega \quad (70) \]

By using Eqs.(43) - Eq.(45), \( \lambda_i(t_e, X) \neq 0 \), and \( \lambda_i(t_e, X) \neq 0 \), we get:

\[ \frac{\partial u_i(t_e, X)}{\partial x} - \frac{\partial u_i(t_e, X)}{\partial x} = 0 \in \mathbb{R}^1 \quad \text{in} \quad \Omega \quad (71) \]

In order to satisfy the above equation, the value of the state variable at the end time \( t_e \) must equal the value at the start time \( t_i \). The state variable at the start time and the end time is set as:

\[ u_i(t_i, X) - u_i(t_e, X) = 0 \in \mathbb{R}^1 \quad \text{in} \quad \Omega \quad (72) \]

The boundary \( \Psi \) term in the above equation is as follows:

\[ \lambda_i \left( \frac{1}{\rho} \frac{\partial p}{\partial x} \frac{\partial}{\partial x} + \nu \left( \frac{\partial u_i}{\partial x} + \frac{\partial u_i}{\partial j} \right) \frac{\partial u_i}{\partial x} \right) + \lambda_i \left( \frac{\partial}{\partial x} + \nu \frac{\partial}{\partial j} \right) \lambda_j \right) d\Omega d\alpha dt = 0 \quad \text{on} \quad \Psi \quad (73) \]

For the boundary \( \Gamma_w \) by using the normal \((n_x, n_y, n_z)=(1, 0, 0)\), we get the following:

\[ \left( \frac{\partial u_i}{\partial x}, \frac{\partial u_i}{\partial x}, \frac{\partial u_i}{\partial x} \right) = (0, 0) \quad \text{on} \quad \Gamma_w \quad (74) \]

By using Eq.(33), we get:

\[ \lambda_i \left( \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial u_i}{\partial x} + \frac{\partial u_i}{\partial j} \right) \frac{\partial u_i}{\partial x} \right) = 0 \quad \text{on} \quad \Gamma_w \quad (75) \]

By using Eqs.(34) - (36), we get:

\[ \frac{\partial u_i}{\partial x} \left( \frac{\partial}{\partial x} + \nu \left( \frac{\partial u_i}{\partial x} + \frac{\partial u_i}{\partial j} \right) \frac{\partial u_i}{\partial x} \right) = 0 \quad \text{on} \quad \Gamma_w \quad (76) \]

By using Eq.(74), Eq.(75) and Eq.(76), we show that Eq.(73) is satisfied. The rest of the boundary conditions are satisfied in a similar way.

In this study, the domain of optimization is the boundary \( \gamma \). The boundary \( \gamma \) satisfies the non-slip condition (Table 1) and Eq.(33). By using these conditions, Eq.(67) is transformed as follows:

\[ \frac{\partial u_i}{\partial x} \left( \frac{\partial}{\partial x} + \nu \left( \frac{\partial u_i}{\partial x} + \frac{\partial u_i}{\partial j} \right) \frac{\partial u_i}{\partial x} \right) = 0 \quad i, j = 1, 2, 3 \quad \text{on} \quad \gamma \quad (77) \]

For the above equations, the sensitivity equations Eq.(47)~Eq.(49) are derived. Boundary conditions \( \Psi \) can be assumed to satisfy Eqs.(34) - (36) by using the boundaries as shown in Table 1, while the boundary \( \gamma \) (the surface) can not be assumed to satisfy Eqs.(34) - (36) in the adjoint analysis because the surface force on the boundary \( \gamma \) is generated.