A Complicated Double Periodicity in a Model of an Artificial Neural Network

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A deterministic dynamical model of an artificial neural network with asymmetric synaptic weights is studied numerically to explore a complicated double periodicity among periodic and chaotic motions. The phenomenon of frequency locking repeats twice as the nonlinear parameter changes. An analogy is pointed out between the presented neurodynamical system, which is related to the Hopfield network model, and the turbulence shell model.

1. INTRODUCTION

A double periodicity in deterministic dynamical systems is known as one of typical regular states which can exist in a transition from periodic to chaotic motions. While a rich store of knowledge has already obtained in previous studies of reduced return maps, we are still far from complete understanding of such a quasiperiodic route to turbulence in nonlinear differential equations. A model of an artificial neural network (ANN) is quite useful for this purpose for several reasons. There are no restrictions on synaptic weights responsible to nonlinear coupling. The degree of freedom (the number of modes) is arbitrary. Once the weights are fixed, the system has only a single parameter which denotes the strength of nonlinearity (or equivalently dissipation). Boundedness of the solutions is guaranteed according to the sigmoidal property of the activation function. The deterministic neurodynamical model considered in this paper is related to the Hopfield network model, but does not belong to the class of the Lyapunov systems.

The author formerly did detailed numerical studies to explore bifurcation structures of the three-mode Langford equation\(^1\),\(^2\) and the six-mode turbulence shell model.\(^3\) In both systems, only a simple torus attractor was found for the double periodicity. On the contrary, a complicated torus attractor was found in the presented five-mode ANN system. These works are initiated to confirm numerically the well-known Ruelle-Takens\(^4\) route to chaos. They are also motivated by the study of the five-mode MHD model by Bekki and Karakissawa (2000).\(^5\)

Here, we adopt a natural usage of the term 'simple torus', which imply that the periodic orbit surrounded by the torus is unknotted and circular. The term 'complicated' means that the torus is not a simple one. The complicated double periodicity may at first sight be related the Ruelle-Takens\(^4\) scenario to chaos via three torus, since in both routes the Hopf bifurcation occurs three times. However, since the frequency locking occurs once in the former case, three torus in the Ruelle-Takens scenario should be considered to be different from the present complicated torus attractor.

2. A MODEL OF AN ARTIFICIAL NEURAL NETWORK

The neurodynamical model considered in this paper is an additive model of a neuron (see Figure 14.7 on page 677 in Haykin 1999\(^6\)), which is equivalent to a coupled system of electronic circuits (neurons) given by

\[
C_j \frac{dv_j}{dt} = -\frac{v_j}{R_j} + \sum_{i=1}^{N} w_{ji} \varphi_i(v_i(t)) + I_j, \quad j = 1, \ldots, N. \tag{1}
\]

Here, \(N\) is the number of neuron, \(x_i(t) = \varphi_i(v_i(t))\) denotes synaptic input (and neural output) of induced local electric potential \(v_i(t)\), the synaptic weight \(w_{ji}\) represents conductance, \(C_j\) is capacitance due to earth leakage, and \(R_j\) is leakage resistance. \(I_j\) denotes the current source and plays a role of the external forcing
or noise. A sigmoidal function (a hyperbolic tangent or logistic function) is often used for the nonlinear activation function $\varphi(\cdot)$.

We consider the situation that nonstationarity is maintained by the term of synaptic weights. Therefore, the term $I_j$ is not relevant and omitted in the following numerical study for the sake of simplicity. The equation (1) can be obtained by applying Kirchhoff's current law to each closed circuit.

The system (1) is closely related to the following system:

$$C_j \frac{dx_j}{dt} = -\frac{x_j}{R_j} + \varphi \left( \sum_{i=1}^{N} w_{ji} x_i(t) \right) + K_j, \quad j = 1, \ldots, N. \quad (2)$$

Equations (1) and (2) are connected under the relation (but not exactly identical)

$$v_k = \sum_{j=1}^{N} w_{kj} x_j(t), \quad (3)$$

and

$$I_k = \sum_{j=1}^{N} w_{kj} K_j(t). \quad (4)$$

According to Pineda (1987)\(^7\), we use (2) instead of (1) in the following study. For simplicity, we assume that all neurons are identical and put $C_j = R_j = 1$. For the nonlinear activation function, we put $\varphi(x) = c \tanh(x)$ where $c$ is assumed to be positive. Then, we obtain the final form of the present neurodynamical model due to Sprott\(^8\) as

$$\frac{dx_j}{dt} = -x_j + c \tanh \left( \sum_{i=1}^{N} a_{ji} x_i(t) \right), \quad j = 1, \ldots, N. \quad (5)$$

If we define the energy of the system by $E = (1/2) \sum_{i=1}^{N} x_i^2(t)$, the temporal evolution of $E$ is bounded as

$$\frac{dE}{dt} \leq - \sum_{i=1}^{N} (x_i^2 - c|x_i|) = - \sum_{i=1}^{N} \left( |x_i| - \frac{c}{2} \right)^2 + \frac{c^2}{4}. \quad (6)$$

Therefore, $dE/dt$ is negative for sufficiently large values of $(|x_i| - c/2)^2$ and the solution is proved to be finite for all positive $c$.

We also assume that the synaptic weights $a_{ij}$ have zero diagonal components, $a_{ii} = 0$ for all $i$. The diagonal component is considered to be expressed by the damping term in (5). Since $\varphi(x) \sim x$ for small $x$, the property of the linear stability of the origin $x = 0$, which is a fixed point of (5), is determined by the eigenvalues $\lambda_k$ of the matrix $a_{ij}$. $\sum_k \lambda_k = 0$ since $a_{ij}$ is traceless.

The main purpose of this paper is to show that this model can have the double periodicity. The cases with $N = 4$ and 5 are considered. Dozens of combinations $a_{ij} = -1, 0, 1$ are selected randomly and bifurcation diagrams for successive local maxima of $x_1(t)$ are plotted. The method to plot bifurcation diagrams is the same as in Umeki (2007)\(^3\).

3. A SIMPLE DOUBLE PERIODICITY FOR THE N=4 CASE

First, we consider the case $N = 4$. If we put the synaptic weights $a_{ij}$ as

$$A = a_{ij} = \begin{pmatrix}
0 & -1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & -1 \\
0 & -1 & 1 & 0
\end{pmatrix}. \quad (7)$$

The eigenvalues $\lambda_k$ is given by two pairs of complex conjugates with one positive and one negative real part. The concrete values are $\lambda_{1,2} = -0.203723 \pm 1.66393i$ and $\lambda_{3,4} = 0.203723 \pm 0.560668i$. There are
relations $\lambda_2 = \lambda_1^\ast$, $\lambda_4 = \lambda_3^\ast$ (the asterisk denotes the complex conjugate), and $\text{Re}(\lambda_1) = -\text{Re}(\lambda_3)$. In this case, we can expect a Hopf bifurcation and a periodic orbit.

The projection of the attractors in the $(x_1, x_2, x_3)$ space is shown in Figure 1. The case $c = 78$ shows a simple torus and $c = 84$ shows chaos. The second Hopf bifurcation shown in Figure 2 is similar to that observed in a turbulence shell model studied in Umeki (2007).\(^3\)

A dozen of combinations of $a_{ij}$ are investigated, and it is found that the torus structure in the projected $(x_1, x_2, x_3)$ space is only simple. Therefore, we should consider the case $N = 5$ in order to find a complicated double periodicity. Comparing with the turbulence model, it is easier for us to investigate various different cases since the restriction on the coefficients $a_{ij}$ is only $a_{ii} = 0$ for $i = 1, \cdots, N$. Since we are interested in the quasiperiodicity, we mainly investigate the cases in which $\lambda_k$ have pairs of complex conjugates with a positive or small negative real part.

![Figure 1: The projection of the attractors in the $(x_1, x_2, x_3)$ space for $N = 4$ and $a_{ij}$ given by (7). The cases $c = 78$ and $c = 84$ show a simple torus and chaos, respectively.](image)
Figure 2: Bifurcation diagrams for the system (5) with $N = 4$ and $a_4$ given by (7). The upper figure shows the period-doubling bifurcation around $c \approx 21$. The second Hopf bifurcation around $c \approx 77$ is shown in the lower figure, which is obtained by increasing $c$ slowly.
4. A COMPLICATED DOUBLE PERIODICITY FOR THE N=5 CASE

Next, we consider the case \( N = 5 \). We keep the components (7) in \( a_{ij} \) for \( N = 5 \) and choose the remaining \( a_{ij} \) for \( i = 5 \) or \( j = 5 \) randomly. We select the synaptic weights \( a_{ij} \) as

\[
A = a_{ij} = \begin{pmatrix}
0 & -1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & -1 \\
1 & 1 & 0 & -1 & -1 \\
0 & -1 & 1 & 0 & 0 \\
-1 & 1 & 0 & -1 & 0
\end{pmatrix}.
\] (8)

The eigenvalues of (8) are

\[
\lambda_1 = -0.383267, \lambda_2,3 = -0.0507991 \pm 1.85323i, \lambda_4,5 = 0.242432 \pm 0.836874i.
\] (9)

Figure 3 shows the bifurcation diagrams. If the parameter \( c \) is small or large enough, a periodic orbit is obtained. The lower figure indicates the detailed structure of the second Hopf bifurcation around \( c \approx 7.7 \), and the second Hopf bifurcation of the knotted periodic orbit leading to a complicated double periodicity around \( c \approx 11.2 \).

Figure 4 shows a Poincaré plot in the \((x_{1,n}, x_{1,n+1})\) plane, where \( x_{1,n} \) is the \( n \)th local maxima of \( x_1(t) \), for the different values of the parameter \( c \) in the region of the complicated double periodicity. The bifurcation scenario is as follows: a periodic orbit \( \rightarrow \) a simple torus \( \rightarrow \) a frequency locking leading to a period 7 periodic orbit \( \rightarrow \) a complicated torus via the Hopf bifurcation of the knotted periodic orbit \( \rightarrow \) a frequency locking leading to a period 13-7 periodic orbit \( \rightarrow \) and so on.

Figure 5 shows the projection of the attractors in the \((x_1, x_2, x_3)\) space, denoting a simple torus for \( c = 9.5 \) and a knotted periodic orbit for \( c = 11.15 \).

5. CONCLUSIONS AND DISCUSSIONS

The presented neurodynamical model with suitable synaptic weights shows a simple double periodicity for \( N = 4 \) and a complicated double periodicity for \( N = 5 \). Since the latter gives more points in the Poincaré plot than the former, a careful consideration with a longer time integration is required in order to distinguish a frequency-locked periodicity from a complicated double periodicity. There are many similarities between this model studied in detail by Sprott\(^8\) and a Gledzer turbulence shell model.\(^3\) Both have a property of the boundedness of solutions, a dissipation term and the external forcing. The degree of freedom of the system is arbitrary. The simple double periodicity is shown in both systems. A cascade process observed in the turbulence model may be investigated similarly in this neurodynamical model if we assign a role of the wavenumber to each neuron by modifying the damping term.

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Figure 3: Bifurcation diagrams for the system (5) with $N = 5$ and $a_{ij}$ given by (8). The upper figure shows the wide range of the parameter $c$. In the lower figure, an interesting parameter region with a complicated double periodicity ($c \approx 11.2$) is enlarged.
Figure 4: The Poincaré plot of the attractors on the plane \((x_{1,n}, x_{1,n+1})\) for \(c = 8, 9.5, 11.15, 11.19, 11.225\) and 11.25. The attractor is a simple torus for \(c = 8, 9.5\), a complicated torus for \(c = 11.19, 11.25\), and a frequency-locked periodic orbit for \(c = 11.15, 11.225\). The number of points for \(c = 11.15\) is seven. The number of points for each small ring for \(c = 11.225\) is thirteen. The Poincaré plots for \(c = 8\) and 9.5 do not resemble each other, but the attractor projection shows that those are both simple tori.
Figure 5: The projection of the attractors in the $(x_1, x_2, x_3)$ space for $N = 5$ and $a_{ij}$ given by (8). The cases $c = 9.5$, $c = 11.5$, and $c = 11.19$ show a simple torus, a frequency locked periodic orbit, and a complicated torus, respectively.
References


