Numerical Computation on Analyticity of the Solution of a Cauchy Problem for the Backward Heat Equation

Hitoshi IMAI and Hideo SAKAGUCHI

Institute of Technology and Science, The University of Tokushima, Tokushima

In the paper a simple problem with two parameters for the backward heat equation is proposed. One parameter provides various situations: an initial and boundary value problem, a continuation problem and an initial value problem. Numerical results show that IPNS (Infinite-Precision Numerical Simulation) is superior in the stability rather than the simple discretization method. Numerical results by IPNS suggest existence and non-existence of the solution.

1. INTRODUCTION

The analytic continuation is a basic and textbook knowledge. Its practical application was investigated by Onishi et al. related with inverse problems. They considered the continuation problem of the Cauchy problem for the Laplace equation$^9$.

In the paper, the continuation problem for the backward heat equation and related problems are investigated. We consider the following simple problem including two parameters $T$ and $\varepsilon$.

Problem 1 For two parameters $T \in [-1, 1]$ and $\varepsilon$, find $u(t, x)$ such that

\begin{align*}
  u_t &= -u_{xx}, \quad -1 < t < 1, \quad -1 < x < 1, \quad (1) \\
  u(-1, x) &= \cos \frac{\pi}{2} x, \quad -1 < x < 1, \quad (2) \\
  u(t, -1) &= 0, \quad -1 \leq t \leq T, \quad (3) \\
  u(t, 1) &= \varepsilon(t + 1), \quad -1 \leq t \leq T, \quad (4) \\
  u(t, \pm 1) &= -u_{xx}(t, \pm 1), \quad T < t < 1. \quad (5)
\end{align*}

Remark A parameter $T$ provides the various situations in this problem as follows:

\begin{align*}
  \begin{cases}
    T = 1 & \cdots \text{an initial and boundary value problem}, \\
    -1 < T < 1 & \cdots \text{a continuation problem}, \\
    T = -1 & \cdots \text{an initial value problem}.
  \end{cases}
\end{align*}
The similar problem for the normal heat equation is investigated. Interesting numerical results are obtained\(^5\). In the paper, the equation is replaced with the backward heat equation. The exact solution of Problem 1 for \(\varepsilon = 0\) is given as follows:

\[
    u(t, x) = \exp((\frac{\pi}{2})^2(t + 1)) \cos \frac{\pi}{2} x.
\]  
(6)

Fig. 1. Profile of the exact solution for \(\varepsilon = 0\).

We have tried direct numerical computation for the backward heat equation\(^6,11,12\). It is very difficult. Simple discretization, e.g. FDM@Euler(2nd order FDM for \(x\), explicit Euler method for \(t\)) easily fails. This is because numerical errors grow up exponentially. To avoid such difficulty, we developed IPNS(Infinite-Precision Numerical Simulation)\(^7\). Numerical results by both methods(FDM@Euler and IPNS) for Problem 1 with \(T = 1\) and \(\varepsilon = 0\) are as follows.

(a) Solution profiles by FDM@Euler
(b-1) $N = 10$  
(b-2) $N = 30$  
(b-3) $N = 50$  
(b-4) Maximum error

(b) Solution profiles and errors by IPNS

Fig. 2. Numerical results($T = 1, \varepsilon = 0, 200\text{digits}$).

In FDM+Euler, for a given positive integer $N_x$, $\Delta x = 2/N_x$, $\Delta t = 0.2(\Delta x)^2$, $N_t = 2/\Delta t$. This setting works well for the normal heat equation and it is used here. In IPNS, the Chebyshev-Gauss-Lobatto collocation method\(^1\) and the multiple precision arithmetic library: exflib\(^2\) are used. For the simplicity we set $N = N_x = N_t$ where $N_x$ and $N_t$ in IPNS denote the approximation order about $x$ and $t$, respectively. Maximum error in Fig. 2 (b-4) is defined as follows:

$$Err = \max_{0 \leq j \leq N, 0 \leq i \leq N-1} \left| u_{ij} - u(t_j, x_i) \right|, \quad t_j = \cos \frac{j\pi}{N}, \quad x_i = \cos \frac{i\pi}{N},$$

(7)

where $u_{ij}$ and $u(t_j, x_i)$ represent the numerical solution and the exact solution (6), respectively.

In the paper, other situations for various $T$ and $\varepsilon$ are computed. The problem and the numerical methods are simple. However, numerical results are interesting and they suggest a new possibility of numerical simulation. It means the numerical simulation of non-existence of the solution. In the previous work\(^{10}\) IPNS was performed to the following inverse problem: For $\delta \geq 0$ find $u(x, y)$ such that

\begin{align*}
\Delta u(x, y) &= 0, \quad \text{in} \quad (0, 1) \times (0, 1), \\
\frac{\partial u}{\partial y}(x, 0) &= f_\delta(x), \quad 0 \leq x \leq 1
\end{align*}

(8)  
(12)
where

\[
f_\delta(x) = \begin{cases} 
    \frac{1}{\pi} \sin \left( \frac{x - \frac{\delta}{2}}{1 - \delta} \pi \right), & 0 \leq x < \frac{\delta}{2}, \\
    0, & \frac{\delta}{2} \leq x \leq 1 - \frac{\delta}{2}, \\
    0, & 1 - \frac{\delta}{2} < x \leq 1.
\end{cases}
\]  

(13)

For \( \delta = 0 \) there is an exact solution as follows:

\[
u(x, y) = \frac{1}{\pi^2} \sin (\pi x) \sinh (\pi y).
\]  

(14)

In this case numerical solutions by IPNS converge to the exact solution (14) rapidly as shown in Fig. 3. Here \( N \) denotes the approximation order for \( x \) and \( y \) in the Chebyshev-Gauss-Lobatto collocation method.

\[\text{Fig. 3. Numerical Results by IPNS for } \delta = 0 \text{ (120 digits)}^{10}.\]

For \( \delta > 0 \) the solution does not exist\(^9\). In this case numerical solutions by IPNS involve oscillation and they do not converge as shown in Fig. 4.

\[\text{Fig. 4. Solution profiles by IPNS for } \delta = 0.0001 \text{ (120 digits)}^{10}.\]
This shows that non-converging numerical solutions by IPNS suggest non-existence of the solution. The similar results are obtained in Problem 1 as mentioned in §2. However, non-existence of the solution is not obvious theoretically. This means that our results may be useful for theoretical analysis and develop numerical simulation of non-existence of the solution.

2. NUMERICAL RESULTS

In this section the various cases for Problem 1 are computed. Numerical computation by FDM+Euler is unstable and meaningless as in Fig. 2. So, it is omitted. The same IPNS as in Fig. 2 is used.

2.1 In the case of $T = 1$ and $\varepsilon = 1$

For $T = 1$ and $\varepsilon = 0$ there exists the exact solution (6). So, here $\varepsilon$ is changed to be 1 and numerical computation is carried out. We are not sure that the exact solution does exist or not. Numerical results are shown in Fig. 5. Numerical solutions do not converge. Numerical solutions for $\varepsilon = 10^{-10}$ which are not shown here do not also converge. From our experience mentioned in §1 they suggest non-existence of the solution for $\varepsilon \neq 0$.

![Graphs of solution profiles for different $N$ values](image)

(a) $N = 10$  
(b) $N = 30$  
(c) $N = 50$

Fig. 5. Solution profiles by IPNS($T = 1$, $\varepsilon = 1$, 200digits).

At first, we considered that $u_t \neq -u_{xx}$ at $(t, x) = (-1, 1)$ was the reason for non-convergence of numerical solutions. So, we carried out additional numerical computation to the following problem where the boundary condition (4) is replaced with (18).
Problem 2  Find $u(t, x)$ such that

\[ u_t = -u_{xx}, \quad -1 < t < 1, \quad -1 < x < 1, \quad (15) \]
\[ u(-1, x) = \cos \frac{\pi}{2} x, \quad -1 < x < 1, \quad (16) \]
\[ u(t, -1) = 0, \quad -1 \leq t < 1, \quad (17) \]
\[ u(t, 1) = \varepsilon (1 + \sin \frac{\pi}{2} t), \quad -1 \leq t < 1. \quad (18) \]

In this problem, $u_t = -u_{xx}$ at $(t, x) = (-1, 1)$. Numerical results are following.

Fig. 6. Solution profiles by IPNS($\varepsilon = 1$, 200digits).

Numerical solutions do not converge again. Fig. 6 also suggests non-existence of the solution. This means that our first consideration was wrong and there is a different and highly mathematical reason. It may concern with the relationship between analyticity and existence.

In some problems existence of the solution means analyticity of the solution\(^9\). If the solution is analytic IPNS captures it very accurately. If this case corresponds with this situation numerical results here show non-existence of the solution.

2.2 In the case of $T = 0$ and $\varepsilon = 0$

In this case Problem 1 becomes a continuation problem. The exact solution (6) of Problem 1 for $\varepsilon = 0$ becomes an exact solution in this case. However, we are not sure about the uniqueness. Numerical results are shown in Fig. 7. $Err$ in Fig. 7 (d) is the same as (7). Numerical solutions
converge to the exact solution (6) of Problem 1 for $\varepsilon = 0$. IPNS succeeds to capture at least one analytic solution as an exact solution. If there are some solutions the numerical method tends to choose the most regular solution\(^4\). In this case an analytic function (6) is the most regular solution. So, even if the uniqueness of the solution is not guaranteed in this problem, IPNS captures (6) as a solution. The similar results are obtained for the normal heat equation\(^5\).

\[ u(x,t) \]
\[ u(x,t) \]
\[ \text{(a) } N = 10 \]
\[ \text{(b) } N = 30 \]
\[ \text{(c) } N = 50 \]
\[ \text{(d) Maximum error} \]

Fig. 7. Solution profiles and errors by IPNS($T = 0$, $\varepsilon = 0$, 200digits).

2.3 In the case of $T = 0$ and $\varepsilon = 1$

In this case Problem 1 becomes a continuation problem. When $\varepsilon = 0$ IPNS succeeds to capture an exact solution(Fig. 7). So, here $\varepsilon$ is changed to be 1. Numerical results are shown in Fig. 8 and they are quite different from Fig. 7. Numerical solutions for $\varepsilon = 10^{-10}$ which are not shown here do not also converge. They suggest non-existence of the solution for $\varepsilon \neq 0$ from the same reason discussed in §1 and §2.1.

\[ u(x,t) \]
\[ u(x,t) \]
\[ \text{(a) } N = 10 \]
\[ \text{(b) } N = 30 \]
2.4 In the case of $T = -1$

In this case Problem 1 becomes an initial value problem and there exists an exact solution (6) as in the case of $\varepsilon = 0$. If the equation is a normal heat equation non-uniqueness of the solution is well known\(^5\). However, we are not sure about uniqueness of the solution in this case. Numerical results are shown in Fig. 9. Numerical solutions converge to the exact solution (6). IPNS succeeds to capture at least one analytic solution as an exact solution. The reason why IPNS captures this function is discussed in §2.2.

Fig. 8. Solution profiles by IPNS($T = 0$, $\varepsilon = 1$, 200 digits).

Fig. 9. Solution profiles and errors by IPNS($T = -1$, 200 digits).
3. CONCLUSION

We proposed a simple problem for the backward heat equation which has parameters $T$ and $\varepsilon$. $T$ determines several situations: an initial value problem, an initial and boundary value problem and a continuation problem. We carried out numerical computation following IPNS. For $\varepsilon = 0$ there exists an analytic function as an exact solution. Numerical solutions by IPNS converge to this analytic function. This is because even if the uniqueness is not guaranteed numerical methods tend to choose the most regular solution which is in our problem an analytic solution. Numerical results by IPNS suggest non-existence of the solution for $\varepsilon \neq 0$. Relationship between existence and analyticity of the solution may concern. Its theoretical analysis is our future work. Furthermore, theoretically the proof for non-existence of the solution may be difficult rather than the proof for existence of the solution. From this viewpoint our numerical simulation IPNS may be useful for the proof of non-existence of the solution.

Acknowledgement

This work was partially supported by Grant-in-Aid for Scientific Research(Nos. 16340024, 16340029, 18654023, 18340045).

REFERENCES


