Effect of Deformation Radius on Stability of Flow

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To study effects of deformation radius on stability of flow, we perform a linear stability analysis of a parallel shear flow whose velocity profile has a single maximum in the context of the Charney–Hasegawa–Mima equation. The linear stability analysis shows that as the deformation radius decreases, both the wavenumber and growth rate of the fastest growing mode decrease. To understand these results physically, first, the concept of resonance between neutral waves is applied. The analysis based on this concept well predicts the wavenumber of the fastest growing mode. Next, we introduce a low-degree-of-freedom system which represents a qualitative picture of the linear stability of the parallel shear flow. Combining this system with the concept of wave resonance, we derive a theoretical prediction of the growth rate of the fastest growing mode as a function of deformation radius. This prediction is in good agreement with the growth rate of the fastest growing mode calculated by the linear stability analysis.

1. Introduction

Large scale extratropical atmospheric and oceanic fluid motions are strongly influenced by planetary rotation and density stratification. The Charney-Hasegawa-Mima (CHM) equation (e.g. Pedlosky 1), a simple model for such flows, is the governing equation of motion of a rotating shallow homogeneous fluid layer with a free surface under the quasi-geostrophic approximation:

$$\frac{\partial \psi}{\partial t} + \partial (\psi, q) = 0, \quad q = \nabla^2 \psi - \lambda^2 \psi,$$

(1)

where $\psi(x, y, t)$ is a stream function, $\partial(A, B) \equiv \partial_x A \partial_y B - \partial_y A \partial_x B$ is the Jacobian operator, $\lambda = L/R_d$, $L$ is a typical horizontal scale of the flow, $R_d$ is the Rossby deformation radius and $\nabla^2 \equiv \partial_{xx} + \partial_{yy}$ is the horizontal Laplacian.

(1) can be rewritten as

$$\frac{\partial \psi}{\partial t} = \frac{1}{2\pi} \int \int K_0(\lambda|x - x'|) \partial(\psi(x'), q(x')) \, dx',$$

(2)

where $K_0$ is the modified Bessel function of the second kind of order zero 2. Because $K_0(z) \sim \exp(-z)$ for large values of $z$, the interaction between flow structures in the CHM system becomes weaker as $\lambda$ increases (i.e. the deformation radius is made smaller). Therefore, the deformation radius affects stability properties of flow. The reason is as follows. It is known that there exist cases that hydrodynamic instability occurs when neutral waves resonate 3. In the case of small deformation radius, only long waves can interact with each other because the interaction range of the CHM system is short. Therefore, it is expected that as the deformation radius is made smaller, a wavelength of neutral waves which resonate to produce instability becomes longer.
As an empirical fact, in stability problem, a flow pattern is dominated by the fastest growing mode, so the maximum growth rate is important. Although it is not difficult to solve an eigenvalue problem in linear stability problem, a concrete expression for a dependence of the maximum growth rate of the disturbance field on deformation radius has not been obtained. This is one of our motivation for this study.

Another motivation is as follows. In linear case, according to the concept of wave resonance \(^3\), the resonance of two waves, instead of interaction of various flow structures such as vortex, waves and mean flow, produces instability. This raises a question whether we can construct a low-degree-of-freedom model which represents an essence of instability and gives a prediction of dependence of maximum growth rate on deformation radius.

In this paper, we study linear stability of a parallel shear flow in the context of the CHM equation to address the above motivations. In addition, we obtain a low-degree-of-freedom system which represents a qualitative picture of linear instability of shear flow in the case of finite deformation radius. This system gives a theoretical prediction of disturbance growth rate. Combining this with the concept of wave resonance, we obtain a theoretical prediction of maximum growth rate which is dependent on the deformation radius. This prediction is in good agreement with the maximum growth rate calculated by the linear stability analysis.

2. Linear stability analysis

Before giving the basic state, we derive a normal mode equation for parallel shear flows. Consider a solution to the CHM equation in the form

\[ \psi(x, y, t) = \Psi(y) + \tilde{\psi}(x, y, t) \]  

where \( \Psi(y) \) is the steady solution to the CHM equation satisfying

\[ \partial \langle \Psi, Q \rangle = 0, \quad Q = \nabla^2 \Psi - \lambda^2 \Psi = \Psi_{yy} - \lambda^2 \Psi \]  

and the disturbance stream function \( \tilde{\psi} \) is small. Substituting (3) to (1) and neglecting quadratic terms in the disturbance stream function \( \tilde{\psi} \), we obtain the linearized equation

\[ \tilde{q}_t - \Psi_y \tilde{q}_x + \tilde{\psi}_y Q_y = 0, \quad \tilde{q}(x, y, t) = \nabla^2 \tilde{\psi} - \lambda^2 \tilde{\psi}. \]  

Assuming \( \tilde{\psi} = \tilde{\psi}(y) \exp[i k(x - ct)] \), (5) becomes

\[ (U - c)\{\tilde{\psi}_{yy} + (-\lambda^2 - \lambda^2)\tilde{\psi}\} + Q_y \tilde{\psi} = 0, \]  

where \( U(y) = -\Psi_y \) is the steady \( x \)-direction velocity.

In geophysical fluid dynamics, there have been many studies of stability of flows. One of central issues in such field is the stability of jet. In this paper, we focus on the linear stability of a jet-like basic flow. In addition, to solve (6) analytically, we set this basic flow to give a piecewise constant potential vorticity basic state. The basic flow considered is

\[ U(y) = \begin{cases} 
0 & (-\infty < y \leq -1) \\
-A_1 \lambda e^{\lambda y} + A_2 \lambda e^{-\lambda y} & (-1 \leq y \leq 0) \\
A_2 \lambda e^{\lambda y} - A_1 \lambda e^{-\lambda y} & (0 \leq y \leq 1) \\
0 & (1 \leq y < \infty) 
\end{cases} \]
where \( A_1 = e^{2\lambda}A_2 \) and \( A_2 = \{\lambda (e^{2\lambda} + 1)\}^{-1} \). For \( \lambda > 0 \), the velocity profile (7) has a single maximum at \( y = 0 \) and gives \( U(0) = 1 \). The stream function and potential vorticity corresponding to (7) are given by

\[
\Psi(y) = \begin{cases} 
-\lambda^{-2} Q_1 & (-\infty < y \leq -1) \\
A_1 e^{\lambda y} + A_2 e^{-\lambda y} - \lambda^{-2} Q_2 & (-1 \leq y \leq 0) \\
-A_2 e^{\lambda y} - A_1 e^{-\lambda y} + \lambda^{-2} Q_2 & (0 \leq y \leq 1) \\
\lambda^{-2} Q_1 & (1 \leq y < \infty)
\end{cases}
\]  

and

\[
Q(y) = \begin{cases} 
Q_1 & (-\infty < y \leq -1) \\
Q_2 & (-1 \leq y \leq 0) \\
-Q_2 & (0 \leq y \leq 1) \\
-Q_1 & (1 \leq y < \infty)
\end{cases}
\]  

respectively, where \( Q_1 = \lambda^2(1 + e^{2\lambda} - 2e^\lambda)A_2 \) and \( Q_2 = \lambda^2(1 + e^{2\lambda})A_2 \). Fig. 1 shows \( U(y) \) and \( Q(y) \) for \( \lambda = 1.0 \).

In each continuous interval where potential vorticity is constant, (6) reduces to

\[
\ddot{\psi}(y) + (\alpha^2 - \lambda^2)\dot{\psi} = 0.
\]  

Assuming the boundary condition \( \dot{\psi} \to 0 \) as \( z \to \pm \infty \), it follows from (10) that

\[
\ddot{\psi}(y) = \begin{cases} 
B_1 e^{\alpha y} & (-\infty < y \leq -1) \\
B_2 e^{\alpha y} + B_3 e^{-\alpha y} & (-1 \leq y \leq 0) \\
B_4 e^{\alpha y} + B_5 e^{-\alpha y} & (0 \leq y \leq 1) \\
B_6 e^{-\alpha y} & (1 \leq y < \infty)
\end{cases}
\]  

where \( \alpha = \sqrt{\lambda^2 + k^2} \), and \( B_1, B_2, B_3, B_4, B_5 \) and \( B_6 \) are constants.
In this problem, two matching conditions are needed. The first matching condition is continuity of \( \psi \) at \( y = 0, \pm 1 \), which requires

\[
B_1 e^{-\alpha} = B_2 e^{-\alpha} + B_3 e^\alpha, \tag{12}
\]
\[
B_2 + B_3 = B_4 + B_5, \tag{13}
\]
\[
B_4 e^\alpha + B_5 e^{-\alpha} = B_6 e^{-\alpha}. \tag{14}
\]

Replacing the derivative with respect to \( y \) in (6) by the finite difference, we derive the second matching condition:

\[
\lim_{\varepsilon \to 0} \left[ (U - c) \left\{ \frac{1}{2\varepsilon} \left[ \psi_{y}^{y+\varepsilon} - \alpha^{2} \psi \right] + \frac{1}{2\varepsilon} \left[ Q_{y}^{y+\varepsilon} - \psi \right] \right\} \right] = 0 \quad \text{at} \quad y = 0, \pm 1. \tag{15}
\]

It follows from (11) and (15) that

\[
\alpha (B_2 e^{-\alpha} - B_3 e^\alpha - B_4 e^{-\alpha} - B_5 e^\alpha) - \frac{D_1 B_1 e^{-\alpha}}{c} = 0, \tag{16}
\]
\[
\alpha (B_4 - B_5 - B_2 + B_3) + \frac{D_2 (B_2 + B_3)}{U_0 - c} = 0, \tag{17}
\]
\[
\alpha (-B_6 e^{-\alpha} - B_4 e^\alpha + B_5 e^{-\alpha}) - \frac{D_1 B_0 e^{-\alpha}}{c} = 0. \tag{18}
\]

where \( U_0 = U(0) = 1, D_1 = Q_2 - Q_1 \) and \( D_2 = -2Q_2 \). When a set of six equations (12)-(14) and (16)-(18) has a nontrivial solution, \( c \) satisfies

\[
E_1 c^2 + E_2 c + E_3 = 0, \tag{19}
\]

where

\[
E_1 = \frac{-4\alpha^2}{D_1}, \tag{20}
\]
\[
E_2 = 2\alpha \left( -1 - e^{-2\alpha} + \frac{2\alpha U_0}{D_1} - \frac{D_2}{D_1} \right), \tag{21}
\]
\[
E_3 = 2\alpha U_0 (1 + e^{-2\alpha}) + D_2 (-1 + e^{-2\alpha}). \tag{22}
\]

If \( E_2^2 - 4E_1 E_3 < 0 \), instability occurs and the growth rate of the perturbation field is given by \( kc_f \) where

\[
c_f = \frac{\sqrt{E_2^2 - 4E_1 E_3}}{2E_1}. \tag{23}
\]

3. Results and Discussions

Fig. 2 shows dispersion curves and growth rate as a function of \( k \) for \( \lambda = 1.0 \). Next, we shall investigate the effect of deformation radius on stability of flow. Since a flow pattern is dominated by the fastest growing mode in stability problem, we give attention to this mode. Fig. 3 shows a growth rate of the fastest growing mode as a function of \( \lambda \) calculated by the linear stability analysis. The maximum growth rate calculated by linear stability analysis decreases as \( \lambda \) increases. In addition, the wavenumber of the fastest growing mode calculated by linear stability analysis becomes smaller as \( \lambda \) becomes larger, as shown in Fig. 4.
To understand these results physically, we shall interpret these results in terms of wave resonance. We begin by calculating the dispersion curves for the following two systems:

\[
U(y) = \begin{cases} 
-A_1 \lambda e^{\lambda y} + A_2 \lambda e^{-\lambda y} & (-\infty < y \leq 0) \\
A_2 \lambda e^{\lambda y} - A_1 \lambda e^{-\lambda y} & (0 \leq y < \infty)
\end{cases}
\]  \hspace{1cm} (24)

and

\[
U(y) = \begin{cases} 
A_2 \lambda e^{\lambda y} - A_1 \lambda e^{-\lambda y} & (-\infty < y \leq 1) \\
0 & (1 \leq y < \infty)
\end{cases}
\]  \hspace{1cm} (25)

For (24) and (25), we find the dispersion curves

\[c_1 = U_0 - \frac{D_2}{2\alpha} \quad \text{and} \quad c_2 = -\frac{D_1}{2\alpha},\]

respectively. For \(\lambda = 1.0\), the growth rate achieves its maximum near the wavenumber where two dispersion curves for (24) and (25) intersect (i.e., two waves resonate) in Fig. 2. Therefore, for \(\lambda = 1.0\), the maximum instability is produced by wave resonance.
Fig. 3: Solid line indicates the growth rate of the fastest growing mode for each \( \lambda \) calculated by the linear stability analysis. Circle indicates \( k_x c_\sigma \) given by (42), which is the growth rate of the solution of the simplified system (34)-(36).

\[
\begin{align*}
\alpha = \sqrt{k_x^2 + \lambda^2} &\equiv \alpha_s, \\
k &= \left\{ \left( \frac{D_2 - D_1}{2U_0} \right)^2 - \lambda^2 \right\}^{1/2} \equiv k_x
\end{align*}
\]

(27)

which is a theoretical prediction of the wavenumber of the fastest growing mode. As shown in Fig. 4, \( k_x \) approximately agrees with the wavenumber of the fastest growing mode for each \( \lambda \) calculated by the linear stability analysis. Therefore, we conclude that the maximum instability is produced by wave resonance for \( \lambda = 1.0 \sim 4.0 \).

From the viewpoint of the wave resonance, the results concerning the maximum growth rate derived by linear stability analysis in Fig. 3 can be explained as follows. In this case, instability is produced by the interaction of the waves. As the deformation radius is made smaller (i.e.
λ is made larger), the interaction range of the CHM system becomes shorter and therefore interaction between waves becomes weaker. Therefore, the maximum growth rate decreases as the deformation radius decreases.

In the case of small deformation radius, only the long waves can interact with each other because the interaction range of the system is short. Therefore, as the deformation radius decreases, the wavenumber of the fastest growing mode decreases, as shown in Fig. 4.

As we discussed, instability can be understood in terms of the resonance between waves at interfaces where the basic state potential vorticity is discontinuous. Therefore, we can expect that there exists a low-degree-of-freedom model which represents a qualitative picture of instability of flow. Here we shall construct a such model. The method used here is similar to that described by Vallis in the context of the 2D Euler equation, which has infinite deformation radius. Although Vallis started from the equation of motion, we start from the linearized potential vorticity equation (5). In this problem, the derivation is based on the fact that the instability is produced by the interaction of the waves on the interface at \( y = 0 \) and \( y = ±1 \).

\[
\hat{\psi}_0 = \begin{cases} 
\hat{\psi}_0(t)e^{-\alpha y}e^{ikx} & (y > 0) \\
\hat{\psi}_0(t)e^{\alpha y}e^{ikx} & (y < 0)
\end{cases}
\]  
(28)

\[
\hat{\psi}_+ = \begin{cases} 
\hat{\psi}_+(t)e^{-\alpha(y-1)}e^{ikx} & (y > 1) \\
\hat{\psi}_+(t)e^{\alpha(y-1)}e^{ikx} & (y < 1)
\end{cases}
\]  
(29)

\[
\hat{\psi}_- = \begin{cases} 
\hat{\psi}_-(t)e^{-\alpha(y+1)}e^{ikx} & (y > -1) \\
\hat{\psi}_-(t)e^{\alpha(y+1)}e^{ikx} & (y < -1)
\end{cases}
\]  
(30)

Our starting point is the linearized potential vorticity equation (5). However, we omit the term \(-\Psi_y\hat{\psi}_x = U\hat{\psi}_x\) in (5) because this term represents the advection of the potential vorticity by the basic flow \( U \) and does not contribute to the growth of disturbance:

\[
\frac{\partial}{\partial t} \left[ \hat{\psi}_x - \hat{\psi}_y - \lambda^2\hat{\psi}_x + \hat{\psi}_y Q_y = 0, \right.
\]  
(31)

where \( \hat{\psi}_x = -\hat{\psi}_y \) and \( \hat{\psi} = \hat{\psi}_x \). Replacing the derivative with respect to \( y \) in (31) by the finite difference, we obtain

\[
\lim_{\epsilon \to 0} \frac{\partial}{\partial t} \left[ \hat{\psi}_x - \frac{1}{2\epsilon} \left[ -\hat{\psi}_y^{+\epsilon} - \lambda^2\hat{\psi}_x \right] + \frac{1}{2\epsilon} Q_{y-\epsilon}^{+\epsilon} = 0. \right.
\]  
(32)

It follows from (32) that

\[
\frac{\partial}{\partial t} \left[ -\hat{\psi}_y^{+\epsilon} + \hat{\psi}_y Q_{y-\epsilon}^{+\epsilon} = 0 \right. \quad \text{at} \quad y = 0, ±1.
\]  
(33)

At \( y = 1 \), the effect of \( \hat{\psi}_- \) is very small compared to that of \( \hat{\psi}_0 \). (In fact, \( \hat{\psi}_- \) is proportional to \( e^{-\alpha y} \)) Therefore, the effect of \( \hat{\psi}_- \) is negligible at \( y = 1 \). Similarly, the effect of \( \hat{\psi}_+ \) is negligible at \( y = -1 \). Therefore, we obtain a simplified system

\[
\frac{\partial}{\partial t} \left[ -\hat{u}_0(0) + \hat{u}_0(0) \right] + (\hat{v}_0(0) + \hat{v}_-(0)) |Q|^{+0}_{-0} = 0,
\]  
(34)

\[
\frac{\partial}{\partial t} \left[ -\hat{u}_0(1) + \hat{u}_0(1) \right] + (\hat{v}_0(1) + \hat{v}_-(1)) |Q|^{+0}_{1-0} = 0,
\]  
(35)

\[
\frac{\partial}{\partial t} \left[ -\hat{u}_0(-1) + \hat{u}_0(-1) \right] + (\hat{v}_0(-1) + \hat{v}_-(1)) |Q|^{-0}_{-1-0} = 0.
\]  
(36)
where
\[
\tilde{u}_a(b) = -\left. \frac{\partial \tilde{\psi}_a}{\partial y} \right|_{y=b}, \quad \tilde{v}_a(b) = \left. \frac{\partial \tilde{\psi}_a}{\partial x} \right|_{y=b}, \quad a = 0, \pm .
\] (37)

Substitution of (28)-(30) into (34)-(36) yields
\[
-2\alpha \frac{d\hat{\psi}_0}{dt} + ike^{-\alpha}(\hat{\psi}_+ + \hat{\psi}_-) |Q|_{+0} = 0,
\] (38)
\[
-2\alpha \frac{d\hat{\psi}_+}{dt} + ike^{-\alpha} \hat{\psi}_0 |Q|_{1+0} = 0,
\] (39)
\[
-2\alpha \frac{d\hat{\psi}_-}{dt} + ike^{-\alpha} \hat{\psi}_0 |Q|_{-1+0} = 0.
\] (40)

It follows from (38)-(40) that
\[
\hat{\psi}_0 \sim \exp \left[ k \left( (2\alpha)^{-1} e^{-\alpha} \sqrt{-[Q]_{+0}}(\sqrt{|Q|_{+0} + |Q|_{-1+0}}) \right) t \right]
= \exp \left[ k \left( \alpha^{-1} e^{-\alpha} \sqrt{-Q(1+0)(Q(1+0) - Q(1-0))} \right) t \right]
= \exp \left[ k \left( \alpha^{-1} e^{-\alpha} \sqrt{Q(1+0)} \right) t \right]
= \exp \left[ k \left( \alpha^{-1} e^{-\alpha} \sqrt{-D_1D_2/2} \right) t \right].
\] (41)

where we used (9). Next, we shall investigate whether this growth rate agrees with the maximum growth rate calculated by linear stability analysis. Setting \( k = k_s \) and \( \alpha = \alpha_s \) in (41), we obtain a theoretical prediction of maximum growth rate:
\[
k_s c_s = k_s \alpha_s^{-1} \exp(-\alpha_s) \sqrt{-D_1D_2/2}.
\] (42)

As shown in Fig. 3, \( k_s c_s \) is in good agreement with the growth rate of the fastest growing mode calculated by the linear stability analysis. Therefore, we conclude that the system (34)-(36) represents a good qualitative picture of linear instability of parallel shear flow in the case of finite deformation radius.

4. Summary

We have investigated the linear stability of a parallel shear flow governed by the CHM equation. The basic state considered is a velocity profile which has a single maximum. The linear stability analysis shows that as the deformation radius decreases, both the wavenumber and growth rate of the fastest growing mode decrease. To understand these results physically, the concept of resonance between neutral waves is applied. The analysis based on this concept well predicts the wavenumber of the fastest growing mode.

We obtained a low-degree-of-freedom system which represents a qualitative picture of linear instability of shear flow in the case of finite deformation radius. This system gives a theoretical prediction of disturbance growth rate. Combining this with the concept of wave resonance which gives a theoretical prediction of the wavenumber of the fastest growing mode, we derived a theoretical prediction of maximum growth rate. This is in good agreement with the maximum growth rate calculated by linear stability analysis.

Once instability occurs, disturbance grows and then nonlinear terms are not negligible even if the initial disturbance is very small. Although the concept of wave resonance succeeded to
elucidate the mechanism of instability in linear case (e.g. Cairns \(^3\), Hayashi and Young \(^5\), Sakai \(^6\), Iga\(^7\), Iga and Matsuda\(^8\), Taniguchi and Ishiwatari\(^9\)), the theory has not been generalized to be applicable to nonlinear case. As we said, a flow pattern is dominated by the fastest growing mode. Our simplified model represents the character of such mode. Therefore, our model can be expected to give information which is useful for a generalization of the theory to nonlinear case because a low-degree-of-freedom system obtained from a given fluid system helps us to gain insight into the physics of phenomenon.

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