Effect of Timoshenko Coefficient in Wave Propagation Analysis of a Three-Dimensional Frame Structure

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The Timoshenko theory is usually used in the vibration analysis of a beam because it can yield a relatively good approximation in the high-frequency domain. This theory uses the so-called Timoshenko coefficient to couple the bending and shear deformations. This coefficient is calculated using the shear deformation distribution within the cross section that arises from the bending deformation of a beam. The shear deformation distribution is calculated based on certain assumptions. In this study, we reexamine Cowper’s theory that introduces the least assumptions in computing the Timoshenko coefficient in the view point of dynamic behavior. The relationship between the Timoshenko coefficient and the dispersion relation is investigated. The evaluation of the shear stress distribution within this cross section is attempted using numerical simulation. Finally the effect of the Timoshenko coefficient on the wave propagation analysis of a beam is examined.

1. INTRODUCTION

It is necessary to perform maintenance management experiments and aging inspection experiments under real conditions for the highly reliable inspection of the safety and conservation of nuclear plants. However, it actually requires a long time with enormous costs. In addition, it may sometimes be very dangerous. Therefore, at our center, a trial investigation was recently performed on the secure and effective integrity evaluation of nuclear plants using computer science technology. In the first step of this trial, a three-dimensional vibration simulation system of the entire scale of the nuclear plant, which cannot be vibrated on a real shaking table, is researched and developed. As one of the technology components, the physical modeling for the simulation of vibration energy that propagates between the enormous parts of nuclear plants is performed for frame structures such as piping system, which are considered to be among the weakest and most vulnerable components of nuclear plants.

Recently, an analytical method that uses Fourier transformation and the finite element method have been used to estimate the natural frequencies and wave propagation properties of the beam structure\(^{12}\). This method is called the spectral finite element method (SFEM) or spectral element method (SEM), etc.; this method has a great advantage that can compute wave propagation phenomenon of structures. Also it has an advantage of computational speed with regard to its ability to use the FFT algorithm. In reference 3), the authors applied this SFEM to some kinds of lattice-type shell structures. In this paper, we are planning to use SFEM for analysis of piping structures in a nuclear plant.

In order to treat a bending deformation in vibration analysis of a beam, the Bernoulli-Euler theory or the Timoshenko theory is generally used. The Bernoulli-Euler theory is frequently used for the vibration analysis of a beam in the low-frequency domain such as that in earthquakes. In reference 3), the Bernoulli-Euler theory is used. However, the Timoshenko theory is widely used for the mechanical
vibration analysis or collision analysis of facilities and machinery for obtaining a comparatively good approximation in the high-frequency domain. Therefore we introduced the Timoshenko theory for SFEM\(^9\). The Timoshenko coefficient is introduced by taking account of the interactions between the bending and the shear deformations in the Timoshenko theory. This coefficient is calculated by the shear deformation distribution in a section that is generated by the bending deformation of the beam. In reference 4), the Timoshenko coefficient is assumed as constant value. However, the shear deformation distribution in a section must be calculated on the basis of plausible assumption. Various assumptions or theories have been suggested to date\(^5\)-\(^9\). The theory by Cowper\(^9\) introduces the least assumptions in computing the Timoshenko coefficient and is widely used. It is said that the assumption is good for the problem of static loading. However, the references that discussed for the problem of dynamic loading, especially wave propagation problem, are very few as far as we examined. Therefore, the theory by Cowper\(^9\) is reexamined in this study to introduce in SFEM for analysis of three-dimensional piping structure in the viewpoint of dynamic behavior. The examination of the influence of the dispersion characteristics of the phase velocity of the beam and the investigation of the sectional stress distribution by the Timoshenko coefficient are shown in this paper. Some numerical simulations by using the Timoshenko coefficient obtained by Cowper's theory are shown and the effects of the Timoshenko coefficient are considered.

2. FORMULATION FOR CALCULATION OF TIMOSHENKO COEFFICIENT

In several types of steel tubes, the Timoshenko coefficients are calculated by introducing the ratio of the inner diameter \(b\) to the outer diameter \(a\), i.e., \(m = b/a\). The calculation of the Timoshenko coefficient of a beam has been often performed to date\(^5\)-\(^9\). Here, the formulation by Cowper is considered as it provides a closest sectional stress distribution to the exact solution. The Cowper's formulation for the Timoshenko coefficients for a hollow circular section is as follows:\(^9\)

\[
K = \frac{6(1+\nu)(1+m^2)}{(7+6\nu)(1+m^2) + (20+12\nu)m^2}
\]  

where, \(\nu\) is Poisson's ratio. Similarly, the formulation for a rectangular section is as follows:

\[
K = \frac{10(1+\nu)}{12+11\nu}
\]

The calculation results obtained using formula (1) are shown in Table 1 along with the results obtained using the formula in reference 10), which is generally used for a simple calculation of the Timoshenko coefficient.

<table>
<thead>
<tr>
<th>External diameter (a) [mm]</th>
<th>Internal diameter (b) [mm]</th>
<th>(m = b/a)</th>
<th>Sectional area (A) [m²]</th>
<th>Moment of inertia (I) [m⁴]</th>
<th>(i = \sqrt{I/A}) [m]</th>
<th>Timoshenko coefficient (Cowper)(K)</th>
<th>Timoshenko coefficient (Kondo)(K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>101.6</td>
<td>95.2</td>
<td>0.937</td>
<td>0.0009886</td>
<td>1.19238E-06</td>
<td>0.03476</td>
<td>0.53151</td>
<td>0.66791</td>
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<td>94.6</td>
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<td>0.0010758</td>
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<td>0.03466</td>
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<tr>
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<td>0.0012817</td>
<td>1.5188E-06</td>
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<td>0.532195</td>
<td>0.66915</td>
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<tr>
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<td>0.0017128</td>
<td>1.9709E-06</td>
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<td>0.0023730</td>
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<td>0.536968</td>
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<tr>
<td>165.2</td>
<td>143.2</td>
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<td>0.0055147</td>
<td>1.5835E-05</td>
<td>0.05458</td>
<td>0.534938</td>
<td>0.67287</td>
</tr>
<tr>
<td>355.6</td>
<td>317.6</td>
<td>0.893</td>
<td>0.0200388</td>
<td>2.8395E-04</td>
<td>0.11904</td>
<td>0.533323</td>
<td>0.67041</td>
</tr>
</tbody>
</table>
3. RELATION BETWEEN PHASE VELOCITY AND TIMOSHENKO COEFFICIENT

The phase velocity–wave number relationship for the bending wave of the Timoshenko beam when the Timoshenko coefficient changes is shown in Fig. 1, where \( k' \) is normalized wavelength by \( i = \sqrt{I/A} \). It is obtained by dividing the phase velocity by the longitudinal phase velocity \( C_s = \sqrt{E/\rho} \) to examine the influence of the Timoshenko coefficient on the dispersion characteristics of the phase velocity of the beam. The relationships for five kinds of Timoshenko coefficients are shown in Fig. 1, and the meaning that Timoshenko coefficient is equal to 100 means that shear deformation hardly occurs and only rotary inertia is considered. It is found that this influence cannot be ignored because the convergence values of the phase velocities considerably vary with the Timoshenko coefficients.

![Dispersion relation for the phase velocity of the Timoshenko beam](image)

**Fig. 1** Dispersion relation for the phase velocity of the Timoshenko beam

4. SHEAR STRESS DISTRIBUTION IN A SECTION

The shear stress distributions in cross sections of a hollow circular bar and a rectangular bar obtained using Cowper’s theory in reference 5 will be shown here. One of the coordinates in the cross section is shown in Fig. 2. The shear stress representations along the two directions in this cross section are given by the following formulae

\[
\sigma_u = -\frac{Q}{2(1+\nu)I} \left[ \frac{\partial \chi}{\partial x} + \frac{\nu x^2}{2} + \frac{(2-\nu) y^2}{2} \right]
\]

\[
\sigma_v = -\frac{Q}{2(1+\nu)I} \left( \frac{\partial \chi}{\partial y} + (2+\nu)xy \right)
\]

where \( Q \) is the shear force, and \( \chi \) is given by the following formulae

a) Hollow circular section \((a: \text{outer diameter}, b: \text{inner diameter})\):

\[
\chi = -\left( \frac{3}{4} + \frac{1}{2} \left[ (a^2 + b^2)x + a^2b^2 - \frac{x}{x^2 + y^2} \right] + \frac{1}{4} \left( x^2 - 3xy \right) \right)
\]

b) Rectangular section \((2a: \text{length along the bending direction}, 2b: \text{length along the vertical direction for bending})\):
\[ x = \left[ -(1 + \nu) \frac{a^2 + \nu b^2}{3} + \frac{2 + \nu}{3} (x^2 - 3xy^2) \right] / b \\
+ 4\pi \beta \sum_{k=1}^\infty \frac{(-1)^k \sinh(\pi x / b)}{\pi \cosh(\pi \alpha / b)} \cos(\pi \alpha y / b) \\
(6) \]

The shear stress distributions in the hollow circular and rectangular sections are shown in Fig.3 and Fig.4, respectively. These stress values are normalized by \( Q / \left[ 2 \left( 1 + \nu \right) I \right] \), where \( I \) is the moment of inertia, and

\[ Q = \int \sigma_{xy} \, dx \, dy. \]

The shear stress occurs not only along the bending direction but also along the vertical direction of bending for the bending force in one direction. It is confirmed that the shear stress distribution in a section estimated by using Cowper’s theory yields results similar to that obtained from the three-dimensional exact solution.

![Fig.2 Coordinates](image)

![Fig.3 Shear stress distribution of beam in a hollow circular section](image)  
\( m = 0.8 \)

![Fig.4 Shear stress distribution of beam in a rectangular section](image)  
\( a = 0.5, \ b = 0.3 \)
5. NUMERICAL SIMULATION

5.1 One-dimensional simulation
As an example, numerical simulation was performed for the model shown in Fig.5, which is a part of a piping system, by using the one-dimensional finite difference method. The dimensions of the steel tube are as follows: outer diameter (101.6 mm) and inner diameter (95.2 mm). The values in Table 1 are used as the Timoshenko coefficient for this model. The full length of the model is 5 m. Both ends of the model are free and the two supports are positioned 0.1 m from each free end. An impact force was loaded at the center of the steel tube in the vertical direction. The time history of the force is shown in Fig.6 and the conditions for the calculations are shown in Table 2. The displacement wave propagation in the steel tube at \( t = 0.4, 1.0, 1.6, 2.2 \) msec is shown in Fig.7. Reflection and transmission waves can be confirmed at the boundary parts.

The time responses of the displacement wave at impact points A and B are shown in Fig.8. Point A is just impacted point and point B is \( \frac{L}{4} \) m apart from the point A. The responses of difference time period at point A are shown in Fig.8 (a) and (b), and those at point B are shown in Fig.8 (c) and (d). It is confirmed from Fig.8 (a) and (c) that the wave propagation phenomenon occurred with many reflections from the support points as well as the vibration phenomenon, which is equivalent to the natural period of the structure supported by a 4.8 m span. By comparing Fig.8 (b) and (d), it can be confirmed that the first arrival time of each wave front is different by the Timoshenko coefficient, and the results are appropriate in comparison with predicted arrival time by using the phase velocity of each case in Fig.1.

![Fig.5 Analytical model [unit: m]](image)

![Fig.6 Input force](image)

<table>
<thead>
<tr>
<th>Table 2 Calculation conditions</th>
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</thead>
<tbody>
<tr>
<td>Space mesh interval</td>
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<tr>
<td>Total space mesh</td>
</tr>
<tr>
<td>Time step</td>
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<td>Input force</td>
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<td>Boundary condition</td>
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<td>Support condition</td>
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<tr>
<td>Support position</td>
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<tr>
<td>Young’s modulus</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
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<tr>
<td>Mass density</td>
</tr>
</tbody>
</table>

![Fig.7 Displacement wave propagation](image)

\( (t = 0.4, 1.0, 1.6, 2.2 \text{ msec}, a = 101.6 \text{ mm}, b = 95.2 \text{ mm}) \)
5.2 Three-dimensional simulation

In order to confirm the three-dimensional shear stress distribution obtained from Cowper’s formula, another numerical simulation was performed on the model shown in Fig.9 by using the three-dimensional finite difference method, the Finite Difference Time Domain (FDTD) method. In order to carry out accurate simulation, computations were done through a compact finite difference scheme\(^{12,13}\). The dimensions of the steel tube are same as that shown in Fig.5 and the conditions for the calculations are shown in Table 3. An impact force was loaded at the center of the steel tube in the vertical direction. The shear stress distribution obtained from Cowper’s theory is shown in Fig.10. The results of the numerical simulation are shown in Fig.11. The stress distributions of \(t=0.5, 1.0, 1.5, 2.0, 2.5\) msec at points A and B are shown. The stress distribution at point A, which is just an impacted point, is very different from Cowper’s distribution. On the other hand, the stress distribution at point B is fairly similar to Cowper’s distribution until reflection waves reached. After \(t=2.0\) msec, the stress distribution at point B is changed dynamically. Consequently, it is confirmed that Cowper’s theory can almost express the three-dimensional shear stress distribution only until reflection waves from the boundaries reached at the observed point, except near the external loaded points.
Fig. 9 Analytical model [unit: m]

Fig. 10 Shear stress distribution obtained from Cowper’s theory (σw)

<table>
<thead>
<tr>
<th>Table 3 Calculation conditions (FDTD method)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Space mesh interval (x)</strong></td>
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<tr>
<td><strong>Space mesh interval (y)</strong></td>
</tr>
<tr>
<td><strong>Space mesh interval (z)</strong></td>
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<tr>
<td><strong>Number of space mesh (x / y / z)</strong></td>
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<tr>
<td><strong>Time step</strong></td>
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<td><strong>Input force</strong></td>
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<tr>
<td><strong>Boundary condition</strong></td>
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<td><strong>Young’s modulus</strong></td>
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<td><strong>Poisson’s ratio</strong></td>
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<td><strong>Mass density</strong></td>
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</table>
Fig. 11 Shear stress distribution by the FDTD method
(t=0.5, 1.0, 1.5, 2.0, 2.5 msec)
6. CONCLUSION REMARKS

The introduction of the Timoshenko theory is effective for a beam in which shear deformation is dominant for a system in which the effects of high frequency are significant. It was shown that the convergence value of the phase velocity at high frequency, i.e. short wave length limit, is influenced by Timoshenko coefficient. This effect could not be ignored in the accurate evaluation of wave propagation or vibration. In addition, it was shown that Cowper’s formula was effective for the evaluation of the Timoshenko coefficient until reflection waves reached by showing the shear stress distribution in a cross section. As an example, the wave propagation phenomenon and shear stress distribution were analyzed by numerical simulations for an example of the piping system, and it was able to be a pre-examination for the analysis of the wave propagation phenomenon in a real complex piping system, which is planned to be analyzed in the near future.

REFERENCES