Stability of Hexagonal Patterns in a Chemotaxis System

Takashi OKUDA

Department of Mathematical Sciences, School of Science and Technology, Kwansei Gakuin University, Sanda, Hyogo

In this paper, dynamics of a chemotaxis system is studied. Applying the invariant manifold theory for parabolic equations, we will show that the hexagonal pattern in chemotaxis system is destabilized with pure imaginary eigenvalues by a suitable choice of parameters.

1 Introduction

In this paper, we consider the chemotaxis-diffusion-growth system:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= a \Delta u - b \nabla \cdot (u \nabla v) + pu(1 - u), \quad (t, x) \in (0, \infty) \times \Omega, \\
\frac{\partial v}{\partial t} &= d \Delta v + f u - gv, \quad (t, x) \in (0, \infty) \times \Omega.
\end{align*}
\] (1.1)

This model is introduced by Mimura and Tsujikawa \(^{(10)}\) as a model for chemotactic aggregation of biological individuals. \(^{(4,5)}\) (It should be noted that the reaction term is given as a bistable type in original problem. As shown in (1.1), we replace it with logistic source: \(pu(1 - u)\). Here, \(u(t, x)\) and \(v(t, x)\) denote the population density and the concentration of chemotactic substance at time \(t\) and position \(x \in \Omega \subset \mathbb{R}^n, \ n = 1, 2\), respectively. In addition, all coefficients in (1.1) are positive, \(\Delta\) and \(\nabla\) denote Laplacian and gradient operator, respectively.

Kurata, et al. \(^{(5)}\) studied the system (1.1) from the point of view of pattern formation problem. They observed the time periodic solutions to chemotaxis system (1.1) in one dimensional case. In the left of figure 1, we reproduce the bifurcation diagram to the one dimensional problem: \(\Omega = (0, L) \subset \mathbb{R}\) by AUTO, which is a software package for bifurcation analysis \(^{(7)}\). One can see that there is a Hopf-bifurcation point which is symbolized by black box on a non-trivial branch. In two dimensional case, existence of the locally stable hexagonal pattern to (1.1) in a rectangular domain is already proved \(^{(11)}\) (see figure 1, right). Moreover, we can observe the oscillatory hexagon numerically (see figure 2). Motivated by these results, we study the stability of non-trivial solutions in two dimensional case, especially hexagonal pattern. Our analysis is based on invariant manifold theorems for the parabolic partial differential equations. To discuss the stability of the hexagonal pattern, we will derive a reduced system on the local invariant manifolds of (1.1). Using the reduced system, we can show that the hexagonal pattern is destabilized with pure imaginary eigenvalues by a suitable choice of parameters, i.e., it will be proved that one of a necessary condition for the Hopf-bifurcation holds.
Figure 1: (Left) : The bifurcation diagram to the chemotaxis system (1.1) in one dimensional case under Neumann boundary conditions on $\Omega = (0, \pi/(0.11))$. Coefficients are $a = 1/4, d = 16, f = 1, g = p = 1/16$. The horizontal and vertical axes correspond to $b$ and $\| (u - 1, v - f/g) \|_{L^2},$ respectively. (Right): The stable hexagonal pattern of (1.1) in two dimensional case. The figure shows values of $u(t, x)$ on $\Omega = (0, 2\pi/\sqrt{3}) \times (0, 2\pi/3)$, and the coefficients are $a = 1/16, b = 98.01, d = 16, f = 1, g = 32, p = 9/2$.

Figure 2: Snap-shots of oscillatory hexagon in (1.1). Figures show the values of $u(t, x)$ on $\Omega = (0, 2\pi/\sqrt{3}) \times (0, 2\pi)$, and coefficients are $a = 1/16, b = 103, d = 16, f = 1, g = 32, p = 9/2$. (Above) : Snap-shots at $t = 969, 970, 971$ from the left. (Middle): Snap-shots at $t = 972, 973, 974$ from the left. (Below) : Snap-shots at $t = 975, 976, 977$ from the left.

This paper is organized as follows. In section 2, we show the general theories of invariant-manifolds for parabolic partial differential equations. In section 3, (1.1) is transformed to a system of ordinary differential equations in Fourier space. It is convenient to consider the dynamics on the invariant manifolds. In section 4, the one dimensional case is considered. We analyze the stability of pure mode solution. In section 5, we analyze the stability of the hexagonal pattern in (1.1) using a reduced system on the invariant manifolds.

2 Preliminaries

In this section, we shall show the general theories of invariant-manifolds for parabolic partial differential equations by imitating well-known arguments $^6, 13, 14, 15$.

Let $X$, $Y$, and $Z$ be Banach spaces, with $X$ continuously embedded in $Y$, and $Y$ continuously embedded in $Z$. Let $L \in C(X, Z)$ and $N \in C^k(X; Y)$ for some $k > 1$ be a linear operator and a nonlinear function, respectively. We suppose $N(0) = 0$ and $DN(0) = 0$, where $D(\cdot)$ denotes Fréchet derivative. Let us
consider the following differential equation:

$$\dot{u} = Lu + N(u).$$  \hspace{1cm} (2.1)

In this section, we assume the following:

- The initial-value problem for (2.1) with initial condition $u(0) = u_0$, $u_0 \in X$ has a unique solution $S(t)u_0$ for all $t \geq 0$;
- There exists a smooth cut-off function $\chi(u)$ which maps $X$ into $[0, 1] \subset \mathbb{R}$:

$$\chi(u) = \begin{cases} 
1, & \|u\|_X \leq 1, \\
0, & \|u\|_X \geq 2.
\end{cases}$$

It should be noted that to prove the existence of local-center-unstable manifold of (2.1), it is sufficient to consider the following equation in $X$:

$$u_t = Lu + N^\varepsilon(u),$$  \hspace{1cm} (2.2)

where $N^\varepsilon(u) = N(u\chi(u/\varepsilon))$, $\varepsilon > 0$. We also impose the following assumptions on a linear operator $L$:

- For each $j = 1, 2$, there exists a continuous projection $P_j \in \mathcal{L}(Z_j; X)$ on to a finite-dimensional subspace $Z_j = X_j \subset X$ such that

$$P_jLu = LP_ju, \forall u \in X;$$

- $\text{Re} \text{spec}(L_1) \subset \mathbb{R}_+$, $\text{spec}(L_2) \subset i\mathbb{R}$ and $\text{Re} \text{spec}(L_3) \subset \mathbb{R}_-$, where

$$L_j = L|_{X_j} \in \mathcal{L}(X_j), j = 1, 2, L_3 = L|_{X_3} \in \mathcal{L}(X_3, Z_3),$$

$$Z_3 = P_3(Z), X_3 = P_3(X).$$

Then, we can split the equation (2.2) into

$$\ddot{u}_j = L_ju_j + N_j^\varepsilon(u_1, u_2, u_3), \quad (u_j = P_ju \in X_j), j = 1, 2, 3.$$  \hspace{1cm} (2.3)

We put

$$\lambda_+ := \inf \text{Re spec}(L_1),$$

$$\lambda_- := \inf |\text{Re spec}(L_3)| = |\sup \text{Re spec}(L_3)|.$$

Then, there exist positive constants $C_j$ such that the following hold:

$$\|e^{L_1t}w\|_X \leq C_1e^{\lambda_+t}\|w\|_X, \forall w \in X_1, \forall t < 0,$$

$$\|e^{L_3t}w\|_X \leq C_3e^{-\lambda_-t}\|w\|_X, \forall w \in X_3, \forall t > 0.$$ 

In addition, for each $r > 0$, there exists a positive constant $C_2(r)$ such that the following holds:

$$\|e^{L_2r}w\|_X \leq C_2(r)e^{rt}\|w\|_X, \forall w \in X_2, \forall t \in \mathbb{R}.$$  

Let $E$ and $F$ be Banach space, $V \subset E$ an open subset, $k \in \mathbb{N} \cup \{0\}$. We define a space of maps $C^k_b(V; F)$ as follows:

$$C^k_b(V; F) := \left\{ w \in C^k(V; F) : \|w\|_{j,V} := \sup_{x \in V} \|D^jw(x)\| < \infty, 0 \leq j \leq k \right\}.$$
or more specifically, for a given \( w \in C^k(V; F) \), \( D^jw(x), (0 \leq j \leq k) \), belongs to a space of N-linear maps (for instance, see [8]): \( \mathcal{L}(V, \ldots, V; F) \) with the norm

\[
\|D^jw(x)\| := \sup_{\|x\|_V \leq 1} \|D^jw(x; \tilde{x}, \ldots, \tilde{x})\|_F.
\]

Then \( C^k_b(V; F) \) is a Banach space with the norm

\[
\|w; C^k_b(V; F)\| := \max_{0 \leq j \leq k} |w|_{j,V}.
\]

We also define

\[
C^{k,1}_b(E; F) := \left\{ w \in C^{k,1}(E; F); |w|_{j,Lip} := \sup_{x,y \in E, x \neq y} \frac{\|D^jw(x) - D^jw(y)\|}{\|x - y\|_E} < \infty, 0 \leq j \leq k \right\}
\]

which is a Banach space with the norm

\[
\|w; C^{k,1}_b(E; F)\| := \|w; C^k_b(E; V)\| + \max_{0 \leq j \leq k} |w|_{j,Lip}.
\]

Then the following hold [1]:

\[
C^{k+1}_b(V; F) \subset C^k(V; F),
C^{k,1}_b(E; F) \subset C^k_b(V; F).
\]

In case \( V = E \) we write \( |w|_j \) for \( |w|_{j,E} \). Moreover, for \( j = 1, 2, 3 \), we assume

\[
N^j_f(u_1, u_2, u_3) \neq 0, \quad u_j \in X_j.
\]

Then, we have the following theorem.

**Theorem 1** If \( N \in C^0_b(X; Y) \) and \( \lambda_- > \lambda_+ \), then there exists a positive constant \( \varepsilon \) such that if \( \|u_1\|_X, \|u_2\|_X < \varepsilon \), then there is a unique mapping \( h \in C^0_b(X_1 \oplus X_2; X_3) \) satisfying the following: The local manifold

\[
\mathcal{W}_b^{\varepsilon,u} := \{(u_1, u_2, u_3) \in X; u_3 = h(u_1, u_2)\}
\]

is a locally invariant under the flow of (2.1). In addition, \( h(0, 0) = 0 \).

**Proof** It is sufficient to consider the system

\[
\begin{align*}
\dot{u}_1 &= L_1u_1 + N^1_f(u_1, u_2, u_3), \quad (u_1 = P_1u \in X_1), \quad \text{(2.4)} \\
\dot{u}_2 &= L_2u_2 + N^2_f(u_1, u_2, u_3), \quad (u_2 = P_2u \in X_2), \quad \text{(2.5)} \\
\dot{u}_3 &= L_3u_3 + N^3_f(u_1, u_2, u_3), \quad (u_3 = P_3u \in X_3). \quad \text{(2.6)}
\end{align*}
\]

to prove Theorem 1.

For a given \( \psi \in C^{0,1}_b(X_1 \oplus X_2; X_3) \) satisfying \( \psi(0, 0) = 0 \), let \( (w_1(t, w^0_1, w^0_2, \psi), w_2(t, w^0_1, w^0_2, \psi)) \) be a solution of

\[
\begin{align*}
\dot{w}_1 &= L_1w_1 + N^1_f(w_1, w_2, \psi(w_1, w_2)), \\
\dot{w}_2 &= L_2w_2 + N^2_f(w_1, w_2, \psi(w_1, w_2))
\end{align*}
\]

with the initial conditions

\[
w_j(0) = w_j(0, w^0_1, w^0_2, \psi) = w^0_j, \quad j = 1, 2.
\]
We define the map $T$:

$$(T\psi)(w_0^0, u_0^0) = \int_{-\infty}^{0} e^{-\lambda s} \left[ \frac{N_j^0(w_1(s), u_1^0, u_2^0, \psi), w_2(s, u_1^0, u_2^0, \psi), \psi(w_1, w_2))}{ds} \right].$$

If $\psi$ is a fixed point of $T$, then $\psi = \hat{h}$ is a local invariant $C^0$-manifold of (2.1). We prove that for sufficiently small $\varepsilon > 0$, $T$ is a contraction mapping on $C^0_0(X_1 \oplus X_2; X_3)$. Using definition of $N_j^0$, for given $w_j \in X_j$, $j = 1, 2, 3$ with $||w_j||_X < \varepsilon$, there exists a continuous function $\kappa(\varepsilon)$ satisfying $\kappa(\varepsilon) \neq 0$ and $\kappa(0) = 0$ such that

$$\sum_{j=1}^{3} ||N_j^0(w_1, w_2, w_3)||_{Y_j} \leq \varepsilon \kappa(\varepsilon),$$

$$||N_j^0(w_1, w_2, w_3) - N_j^0(\tilde{w_1}, \tilde{w_2}, \tilde{w_3})||_{Y_j} \leq \kappa(\varepsilon) \sum_{j=1}^{3} ||w_j - \tilde{w}_j||_X.$$ 

It also follows that there are positive constants $p$ and $p_1$ such that the following hold:

$$|\psi|_p < p, \quad ||\psi(w_1, w_2) - \psi(\tilde{w_1}, \tilde{w_2})||_X < p_1 \sum_{j=1}^{3} ||w_j - \tilde{w}_j||_X.$$ 

Then, we have

$$|T\psi(\psi_0^0, u_0^0)|_0 \leq C_3 \varepsilon \kappa(\varepsilon) \int_{-\infty}^{0} e^{-\lambda s} ds \leq C_3 \varepsilon \kappa(\varepsilon) \lambda^{-1}. $$

For $t \leq 0$, putting $\tilde{w}_j(t) = w(t, \tilde{w}_1^0, \tilde{w}_2^0, \psi)$, we get the following estimates:

$$||w_1(t, w_0^0, u_0^0, \psi) - w_1(t, w_1^0, w_2^0, \psi)||_X \leq C_1 e^{-\lambda t} ||w_1^0 - \tilde{w}_1^0||_X$$

$$+ C_1 e^{-\lambda t} \int_{-\infty}^{0} e^{\lambda(s-t)} \{||w_1 - \tilde{w}_1||_X + ||w_2 - \tilde{w}_2||_X \} ds,$$

$$||w_2(t, w_0^0, u_0^0, \psi) - w_2(t, \tilde{w}_1^0, \tilde{w}_2^0, \psi)||_X \leq C_2 e^{-\lambda t} ||w_2^0 - \tilde{w}_2^0||_X$$

$$+ C_2 e^{-\lambda t} \int_{-\infty}^{0} e^{\lambda(s-t)} \{||w_1 - \tilde{w}_1||_X + ||w_2 - \tilde{w}_2||_X \} ds.$$

Then, we have

$$||w_1 - \tilde{w}_1||_X + ||w_2 - \tilde{w}_2||_X \leq 2 \max \{C_1, C_2(\gamma)\} e^{-\max\{\lambda_+, \gamma\} t} \{||w_1^0 - \tilde{w}_1^0||_X + ||w_2^0 - \tilde{w}_2^0||_X \}$$

$$+ 2 \max \{C_1, C_2(r)\} \kappa(\varepsilon)(1 + p_1) \int_{-\infty}^{0} e^{-\lambda(s-t)} \{||w_1 - \tilde{w}_1||_X + ||w_2 - \tilde{w}_2||_X \} ds.$$ 

Here, we used $e^{\lambda t} \leq e^{\lambda |t|} \leq e^{-\lambda t}$, $(t \leq 0)$. By Gronwall’s inequality, it yields

$$||w_1 - \tilde{w}_1||_X + ||w_2 - \tilde{w}_2||_X \leq 2C_4 \{||w_1^0 - \tilde{w}_1^0||_X + ||w_2^0 - \tilde{w}_2^0||_X \} e^{-\gamma t},$$

where $C_4 = \max \{C_1, C_2(r)\}$ and $\gamma = \max \{\lambda_+, \gamma\} + 2\kappa(\varepsilon)(1 + p_1) C_4$. If $\lambda_+ > \gamma$, we have

$$|T\psi(\psi_0^0, u_0^0)|_0 - T\psi(\tilde{\psi}_0^0, \tilde{\psi}_0^0)$$

$$\leq \int_{-\infty}^{0} C_3 e^{-\lambda s} ||N_j^0(w_1, w_2, \psi) - N_j^0(\tilde{w}_1, \tilde{w}_2, \psi)||_X ds$$

$$\leq C_3 \kappa(\varepsilon)(C_4 + p_1)(\lambda_+ - \gamma)^{-1} \{||w_1^0 - \tilde{w}_1^0||_X + ||w_2^0 - \tilde{w}_2^0||_X \}. $$
This implies that by choosing \( r < \lambda_+ \), there exists a positive constant \( \varepsilon \) such that \( |T\psi|_{Lip} \leq p_1 \) holds. For given \( \psi_1, \psi_2 \in C^{0,1}_b(X_1 \oplus X_2 \oplus X_3) \) satisfying \( |\psi_1|_{Lip} < p_1 \), we have

\[
|T\psi_1 - T\psi_2|_0 \leq C_3 \kappa(\varepsilon) |\psi_1(w_1, w_2) - \psi_2(w_1, w_2)|_0 + I_1,
\]

where

\[
I_1 := \int_{-\infty}^{0} e^{\lambda \tau} \sum_{j=1,2} \|w_j(s, w_1^0, w_2^0, \psi_1) - w_j(s, w_1^0, w_2^0, \psi_2)\|_X \, ds.
\]

Here, we used

\[
\|\psi_j(w_1, w_2)\|_X \leq \sup_{w_1 \in X_1, w_2 \in X_2} \|\psi_j(w_1, w_2)\|_X = |\psi_j|_0, \quad j = 1, 2.
\]

Putting \( w_j(t, \psi_j) = w_j(t, w_1^0, w_2^0, \psi_j) \), we also have the following estimates:

\[
\|w_1(t, \psi_1) - w_1(t, \psi_2)\|_X \leq C_1 \kappa(\varepsilon) |\psi_1 - \psi_2|_0 e^{\lambda \tau}
\]

\[
+ C_2 \kappa(\varepsilon) \int_{t}^{0} e^{\lambda (t-s)} \left\{ \|w_1(s, \psi_1) - w_1(s, \psi_2)\|_X + \|w_2(s, \psi_1) - w_2(s, \psi_2)\|_X \right\} \, ds,
\]

\[
\|w_2(t, \psi_1) - w_2(t, \psi_2)\|_X \leq C_2 \kappa(\varepsilon) |\psi_1 - \psi_2|_0 e^{\lambda \tau}
\]

\[
+ C_2 \kappa(\varepsilon) \int_{t}^{0} e^{\lambda (t-s)} \left\{ \|w_1(s, \psi_1) - w_1(s, \psi_2)\|_X + \|w_2(s, \psi_1) - w_2(s, \psi_2)\|_X \right\} \, ds.
\]

Then, we get

\[
\sum_{j=1,2} \|w_j(s, w_1^0, w_2^0, \psi_1) - w_j(s, w_1^0, w_2^0, \psi_2)\|_X \leq 2C_1 \kappa(\varepsilon) |\psi_1 - \psi_2|_0 (\min\{\lambda_+, r\})^{-1} e^{-\lambda \tau t}
\]

\[
+ 2C_2 \kappa(\varepsilon) (1 + C_3 \kappa(\varepsilon) (\min\{\lambda_+, r\})^{-1} p_1 (1 + p_1)) \sum_{j=1,2} \|w_j(s, w_1^0, w_2^0, \psi_1) - w_j(s, w_1^0, w_2^0, \psi_2)\|_X \, ds.
\]

By Gronwall's inequality, we have

\[
\|w_1(s, \psi_1) - w_1(s, \psi_2)\|_X + \|w_2(s, \psi_1) - w_2(s, \psi_2)\|_X \leq 2C_3 \kappa(\varepsilon) (\min\{\lambda_+, r\})^{-1} |\psi_1 - \psi_2|_0 e^{-\gamma_2 \tau},
\]

where

\[\gamma_2 = \max\{\lambda_+, r\} + 2C_4 \kappa(\varepsilon) (1 + C_4 \kappa(\varepsilon) (\min\{\lambda_+, r\})^{-1} p_1 (1 + p_1)).\]
If $\lambda_+ > \gamma_2$ then the following holds:

$$I_1 \leq C_4 \kappa(\varepsilon)(\min\{\lambda_+, r\})^{-1} |\psi_1 - \psi_2| \int_{-\infty}^{0} e^{(\lambda_- - \gamma_2)s} ds = C_4 \kappa(\varepsilon)(\min\{\lambda_+, r\})^{-1} |\psi_1 - \psi_2|_0 (\lambda_- - \gamma_2)^{-1}.$$ 

Finally, we have

$$|T\psi_1 - T\psi_2|_0 \leq \kappa(\varepsilon)|C_3 + C_4 \min\{\lambda_+, r\} (\lambda_- - \gamma_2)^{-1}| |\psi_1 - \psi_2|_0.$$ 

Then, choosing $r$ satisfying $\lambda_+ > r$, there exists a positive constant $\varepsilon$ such that $T$ is a contraction on $C^0_b(X_1 \oplus X_2; X_3)$ if $\lambda_- > \lambda_+$. In addition, since $u \equiv 0$ is a solution of (2.1), for arbitrarily $\varepsilon > 0$, $(u_1, u_2, u_3) = (0, 0, 0)$ must be a solution of the following system:

$$\begin{align*}
\dot{u}_1 &= N_1^r(u_1, u_2, u_3), \\
\dot{u}_2 &= N_2^r(u_1, u_2, u_3), \\
u_3 &= h(u_1, u_2),
\end{align*}$$

namely, the following holds:

$$N_1^r(0, 0, h(0, 0)) = N_2^r(0, 0, h(0, 0)) = h(0, 0) = 0.$$

This completes the proof.

**Theorem 2** If $N \in C^0_b(X; Y)$ and $\lambda_- > \lambda_+$, then the map $h$ obtained in Theorem 1 satisfies

(i) $h \in C^{1,1}_B(X_1 \oplus X_2; X_3)$;

(ii) $\frac{\partial h}{\partial u_j}(0, 0) = 0$, $j = 1, 2$.

Proof of (i): We prove that the map $h(w_1^0, w_2^0), w_j^0 \in X_j$ is differentiable with respect to $w_1^0$. One can prove differentiability of $h$ with respect to $w_2^0$ by similar way.

Let $(w_1(t, w_1^0, w_2^0, h), w_2(t, w_1^0, w_2^0, h))$ be a solution of

$$\begin{align*}
\dot{w}_1 &= L_1 w_1 + N_1^r(w_1, w_2, h(w_1, w_2)), \\
\dot{w}_2 &= L_2 w_2 + N_2^r(w_1, w_2, h(w_1, w_2))
\end{align*}$$

(2.8)

with the initial conditions

$$w_j(0) = w_j(0, w_1^0, w_2^0, h) = w_j^0, \quad j = 1, 2.$$

Since $\dim X_j < \infty$, $(j = 1, 2)$, the solution $(w_1, w_2)$ is differentiable with respect to initial values $(w_1^0, w_2^0)$. Let us define the map $T^{(1)}$ as follows:

$$(T^{(1)} \psi^{(1)})(w_1^0, w_2^0) = \int_{-\infty}^{0} e^{-L_2 s} N_1^{(1)}(w_1(s, w_1^0, w_2^0, h), w_1(s, w_1^0, w_2^0, h), h, \psi^{(1)}) ds,$$

where

$$N_1^{(1)}(w_1, w_2, h, \psi^{(1)}) = \psi^{(1)} \cdot \frac{\partial N_1^r}{\partial v}(w_1, w_2, v) \bigg|_{v=h} + \sum_{j=1,2} \frac{\partial N_1^r}{\partial u_j}(w_1, w_2, h) \left( \frac{\partial w_j^r}{\partial w_1^0}(w_1, w_2, v) + \psi^{(1)} \cdot \frac{\partial w_j^r}{\partial v}(w_1, w_2, v) \right) \bigg|_{v=h}.$$
One can verify that if $\lambda_+ > \lambda_-$, then the map $T^{(1)}$ is a contraction on $C^1_b(X_1 \oplus X_2; X_3)$ and a fixed point is Lipschitzian, and moreover, for sufficiently small $\epsilon > 0$, $\|T^{(1)}(\cdot)\|_{Lip} \leq \|\psi\|_{Lip}$ holds for $j \geq 1$. For a given $\sigma \in X_1$ and a constant $\alpha > 0$, let us define
\[
\zeta(w_1^0, : \alpha, \sigma) := \{ h(w_1^0 + \alpha \sigma, w_1^0, \cdot) - h(w_1^0, \cdot) \} / \alpha,
\]
\[
\theta(w_1, w_2, h; \alpha, \sigma) := \{ N_2^0(w_1^+, w_2^+, h^+) - N_2^0(w_1, w_2, h) \} / \alpha,
\]
where
\[
h^+ := h(w_1(t, w_1^0 + \alpha \sigma, w_1^0), w_2(t, w_1^0 + \alpha \sigma, w_1^0)),
\]
\[
w_j^+ := w_j(t, w_1^0 + \alpha \sigma, w_1^0, h^+), \quad j = 1, 2.
\]
Then, we have
\[
(\zeta(w_1^0, : \alpha, \sigma) = \int_{-\infty}^{0} e^{-L s} \{ \theta(w_1, w_2, h; \alpha, \sigma) + \alpha N_1^0(w_1, w_2, h, \zeta / \sigma) - \sigma N_1^0(w_1, w_2, h, \zeta / \sigma) \} \, ds.
\]
Since $N \in C^2_b(X; Y)$ and $h$ are Lipschitzian (see proof of Theorem 1), it follows that as $\alpha \to 0$
\[
m(\alpha) := \sup_{t \in [0, \infty)} \| \theta(w_1, w_2, h; \alpha, \sigma) - \sigma N_1^0(w_1, w_2, h, \zeta / \sigma) \|_X \to 0.
\]
Then, there exists a positive constant $C_5$ (which is independent of $\alpha$) such that the following holds:
\[
G(\alpha) := \sup_{\|w_1^0\| \leq \epsilon} \| \zeta(w_1^0, w_2^0, : \alpha, \sigma) - \sigma \psi_1(w_1^0, w_2^0) \|_X
\]
\[
\leq C_5 \int_{-\infty}^{0} e^{-\lambda s} \| \sigma N_1^0(w_1, w_2, h, \zeta / \sigma) - \sigma N_1^0(w_1, w_2, h, \psi_1) \|_X \, ds + \frac{C_3}{\lambda_-} m(\alpha)
\]
\[
\leq C_5 G(\alpha) + \frac{C_3}{\lambda_-} m(\alpha).
\]
This implies $G(\alpha) \to 0$ as $\alpha \to 0$ (i.e., $\psi_1$ is the Gâteaux derivative of $h(w_1^0, w_2^0)$ with respect to $w_1^0$).
Since $\psi_1(w_1^0, w_2^0)$ is a Lipschitzian, this yields
\[
\frac{\partial h}{\partial w_1}(w_1^0, w_2^0) = \psi_1(w_1^0, w_2^0).
\]
Proof of (ii): Since $w_j(s, 0, 0, h(0, 0)) = 0$, $j = 1, 2$, and $DN(0) = 0$ holds, using the fact that $\psi_1$ is a fixed point of the map $T^{(1)}$, we have
\[
\frac{\partial h}{\partial w_1}(0, 0, h(0, 0)) = 0.
\]
Similarly, we can prove that the derivative of $h(w_1^0, w_2^0)$ with respect to $w_2^0$ is vanished at an origin. \qed

Using the similar arguments of proof of Theorem 2, one can prove that if $N \in C^k_b(X_1 \oplus X_2; X_3)$, then the map $h$ belong to $C^{k-1}_b(X_1 \oplus X_2; X_3)$. Let $h \in C^k_b(X_1 \oplus X_2; X_3)$, $k \geq 2$ be a map such that $W^{k, 0}_{loc}$ is a local invariant manifold of (2.1). Then the following holds.

**Theorem 3** If $w_1(t), w_2(t) \in X_1 \oplus X_2$ is a solution of the system of ordinary differential equations:

\[
\dot{w}_1 = L_1 w_1 + N_1^0(w_1, w_2, h(w_1, w_2)),
\]
\[
\dot{w}_2 = L_2 w_2 + N_2^0(w_1, w_2, h(w_1, w_2)),
\]
(2.9)
then there exists a solution \( u_j (j = 1, 2, 3) \) of (2.3) such that as \( t \to \infty \)

\[
\begin{align*}
    u_j(t) &= w_j(t) + O(e^{-\mu t}), \\ u_3(t) &= h(u_1(t), u_2(t)) + O(e^{-\mu t}),
\end{align*}
\]

where \( \mu > 0 \) is a positive constant.

**Proof** Since the system (2.9) is a system on finite dimensions, uniqueness of solutions is valid. We notice again that the center-unstable manifold is also unique. Putting \( v(t) := u_3(t) - h(u_1(t), u_2(t)) \in X_3 \), we have

\[
    \dot{v} = L_3 v + Q(u_1, u_2, v),
\]

where

\[
    Q(u_1, u_2, v) = \sum_{j=1,2} D_u h(u_1, u_2) \{ N_j^e(u_1, u_2, h(u_1, u_2)) - N_j^f(u_1, u_2, v + h(u_1, u_2)) \} + N_3^e(u_1, u_2, v + h(u_1, u_2)) - N_3^f(u_1, u_2, h(u_1, u_2)).
\]

Since \( h \in C_0^k(X_1 \oplus X_2; X_3), k \geq 2, \) for given \( u_j, \tilde{u}_j \in X_j, j = 1, 2, 3, \) there exists a constant \( \delta(\varepsilon) \) with \( \delta(0) = 0 \) such that the following holds:

\[
    \| Q(u_1, u_2, u_3) - Q(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \|_Y = \delta(\varepsilon) \sum_{j=1,2,3} \| u_j - \tilde{u}_j \|_X.
\]

Therefore, we have

\[
    \| v(t) \|_X \leq C_3 \| v(0) \|_X e^{-\lambda_- t} + C_3 \delta(\varepsilon) \int_0^t e^{-\lambda_- (t-s)} \| v(s) \|_X \, ds.
\]

This yields

\[
    \| v(t) \|_X \leq C_3 \| v(0) \|_X e^{-(\lambda_- - C_3 \delta(\varepsilon)) t},
\]

i.e., the following holds:

\[
    \| u_3 - h(u_1, u_2) \|_X \leq C_3 \| u_3(0) - h(u_1(0), u_2(0)) \|_X e^{-(\lambda_- - C_3 \delta(\varepsilon)) t}.
\]

Putting \( \phi_j := u_j - w_j, \) \( j = 1, 2, \) then \( \phi_j \) and \( v \) satisfies

\[
    \begin{cases}
        \dot{\phi}_j = L_j \phi_j + R_j(\phi_1, \phi_2, v), & j = 1, 2, \\
        \dot{v} = L_3 v + Q(\phi_1 + w_1, \phi_2 + w_2, v),
    \end{cases}
\]

where

\[
    \begin{align*}
    R_j(\phi_1, \phi_2, v) &= N_j^e(w_1 + \phi_1, w_2 + \phi_2, v + h(w_1 + \phi_1, w_2 + \phi_2)) \\
    &\quad - N_j^f(w_2, h(w_1, w_2)).
\end{align*}
\]

For \( \eta \geq 0, \) let us define a space of maps as follows

\[
    BC^n(R; E) := \left\{ w \in C^0(V; E); \|w\|_\eta := \sup_{t \in [0, \infty)} e^{\eta t} \| w(t) \|_E < \infty \right\}.
\]

For given \( \phi_j \in BC^n(R; X_j) \) where \( \eta \in (0, \lambda_+) \), we define the map \( T_j \):

\[
    (T_j \phi_j)(t) := -\int_t^\infty e^{-\lambda_+(t-s)} R_j(\phi_1, \phi_2, v) \, ds.
\]
We also define the map \( T \):
\[
[T(\phi_1 + \phi_2)](t) = (T_1 \phi_1)(t) + (T_2 \phi_2)(t).
\]

If \( T \) is a contraction mapping from \( BC^n(\mathbb{R}; X_1 \oplus X_2) \) into itself, the statement of Theorem is true. To prove this fact, we need the following estimates:
\[
\|R_j(\phi_1, \phi_2)\|_X \leq \kappa(\varepsilon) \{ (1 + p_1)(\|\phi_1\|_X + \|\phi_2\|_X) + \|v\|_X \},
\]
\[
\|T_j \phi_i\|_Y \leq \kappa(\varepsilon) \{ (1 + p_1)C_1(\|\phi_1\|_Y + \|\phi_2\|_Y)(\lambda_+ + \eta)^{-1} + C_3\|v(0)\|_X(\lambda_+ + \mu)^{-1} \}.
\]

Since similar estimate to \( T_2 \) can be obtained, \( T \) maps \( BC^n(\mathbb{R}; X_1 \oplus X_2) \) into itself. We can also prove that \( T \) is a contraction on \( BC^n(\mathbb{R}; X_1 \oplus X_2) \) as follows. For given \( \phi_j, \phi_j \in X_j, \quad j = 1, 2 \) and \( v, \tilde{v} \in X_3 \) with \( \|v\|_X > \|\tilde{v}\|_X, v(0) = \tilde{v}(0) \), we have
\[
\|Q(\phi_1, \phi_2, v) - Q(\phi_1, \phi_2, \tilde{v})\|_Y \leq 2\kappa(\varepsilon)\|v\|_X p_2 \sum_{j=1,2} \|\phi_j - \tilde{\phi}_j\|_X + \kappa(\varepsilon)\|v - \tilde{v}\|_X.
\]

Here, \( p_2 \) is a Lipschitz constant of the map \( \sum_{j=1,2} D_{u_j} h(u_1, u_2) \). Then, we have
\[
\|v(t) - \tilde{v}(t)\|_X \leq \kappa(\varepsilon) C_3 \int_0^t e^{-\lambda_-(t-s)} \|v - \tilde{v}\|_X ds + I_2,
\]
where
\[
I_2 := 2\kappa(\varepsilon) C_3 p_2 \int_0^t e^{-\lambda_-(t-s)} \|v(s)\|_X \sum_{j=1,2} \|\phi_j(s) - \tilde{\phi}_j(s)\|_X ds
\]
and the following holds
\[
I_2 \leq 2\kappa(\varepsilon) p_2 C_3^2 \|v(0)\|_X \sum_{j=1,2} \|\phi_j(t) - \tilde{\phi}_j(t)\|_\eta e^{-\lambda_- t}
\]
by choosing \( \varepsilon \) satisfying \( C_3 \delta(\varepsilon) \leq \eta \). Then, we have
\[
\|v(t) - \tilde{v}(t)\|_X e^{\lambda_- t} \leq \kappa(\varepsilon) C_6 \sum_{j=1,2} \|\phi_j(t) - \tilde{\phi}_j(t)\|_\eta t + \kappa(\varepsilon)C_3 \int_0^t e^{\lambda_- s} \|v(s) - \tilde{v}(s)\|_X ds.
\]

Using Gronwall's lemma, we get
\[
\|v(t) - \tilde{v}(t)\|_X e^{\lambda_- t} \leq C_6 C_3^{-1} \sum_{j=1,2} \|\phi_j(t) - \tilde{\phi}_j(t)\|_\eta e^{C_3 \kappa(\varepsilon) t} - 1).
\]

Hence, we have
\[
\|v - \tilde{v}\|_X \leq C_7 \sum_{j=1,2} \|\phi_j(t) - \tilde{\phi}_j(t)\|_\eta e^{C_3 \kappa(\varepsilon) t} - 1) e^{-\lambda_- t}.
\]

Then, the following holds:
\[
\|T_1 \phi_1 - T_1 \tilde{\phi}_1\|_Y \leq \sup_{t \in [0, \infty)} e^{\eta t} \kappa(\varepsilon) C_1 \int_t^\infty e^{\lambda_+(t-s)} \left( \sum_{j=1,2} \|\phi_j(s) - \tilde{\phi}_j(s)\|_X + \|v - \tilde{v}\|_X \right) ds
\]
\[
\leq \kappa(\varepsilon) C_8 \sum_{j=1,2} \|\phi_j - \tilde{\phi}_j\|_\eta,
\]
where
\[
C_8 = C_1 \left( (\lambda_+ + \eta)^{-1} + C_7 [(\lambda_+ + \lambda_- + \eta + \kappa(\varepsilon))^{-1} - (\lambda_+ + \lambda_- + \eta)^{-1}] \right).
\]
Similarly, there exists a positive constant $C_0$ such that the following holds:

$$\|T_2(\phi_2 - \tilde{\phi}_2)\|_{\eta} \leq \kappa(\varepsilon)C_0 \sum_{j=1,2} \|\phi_j - \tilde{\phi}_j\|_{\eta}.$$ 

Finally, we have

$$\|T((\phi_1 + \phi_2) - (\tilde{\phi}_1 + \tilde{\phi}_2))\|_{\eta} \leq \kappa(\varepsilon)(C_8 + C_9)(\|\phi_1 - \tilde{\phi}_1\|_\eta + \|\phi_2 - \tilde{\phi}_2\|_\eta).$$

This implies that for sufficiently small $\varepsilon > 0$, $T$ is a contraction on $BC^0(\mathbb{R}; X_1 \oplus X_2)$. \hfill $\Box$

**Remark** A. Vanderbauwhede and G. Iooss \cite{13} show that if $N$ is class $C^k$, $k = 1, 2, \ldots$, then the invariant manifold belong to class $C^k$ (see also \cite{15}). It is clear that the statements of theorems in this section are also valid replacing the assumption $N \in C^{k+1}_0(X; Y)$ with the following:

- $N \in C^k_b(X; Y)$, and there exists a positive constant $\kappa(\varepsilon)$ with $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that if $\|\alpha\| < \varepsilon$ then $\|D^jN\|_{Lip} \leq \kappa(\varepsilon), 0 \leq j \leq k$.

### 3 Equations on Fourier space

Let us consider the chemotaxis-diffusion-growth system (1.1) in a rectangular domain $x = (x, y) \in \Omega = (0, L_x) \times (0, L_y)$ with periodic boundary conditions. In this section, to study the dynamics of (1.1) on the invariant manifold, we transform the chemotaxis system to the system of infinite number of ordinary differential equations in Fourier space. The system (1.1) have constant stationary states $(u, v) = (0, 0), (1, f/g)$. Changing the variables $(u_*, v_*) = (u - 1, v - f/g)$, we have

$$\begin{align*}
\frac{\partial u_*}{\partial t} &= a\Delta u_* - b\nabla \cdot (u_* \nabla v_*) - pt(u_*(1 + u_*)), \\
\frac{\partial v_*}{\partial t} &= d\Delta v_* + fu_* - gv_*.
\end{align*}$$

(3.1)

We note that the system (3.1) can be represented in the following form:

$$u_t = Lu + N(u), (t, x) \in (0, \infty) \times \Omega,$$

where $u = (u_*, v_*)$,

$$L \begin{pmatrix} u_* \\ v_* \end{pmatrix} = \begin{pmatrix} a\Delta - p & -b\nabla \\ f & \Delta - g \end{pmatrix} \begin{pmatrix} u_* \\ v_* \end{pmatrix},$$

$$N \begin{pmatrix} u_* \\ v_* \end{pmatrix} = \begin{pmatrix} -b\nabla \cdot (u_* \nabla v_*) - ptu_*^2 \\ 0 \end{pmatrix}.$$ 

We can check that $L \in \mathcal{L}(H^1(\Omega)^2; H^2(\Omega)^2)$, $N \in C^0_b(H^1(\Omega)^2; H^2(\Omega)^2)$, and moreover, $N(0) = DN(0) = 0$ holds. In addition, since $H^4(\Omega)$ is continuously embedded in $H^2(\Omega)$, the assumptions of theorems in section 2 are satisfied by settings of the following: Consider the Fourier expansion of the functions $u_*(t, x, y)$ and $v_*(t, x, y)$:

$$u_*(t, x, y) = \sum_{\ell \in \mathbb{Z}^2} u_\ell(t) e^{i(\ell_1 \alpha + \ell_2 \beta y)}, \quad v_*(t, x, y) = \sum_{\ell \in \mathbb{Z}^2} v_\ell(t) e^{i(\ell_1 \alpha + \ell_2 \beta y)}$$

(3.2)
in a function space

\[ X := \left\{ u \in H^4(\Omega) \times H^4(\Omega); u(t, x, y) \text{ satisfies periodic boundary conditions, } \|u\|^2_X = \sum_{t \in \mathbb{Z}^2} (1 + |t|^2)^4|u_t|^2 < \infty \right\}. \]

Here, \( \ell, \alpha \) and \( \beta \) denote \((\ell_1, \ell_2), 2\pi/L_x \) and \( 2\pi/L_y \), respectively. In addition, \( u_\ell \) denotes \((u_\ell, v_\ell)\). Further, since the functions \( u_\ell \) and \( v_\ell \) are real valued, it is required that

\[ u_\ell = \overline{u}_{-\ell}, \quad \ell \in \mathbb{Z}^2, \tag{3.3} \]

where \( \cdot \) denotes the complex conjugate. For the sake of simplicity, we use the notation \((u_{\ell_1, \ell_2}, v_{\ell_1, \ell_2})\) to denote \((u_{(\ell_1, \ell_2)}, v_{(\ell_1, \ell_2)})\). In this situation, the linearized operator of (3.1) is a generator of an analytic semigroup \(^9\). It is convenient to consider the equation (3.1) on the space \( Y \) of Fourier coefficients instead:

\[ Y := \left\{ \hat{u} = \{u_\ell\}_{\ell \in \mathbb{Z}^2}; \|\hat{u}\|^2_Y = \sum_{\ell \in \mathbb{Z}^2} (1 + |\ell|^2)^4|u_\ell|^2 < \infty \right\} \]

which is equivalent to \( X \) by the map \( \mathcal{R} : X \rightarrow Y \), where

\[ \mathcal{R}(u) = \left\{ \frac{1}{\mathcal{M}(\Omega)} \int_{\Omega} u e^{-i(\ell_1 \alpha x + \ell_2 \beta y)} dx \right\}_{\ell \in \mathbb{Z}^2}. \]

Here, \( \cdot \) and \( \mathcal{M}(\cdot) \) denote the Euclidean norm and the Lebesgue measure in \( \mathbb{R}^2 \), respectively. For a given pair of integers \( \mathbf{m} = (m, n) \), we define the projection \( \mathcal{P}_\mathbf{m} : Y \rightarrow Y \) as follows:

\[ \mathcal{P}_\mathbf{m}(\{u_\ell\}_{\ell \in \mathbb{Z}^2}) = \{\delta^\mathbf{m}_\ell\}_{\ell \in \mathbb{Z}^2}, \tag{3.4} \]

where

\[ \delta^\mathbf{m}_\ell = \begin{cases} u_m & (\ell = \mathbf{m}), \\ (0, 0) & (\ell \neq \mathbf{m}). \end{cases} \]

Using Fourier expansions (3.2) and the map \( \mathcal{P}_\mathbf{m} \circ \mathcal{R} \), the flow on \( Y \) of the chemotaxis-diffusion-growth system (3.1) can be characterized as follows:

\[ \begin{pmatrix} \dot{v}_m \\ \dot{v}_m \end{pmatrix} = \mathcal{M}_m \begin{pmatrix} \hat{u}_m \\ \hat{u}_m \end{pmatrix} + \begin{pmatrix} \mathcal{N}_m \\ 0 \end{pmatrix}, \tag{3.5} \]

Here, we put

\[ \mathcal{N}_m := -b \sum_{m_1 + m_2 = m} (m_1 m_2 \alpha^2 + n_1 n_2 \beta^2 - \omega_m^2) u_{m_1} v_{m_2} - p \sum_{m_1 + m_2 = m} u_{m_1} u_{m_2}, \]

\[ \mathcal{M}_m := \begin{pmatrix} -A_m & b \omega_m^2 \\ f & -B_m \end{pmatrix}, \]

where

\[ A_m := a \omega_m^2 + p, \quad B_m := d \omega_m^2 + g, \]

\[ m_j := (m_j, n_j), \quad \omega_m^2 := m^2 \alpha^2 + n^2 \beta^2. \]
It is easy to see that the matrix $M_m$ has a zero eigenvalue if and only if
\[ b = \frac{A_mB_m}{f\omega_m^2}. \]  (3.6)

We introduce new variables $\tilde{u}_m$ and $\tilde{v}_m$ as follows:
\[ \Phi_m^{-1}(\tilde{u}_m, \tilde{v}_m) = (u_m, v_m), \]  (3.7)
\[ \Phi_m = \begin{pmatrix} B_m & -A_m \\ f & f \end{pmatrix}. \]  (3.8)

For given pair of integers $m$ and sufficiently small $\varepsilon_0$, let us take the parameters in (3.5) satisfying $|\det M_m| < \varepsilon_0$. Then, we have
\[ \begin{pmatrix} \tilde{u}_m \\ \tilde{v}_m \end{pmatrix} = \begin{pmatrix} \mu_m & 0 \\ 0 & -A_m - B_m + O(\varepsilon_0) \end{pmatrix} \begin{pmatrix} \tilde{u}_m \\ \tilde{v}_m \end{pmatrix} + \Phi_m^{-1} \begin{pmatrix} N_m \\ 0 \end{pmatrix}, \]  (3.9)
where
\[ \mu_m := \frac{1}{2} \{ - (A_m + B_m) + \sqrt{(A_m + B_m)^2 - 4(A_mB_m - f\omega_m^2)} \}, \]
and it follows that $|\mu_m| < O(\varepsilon_0)$ by Taylor's theorem.

Let us consider the linearized problem of (3.5) around the trivial solution. We can obtain the neutral stability surface for each mode $m \in \mathbb{Z}^2$ as follows:
\[ G_m := \left\{ (b, \alpha, \beta) \in \mathbb{R}^3 ; b = b_m(\alpha, \beta) := \frac{A_mB_m}{f\omega_m^2} \right\}. \]  (3.10)

Furthermore, for a given pair of integers $m$ and $n$, the function $b_{m,n}(\alpha, \beta)$ attains its minimum $b_c := \frac{1}{f} (\sqrt{ab} + \sqrt{cd})^2$ on the curve
\[ C_m = \left\{ (\alpha, \beta) ; m^2\alpha^2 + n^2\beta^2 = \sqrt{ab} \right\}. \]
That is, in a space of parameters, "$(m, n)$-mode" is destabilized on the surface $G_{m,n}$. Moreover, for given two pairs of integers $m = (m, n)$ and $\tilde{m} = (\tilde{m}, \tilde{n})$, with $m \neq \tilde{m}$, the intersection of $C_m$ and $C_{\tilde{m}}$ on the plane $b = b_c$ gives us a multiple critical point.

### 4 One dimensional problem

In this section, we review the results to the system (1.1) in one dimension $^{121}$. We consider (1.1) on $\Omega = (0, L) \subset \mathbb{R}$ under Neumann boundary conditions. Using the Fourier expansion:
\[ u_\ell(t, x) = u_0 + \sum_{\ell \in \mathbb{N}} u_\ell(t) \cos(\ell \alpha x), \quad v_\ell(t, x) = v_0 + \sum_{\ell \in \mathbb{N}} v_\ell(t) \cos(\ell \alpha x) \]  \quad (4.1)
where $\alpha = \pi/L$, the dynamical system of (3.1) on Fourier space is represented as follows:
\[ \begin{cases} 
\begin{pmatrix} \mu_m & 0 \\ 0 & -A_m - B_m + O(\varepsilon_0) \end{pmatrix} \begin{pmatrix} \tilde{u}_m \\ \tilde{v}_m \end{pmatrix} + \Phi_m^{-1} \begin{pmatrix} N_m \\ 0 \end{pmatrix}, & m \in \mathcal{S}_b, \\
\begin{pmatrix} \mu_m & 0 \\ 0 & -A_m - B_m + O(\varepsilon_0) \end{pmatrix} \begin{pmatrix} \tilde{u}_m \\ \tilde{v}_m \end{pmatrix} = \begin{pmatrix} N_m \\ 0 \end{pmatrix}, & m \in (\mathbb{N} \cup \{0\}) \setminus \mathcal{S}_b, 
\end{cases} \]  \quad (4.2)
\[ \mu_m := \frac{1}{2} \left\{ -A_m - B_m + \sqrt{(A_m - B_m)^2 + 4jm^2} \right\}. \]

It can be derived by similar way shown in the previous section. For given natural numbers \( j \) and \( k \), we can see that if

\[ \alpha = \alpha^{jk} := \frac{1}{\sqrt{jk}} \left( \frac{pg}{ad} \right)^{\frac{1}{2}}, \quad b = b^{jk} := \frac{A_j B_j}{f^{\alpha^{jk}}}, \]

then \( \mu_j = \mu_k = 0 \) holds.

Firstly, let us see the 1 : 2 modal interaction in the system (1.1) using the known arguments \(^2\). Let \( W_{loc}^c \) be a center manifold in a neighborhood of \( \mathbb{R}^2 \times Y \) at \((\alpha, b) = (\alpha^{1.2}, b^{1.2})\). Then the following holds.

**Theorem 4** The dynamics of (4.2) on \( W_{loc}^c \) is topologically equivalent to the dynamics of the following system:

\[ \dot{\hat{u}}_j = f_j(\hat{u}_1, \hat{u}_2), \quad j = 1, 2, \quad (4.3) \]

where

\[ f_1(z_1, z_2) = e_{10}z_1z_2 + (\mu_1 + e_{11}z_1^2 + e_{12}z_2^2)z_1, \]
\[ f_2(z_1, z_2) = e_{20}z_1^2 + (\mu_2 + e_{21}z_1^2 + e_{22}z_2^2)z_2. \]

Moreover, the coefficients \( e_{jk}, \quad j = 1, 2, \quad k = 0, 1, 2 \) are dependent on parameters and coefficients in (4.2).

Let us focus attention on Hopf-bifurcation phenomena around the equilibrium of (4.3). Suppose that \((z_1^*, z_2^*) \neq 0\) is an equilibrium of (4.3), and \( M \) is a linearized matrix around \((z_1^*, z_2^*)\). Then, the matrix \( M \) has pure imaginary eigenvalues if and only if \( \det M > 0 \) and trace \( M = 0 \) hold. Solving \( f_1 = f_2 = 0 \), \( z_1 = \rho z_2 \) and trace \( M = 0 \) for \((z_1, z_2, \mu_1, \mu_2)\), we have

\[ z_2 = z_2^* := \frac{\rho^2 e_{20}}{2(e_{11} + e_{22})}, \]
\[ z_1 = z_1^* := \rho z_2^*, \]

and the bifurcation point is

\[ \mu_1 = \mu_{1*} := -\frac{\rho^2 e_{20} \{2e_{10}(\rho^2 e_{11} + e_{22}) + \rho^2 e_{20}(\rho^2 e_{11} + e_{12})\}}{4(\rho^2 e_{11} + e_{22})^2}, \]
\[ \mu_2 = \mu_{2*} := -\frac{\rho^4 e_{20}^2 (2\rho^2 e_{11} + 3e_{22} + \rho^2 e_{21})}{4(\rho^2 e_{11} + e_{22})^2}. \]

Thus, the condition: \( \det M > 0 \) at \((\mu_{1*}, \mu_{2*}, z_1^*, z_2^*)\) is a necessary condition to the Hopf-bifurcation around \((z_1^*, z_2^*)\). Let us see the case where the coefficients in (1.1) are given as follows:

\[ a = 1/16, \quad d = 1, \quad f = 1, \quad g = 32, \quad p = 2. \]

Then, we have

\[ e_{10} = \frac{3972}{17}, \quad e_{11} = \frac{10991616}{4913}, \quad e_{12} = -\frac{1046707104}{24865}, \]
\[ e_{20} = \frac{708}{17}, \quad e_{21} = \frac{46831112}{24865}, \quad e_{22} = -\frac{811098}{17}. \]

Choosing \( \rho = 1 \), the Hopf-bifurcation point is

\[ \mu_{1*} = \frac{4628915091}{63798558760} \approx -0.0725, \quad \mu_{2*} = \frac{271863864}{7974819845} \approx 0.03409. \]

In fact, we can observe the limit cycle which corresponds to the standing wave solution to (1.1) (see Figure 3).
Secondly, let us consider the three modal interaction. Setting \((a, b) = (a^{1.3}, b^{1.3})\), the following holds:
\[
\mu_j < 0 = \mu_1 = \mu_3 < \mu_2 \quad \text{and} \quad \mu_2 < \min |\mu_j|, (j \in \{\mathbb{N} \cup \{0\}\} \setminus \{1, 2, 3\}).
\]

Let \(W^{cu}_{loc}\) be a center-unstable manifold of \((4.2)\) in a neighborhood of \(\mathbb{R}^3 \times Y\) and taking \(g\) and \(p\) small, we have the following theorem.

**Theorem 5** For given positive constants \(a, d\) and \(f\), there exist constants \(a_k, k = 1, ..., 7, b_k, k = 1, ..., 7\) and \(c_k, k = 1, ..., 6\) such that the dynamics of \((4.2)\) on \(W^{cu}_{loc}\) is topologically equivalent to the dynamics of the following system
\[
\dot{z}_j = f_j(\hat{u}_1, \hat{u}_2, \hat{u}_3), \quad z_j \in \mathbb{R}, \quad j = 1, 2, 3. \tag{4.4}
\]

where
\[
\begin{align*}
f_1(z_1, z_2, z_3) &= (a_1 z_1 + a_2 z_2) z_2 + (\mu_1 + a_3 z_1^2 + a_4 z_2^2 + a_5 z_3^2 + a_6 z_1 z_3) z_1 + a_7 z_2^2 z_3, \\
f_2(z_1, z_2, z_3) &= (b_1 z_1 + b_2 z_2) z_1 + (\mu_2 + b_3 z_1^2 + b_4 z_2^2 + b_5 z_3^2 + b_6 z_1 z_3) z_2, \\
f_3(z_1, z_2, z_3) &= c_1 z_1 z_2 + (\mu_3 + c_2 z_1^2 + c_3 z_2^2 + c_4 z_3^2) z_3 + c_5 z_1^2 + c_6 z_2^2.
\end{align*}
\]

Let us study the linearized stability of the equilibriums of \((4.4)\): \((z_1, z_2, z_3) = (0, \pm \sqrt{-\mu_2/b_4}, 0)\). The linearized matrix around these equilibriums are
\[
\mathcal{M} = \begin{pmatrix}
\mu_1 + a_1 z_2 + a_4 z_2^2 & 0 & \pm a_2 z_2 + a_7 z_2^2 \\
0 & -2\mu_2 & 0 \\
\pm c_1 z_2 + c_6 z_2^2 & 0 & \mu_3 + c_3 z_2^2
\end{pmatrix}.
\]

Here, \(z_{2*}\) denotes \(\sqrt{-\mu_2/b_4}\). It can be observed that the necessary conditions to Hopf-bifurcation are \((c_1 \pm c_6 z_{2*})(a_2 \pm a_7 z_{2*}) < 0\) and \(\mu_2 b_4 < 0\) at \((\mu_1, \mu_3) = ((\mp a_1 - a_4 z_{2*}) z_{2*}, -c_3 z_{2*}^2)\). In this case, the eigenvalues of \(\mathcal{M}\) are \(-2\mu_2\) and \(\pm i\omega_{2*}\), where \(\omega_{2*} = -(c_1 \pm c_6 z_{2*})(a_2 \pm a_7 z_{2*})\). Similarly, the equilibriums \((0, 0, \pm z_{3*}) : = (0, 0, \pm \sqrt{-\mu_3/c_1})\) are destabilized with eigenvalues \(-2\mu_3, \pm \sqrt{a_2 b_2}\) at \((\mu_1, \mu_2) = (-a_5 z_{3*}^2, -b_5 z_{3*}^2)\). Thus, if \(a_2 b_2 < 0\) then these equilibriums are destabilized with pure imaginary eigenvalues.

Let us study the following case:
\[
a = 1/4, \quad d = 16, \quad f = 1, \quad g = p = 1/16. \tag{4.5}
\]

Then we have
\[
(a^{1.3}, b^{1.3}) = (275/192, \sqrt{6}/24).
\]
In this case, the coefficients of third order terms are given by the following:

\[
\begin{align*}
\mu_2 &= \mu_2^* := \frac{-77 + \sqrt{6169}}{192} \approx 0.008, \\
\frac{a_1}{520704} &= -\frac{22055}{57856}, \quad \frac{a_2}{a_5} = -\frac{15375}{34728064}, \quad \frac{a_3}{a_5} = \frac{217653469}{400092076691456}, \quad a_4 = -\frac{22711319143975}{7104343523733504}, \\
\frac{a_5}{117356} &= 1322893, \quad a_6 = \frac{91622502625}{159572865024}, \quad a_7 = -\frac{91622502625}{159572865024}, \\
\frac{b_1}{64512} &= \frac{143}{10752}, \quad b_2 = -\frac{475}{10752}, \quad b_3 = -\frac{181109972065}{440092076691456}, \quad b_4 = -\frac{6580325}{51093504}, \\
b_5 &= -\frac{6956901071875}{14139065892864}, \quad b_6 = -\frac{9156412925}{46130208768}, \\
c_1 &= -972288, \quad c_2 = -\frac{330935}{6806016}, \quad c_3 = -\frac{172891631284375}{1220458460479488}, \quad c_4 = -\frac{8715625}{41484288}, \\
c_5 &= -\frac{17943211}{4863465600}, \quad c_6 = -\frac{12108977485}{446945236992}.
\end{align*}
\]

Since \(a_2b_2 > 0\), Hopf-bifurcation around the pure mode equilibriums \((0,0,\pm z_3)\) does not occur. We can see that \((z_1, z_2, z_3) = (0, \sqrt{-\mu_2^2/b_4}, 0) \approx (0, 0.2498, 0)\) holds:

\[(c_1 + c_6z_3)(a_2 + a_7z_3) \approx -0.1235 \times 10^{-2} < 0,\]

and Hopf-bifurcation point is

\[\mu_1 = (a_1 - a_4z_2)z_2 \approx 0.011, \mu_3 = c_3z_2^2 \approx 0.0088.\]

We note that the equilibrium \((0, -\sqrt{-\mu_2^2/b_4}, 0) \approx (0, -0.2498, 0)\) also has pure imaginary eigenvalues at a point

\[\mu_1 = -(a_1 - a_4z_2)z_2 \approx -0.010, \mu_3 = c_3z_2^2 \approx 0.0088.\]

However, in this case, \(\mu_1\) is positive, i.e., this parameter values are away from the region that the invariant manifold is attractive.

5 Stability of the hexagonal patterns

In this section, we study the dynamics of (1.1) in a rectangular domain: \(\Omega = (0, L_x) \times (0, L_y)\) under periodic boundary conditions. Choosing

\[\alpha^2 = \{(p\gamma)/(ad)\}^{1/2}/4, \beta = \sqrt{3}\alpha,\] (5.1)

then the six Fourier modes:

\[\{\pm(2, 0), \pm(1, 1), \pm(1, -1)\}\] (5.2)

are destabilized at \(b = b_c\). More general, the following holds \([11]\):
Theorem 6 For given integers $k$, $k'$ and $j$ which have the same sign and satisfy $k/k' = 2$, the dynamics of (3.5) on the center manifold is topologically equivalent to the dynamics of the following system:

$$
\begin{align*}
\dot{U} &= Q_1 VW + (\mu_1 + \rho_1 |U|^2 + \rho_2 |V|^2 + \rho_3 |W|^2)U, \\
\dot{V} &= Q_2 U W + (\mu_2 + \rho_3 |U|^2 + \rho_4 |V|^2 + \rho_5 |W|^2)V, \\
\dot{W} &= Q_3 U V + (\mu_2 + \rho_3 |U|^2 + \rho_4 |V|^2 + \rho_5 |W|^2)W.
\end{align*}
$$

(5.3)

Here, $U$, $V$ and $W$ denote $\tilde{u}_{k,0}$, $\tilde{u}_{k',j}$ and $\tilde{u}_{k',-j}$, respectively. In addition, the coefficients $Q_j$ and $\rho_j$ are dependent on coefficients and parameters in (3.5).

For the reader's convenience, we represent the proof of this theorem in appendix. It should be noted that Theorem 5 (in section 4) can be proved using the similar arguments of the proof. Let us study the following case:

$$
\alpha = 1/16, \quad \delta = 16, \quad \beta = 32, \quad p = 9/2.
$$

(5.4)

We also take the parameters $\alpha$ and $\beta$ as shown in (5.1). Then, we get the coefficients of the normal form (5.3) as follows:

$$
Q_j = \frac{193536}{131}, \quad j = 1, 2, \rho_1 = \rho_4 = \frac{341999616}{131},
$$

$$
\rho_2 = \rho_3 = \rho_5 = -\frac{672418610496}{2248091}.
$$

We can verify that the system (5.3) has an asymptotically stable equilibrium $(U, V, W) = (r_*, r_*, r_*) \in \mathbb{R}^3, (r_* \neq 0)$ by a suitable choice of parameters $\mu_j$. Note that this equilibrium correspond to locally stable hexagonal pattern in the chemotaxis system (3.1), or (1.1).

Let us consider the case where more than six Fourier modes are destabilized. We define a set $S \subset \mathbb{Z}^2$ as follows:

$$
S := S_h \cup \{\pm(2, 1), \pm(2, -1), \pm(0, 1)\}.
$$

Setting

$$
\alpha = \alpha^* := (\gamma^2/21\alpha d)^{1/4}, \quad \beta^* = \sqrt{3\alpha^*}, \quad b^* := b_{0,1}(\alpha^*, \sqrt{3\alpha^*}),
$$

then the following holds:

$$
\mu_1 < 0 = \mu_{2,1} = \mu_{2,1} < \mu_{m, \ell} \in \mathbb{Z}^2 \setminus S, \quad m, \ell \in S_h.
$$

To study the stability of hexagonal pattern, we need the reduced system with respect to $\tilde{u}_m, m \in S$. However, it is too complicated to analyze the dynamics of reduced system on $C^6 \equiv \mathbb{R}^1$ even if we focus attention on only the problem of stability of hexagonal pattern. For the sake of simplicity, we restrict the problem (1.1) in a function space

$$
X_{even} := \{u = (u, v) \in X; u(t, x, y) = u(t, L_x - x, y), u(t, x, y) = u(t, x, L_y - y)\}.
$$

This restriction correspond to the restriction of the system (3.5) in the real subspace:

$$
Y_R := \{(u)_{m \in \mathbb{Z}^2} \in Y; u_m = \bar{u}_m\}.
$$

It should be noted that the subspace $Y_R \subset Y$ is invariant under the flow of system (3.5) and corresponds to the case of Neumann boundary conditions. Putting

$$
(z_1, z_2, z_3, z_4, z_5, z_6) = Re (\tilde{u}_{2,0}, \tilde{u}_{1,1}, \tilde{u}_{1,-1}, \tilde{u}_{0,1}, \tilde{u}_{2,1}, \tilde{u}_{2,-1}),
$$

we have the following theorem.
Theorem 7 The dynamics of (3.5) on the local invariant manifold $W_{loc}^{\text{sp}}$ is topologically equivalent to the following system:

$$
\dot{z}_j = f_j(z_1, z_2, z_3, z_4, z_5, z_6), \quad z_j \in \mathbb{R}, \quad j = 1, \ldots, 6.
$$

where

$$
f_1 = (\mu_{2, 0} + \sum_{j=1}^{6} d_{1j} z_j^2 + d_{17} z_5 z_6) z_1 + g_{11} z_2 z_3 + g_{12} (z_5 + z_6) z_4,
$$

$$
f_2 = (\mu_{1, 1} + \sum_{j=1}^{6} d_{2j} z_j^2 + d_{27} z_4 z_5) z_2 + g_{21} z_1 z_3 + g_{22} z_3 z_4 + g_{23} z_3 z_5 + g_{24} z_3 z_6 + g_{25} z_3 z_5 z_6 + g_{26} z_3 z_5 z_6,
$$

$$
f_4 = (\mu_{0, 1} + \sum_{j=1}^{6} d_{4j} z_j^2 + d_{47} z_2 z_3) z_4 + g_{41} (z_5 + z_6) z_1 + g_{42} (z_5 + z_6) z_1 + g_{43} (z_5 + z_6) z_1 + g_{44} (z_5 + z_6) z_2 z_3,
$$

$$
f_5 = (\mu_{1, 2} + \sum_{j=1}^{6} d_{5j} z_j^2) z_5 + g_{51} z_1 z_4 + g_{52} z_5^2 z_6 + g_{53} z_1 z_4 + g_{54} z_5 z_6 + g_{55} z_5 z_6 + g_{56} z_5 z_6.
$$

and $f_3, f_6$ are given by the following:

$$
f_3(z_1, z_2, z_3, z_4, z_5, z_6) = f_2(z_1, z_3, z_2, z_4, z_5, z_6),
$$

$$
f_6(z_1, z_2, z_3, z_4, z_5, z_6) = f_5(z_1, z_3, z_2, z_4, z_5, z_6).
$$

Moreover, the coefficients $d_{jk}$ and $g_{jk}$ of (5.5) are dependent on the coefficients and parameters in (3.5).

This theorem can be proved using the similar arguments of proof of Theorem 6 (see appendix).

Let us study the stability of the equilibria of (5.5) corresponding to hexagonal pattern in (1.1). Let $z_\ast = (z_1, z_2, z_3, z_4, 0, 0, 0) \in \mathbb{R}^6$ be a hexagonal equilibrium of (5.5). Then the linearized eigenvalues around this equilibrium are given by the eigenvalues of two matrixes:

$$
M_1 := \begin{pmatrix}
\frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \frac{\partial f_1}{\partial z_3} \\
\frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \frac{\partial f_2}{\partial z_3} \\
\frac{\partial f_3}{\partial z_1} & \frac{\partial f_3}{\partial z_2} & \frac{\partial f_3}{\partial z_3}
\end{pmatrix}(z_\ast) \quad \text{and} \quad M_2 := \begin{pmatrix}
\frac{\partial f_4}{\partial z_1} & \frac{\partial f_4}{\partial z_2} & \frac{\partial f_4}{\partial z_3} \\
\frac{\partial f_5}{\partial z_1} & \frac{\partial f_5}{\partial z_2} & \frac{\partial f_5}{\partial z_3} \\
\frac{\partial f_6}{\partial z_1} & \frac{\partial f_6}{\partial z_2} & \frac{\partial f_6}{\partial z_3}
\end{pmatrix}(z_\ast).
$$

We can verify that all the eigenvalues of $M_1$ are real. Let $P(\lambda)$ be a characteristic polynomial of $M_2$:

$$
P(\lambda) = \lambda^3 + s_2 \lambda^2 + s_1 \lambda + s_0.
$$

Then it has pure imaginary roots if and only if

$$
s_0 - s_2 s_1 = 0 \quad \text{and} \quad s_1 > 0.
$$

This implies that for given coefficients $(a, d, f, g, p)$ in (3.9) (or (1.1)), if there exist positive constants $(a, b, \beta)$ near $(\alpha^*, \beta^*, \theta^*)$ satisfying (5.6) and the assumptions in Theorem 7, then the equilibrium $z_\ast$ is destabilized with pure imaginary eigenvalues.

Let us see the following case:

$$
a = 1/21, \quad d = 16, \quad f = 1, \quad g = 1/4, \quad p = 1/4.
Then, we get $\alpha^* = 1/4, \beta^* = \sqrt{3}/4$ and $b^* = 377/84$. Moreover, we can compute coefficients in (5.5) rigorously. In fact, the hexagonal equilibria of (5.5) are given by roots of the following:

$$-rac{101362426606463}{274377292560} z^2 + \frac{7089}{6064} z + \mu_{2,0} = 0. \tag{5.7}$$

Let $z_*$ be one of a root of (5.7). Choosing $b = b^* + 0.01$, all eigenvalues of the linearized matrix around $z_* = (z_*, z_*, z_*, 0, 0, 0)$ have negative real parts, i.e., (1.1) has a stable hexagonal pattern.

Setting $\mu_{2,1} = 10^{-4}$, we can verify the following fact: there exists a constants $\mu_*$ such that if $\mu_{0,1} = \mu_*$, then (5.6) holds. This implies that $M_2$ has pure imaginary eigenvalues by a suitable choice of parameters.

Notice that the approximate value of $\mu_*$ is $\mu_* \approx 10.5310 \times 10^{-3}$. To check the assumptions with respect to the spectrum (in Theorem 1), we solve the equations $\mu_{2,1}(a, b) = 10^{-4}$ and $\mu_{0,1} = \mu_*$ with $\beta = \sqrt{3} a$ for $(a, b)$. Then, we get $(a, b) \approx (0.3161, 4.6357)$. This yields

$$\min_{m \in S} |\mu_m| = \mu_{2,1} = 10^{-4} < \min_{l \in \mathbb{Z} \setminus S} |\mu_l| = |\mu_{1,0}| \approx 0.3709 \times 10^{-2}.$$  

This implies that the hexagonal pattern is destabilized with pure imaginary eigenvalues.

**Appendix**

We give the proof of Theorem 6. It should be noted that the other theorems with respect to reduced system: Theorem 4, 5, and 7 can be proved similarly. We start our consideration with the system (3.5) on Fourier space $Y$:

$$
\begin{pmatrix}
\dot{u}_m \\
\dot{v}_m 
\end{pmatrix} = M_m 
\begin{pmatrix}
u_m \\
v_m 
\end{pmatrix} + \begin{pmatrix}
N_m \\
0
\end{pmatrix}.
$$

Definitions of $M_m$ and $N_m$ are given in section 3. Consider the case where the following holds:

$$|\det M_m| < O(\varepsilon_0), m \in S_h := \{ \pm (k, 0), \pm (k', j), \pm (k', -j) \}. \tag{5.8}$$

Let us define the projection $P$ for critical modes by

$$P(\{\tilde{u}\}) = \{\gamma_l \}_{l \in \mathbb{Z}^2}$$

where

$$\gamma_l = \begin{cases} 
(\tilde{u}_l, 0) & (l \in S_h), \\
(0, 0) & (l \notin S_h),
\end{cases} \tag{5.9}$$

and define the projection $Q := I - P$. We consider the extended system (3.5) for $m \in \mathbb{Z}^2 \setminus S_h$ and (3.9) for $m \in S_h$ with the trivial equations $\dot{u}_m = 0, m \in S_h$. It follows that the center space of this extended flow is spanned by $(\mu, \tilde{u})$, where

$$
\begin{aligned}
\mu &:= (\mu_{k,0}, \mu_{k',j}, \mu_{k',-j}), \\
\tilde{u} &:= (\tilde{u}_{\pm(k,0)}, \tilde{u}_{\pm(k',j)}, \tilde{u}_{\pm(k',-j)}) \\
&= (\tilde{u}_{k,0}, \tilde{u}_{k',j}, \tilde{u}_{k',-j}).
\end{aligned}
$$

We put $\mathcal{U} = Q(\tilde{u})$ for $\tilde{u} \in Y$. Then, the center manifold theory states that there exists a positive constant $\varepsilon$ such that there is a neighborhood $\mathcal{U}^\varepsilon$ of $\mathbb{R}^3 \times Y \equiv \mathbb{R}^3 \times \mathbb{C}^6 \times QY$: 

$$\mathcal{U}^\varepsilon := \{(\mu, \tilde{u}) : |\mu| + |\tilde{u}| + |\tilde{u}|_Y < \varepsilon\} \tag{5.10}$$
which contains a smooth invariant manifold \( W^c_{\text{loc}} \) of (3.1). Moreover, there exists a smooth map \( \mathcal{H} : \mathbb{R}^3 \times \mathbb{C}^6 \to \mathcal{Q}Y \) satisfying
\[
\frac{\partial \mathcal{H}}{\partial J}(0) = 0, \quad \text{for } J = \mu, \tilde{u}, \tilde{v}, \quad \mathbf{m} \in \mathcal{S}_b
\]
by which \( W^c_{\text{loc}} \) is represented as
\[
W^c_{\text{loc}} = \{ (\mu, \tilde{u}, \tilde{v}) : \tilde{u} = \mathcal{H}(\mu, \tilde{u}) \}.
\]
Furthermore, we define the maps \( \{ h^u_{\mathbf{m}} \} \) and \( \{ h^v_{\mathbf{m}} \} \) which are terms of sequences \( \{ h_{\ell} \}_{\ell \in \mathbb{Z}^2} \) and \( \{ \tilde{h}_{\ell} \}_{\ell \in \mathbb{Z}^2} \) as follows:
\[
\begin{align*}
\{ h_{\ell} \}_{\ell \in \mathbb{Z}^2} &:= \mathcal{P}_m(\mathcal{H}(\mu, \tilde{u})) \quad \text{for } \mathbf{m} \in \mathbb{Z}^2 \setminus \mathcal{S}_b, \\
\{ \tilde{h}_{\ell} \}_{\ell \in \mathbb{Z}^2} &:= \mathcal{P}_m(\mathcal{H}(\mu, \tilde{u})) \quad \text{for } \mathbf{m} \in \mathcal{S}_b,
\end{align*}
\]
\[
h_{\ell} := \begin{cases} 
( h^u_{\mathbf{m}}(\mu, \tilde{u}), h^v_{\mathbf{m}}(\mu, \tilde{u}) ) & \text{if } \ell = \mathbf{m}, \\
(0,0) & \text{if } \ell \neq \mathbf{m},
\end{cases}
\]
\[
\tilde{h}_{\ell} := \begin{cases} 
(0, h^v_{\mathbf{m}}(\mu, \tilde{u}) ) & \text{if } \ell = \mathbf{m}, \\
(0,0) & \text{if } \ell \neq \mathbf{m},
\end{cases}
\]
where \( \mathcal{P}_m \) is defined in (3.4).

It should be noted that the reality condition (3.3) implies that we need only consider the equations for \( \tilde{u}_{k,0}, \tilde{u}_{k',j} \) and \( \tilde{u}_{k',-j} \) on the manifolds \( W^c_{\text{loc}} \).

To obtain the cubic normal form, it is necessary to calculate the quadratic approximation of center manifolds. We present the following lemma.

**Lemma 1** Let \( \mathbf{m}_c \in \mathcal{S}_b \) and \( \mathbf{m}_c \in \mathbb{Z}^2 \setminus \mathcal{S}_b \). The quadratic approximations of the maps \( h^u_{\mathbf{m}_c}, h^v_{\mathbf{m}_c} \) and \( \tilde{h}^v_{\mathbf{m}_c} \) are given by the graph of the functions:
\[
\begin{align*}
u_{\mathbf{m}_c} &= h^u_{\mathbf{m}_c}(\tilde{u}_{k,0}, \tilde{u}_{k',j}, \tilde{u}_{k',-j}), \\
v_{\mathbf{m}_c} &= h^v_{\mathbf{m}_c}(\tilde{u}_{k,0}, \tilde{u}_{k',j}, \tilde{u}_{k',-j}), \\
\tilde{v}_{\mathbf{m}_c} &= \tilde{h}^v_{\mathbf{m}_c}(\tilde{u}_{k,0}, \tilde{u}_{k',j}, \tilde{u}_{k',-j}),
\end{align*}
\]
and which are approximated as follows:
\[
\begin{pmatrix}
h^u_{\mathbf{m}_c}(\tilde{u}_{k,0}, \tilde{u}_{k',j}, \tilde{u}_{k',-j}) \\
h^v_{\mathbf{m}_c}(\tilde{u}_{k,0}, \tilde{u}_{k',j}, \tilde{u}_{k',-j}) \end{pmatrix} = M^{-1}_{\mathbf{m}_c} \begin{pmatrix} -N_{\mathbf{m}_c} \\ 0 \end{pmatrix}, \tag{5.11}
\]
\[
\tilde{h}^v_{\mathbf{m}_c}(\tilde{u}_{k,0}, \tilde{u}_{k',j}, \tilde{u}_{k',-j}) = \frac{-N_{\mathbf{m}_c}}{(A_{\mathbf{m}_c} + B_{\mathbf{m}_c})^2}, \tag{5.12}
\]
where
\[
N_{\mathbf{m}_c} := \sum_{m_1, m_2, m_3, m_4} \left\{ b(m_1m_2\alpha^2 + n_1n_2\beta^2 - \omega_{m_2}^2)u_{m_1}v_{m_2} + pu_{m_1}u_{m_2} \right\}.
\]

**Proof** The center manifold theory states that for given pair of integers \( \mathbf{m}_c \notin \mathcal{S}_b \), \( u_{\mathbf{m}_c} \), and \( v_{\mathbf{m}_c} \), are characterized by the map
\[
\begin{pmatrix}
u_{\mathbf{m}_c} \\
v^u_{\mathbf{m}_c} \end{pmatrix} = \begin{pmatrix}
h^u_{\mathbf{m}_c}(\tilde{u}_{k,0}, \tilde{u}_{k',j}, \tilde{u}_{k',-j}) \\
h^v_{\mathbf{m}_c}(\tilde{u}_{k,0}, \tilde{u}_{k',j}, \tilde{u}_{k',-j}) \end{pmatrix}.
\]
Differentiating with respect to $t$, we have

$$
\left( \sum_{m \in S_\delta} \frac{\partial h_{m, c}}{\partial u_{m, c}} \hat{u}_{m, c} \right) = M_{m, c} \begin{pmatrix} h_{m, c}^u \\ h_{m, c}^v \end{pmatrix} + \begin{pmatrix} N_{m, c} \\ 0 \end{pmatrix}.
$$

(5.13)

By the center manifold theory, for sufficiently small $\delta > 0$, if $|\hat{u}_{m, c}| < O(\delta)$, then it holds that

$$
|h_{m, c}^u| < O(\delta^2) \quad \text{and} \quad |h_{m, c}^v| < O(\delta^2).
$$

Furthermore, since $\hat{u}_{m, c} = \mu_{m, c} \hat{u}_{m, c} + O(\delta^2)$ and (5.8) holds, it follows that $\mu_{m, c} = O(\varepsilon_0)$. Therefore, the left hand side of (5.13) is $O(\delta^2)$ by choosing $\varepsilon_0 \approx \delta$. Finally, since the matrix $M_{m, c}$ is regular, we obtain the quadratic approximations of $h_{m, c}^u$ and $h_{m, c}^v$, as shown in (5.11). The approximation (5.12) can be obtained similarly.

Proof of Theorem 6 Let us consider the equations (3.9) for $\vec{u}_{m, c} \in S_\delta$. Using Lemma 1, we can calculate the nonlinear terms up to third order as follows:

$$
\dot{\vec{u}}_{m, c} = \mu_{m, c} \vec{u}_{m, c} - \frac{1}{A_{m, c} + B_{m, c}} \left\{ \sum_{m_1 + m_2 = m \in S_\delta} \left\{ (m_1 m_2 \alpha^2 + n_1 n_2 \beta^2 - \omega_{m_2}^2) u_{m_1} \hat{u}_{m_2} + p \mu_{m_1} u_{m_2} \right\} \right. \\
+ b \sum_{m_1 + m_2 = m \in S_\delta} \left\{ (m_1 m_2 \alpha^2 + n_1 n_2 \beta^2 - \omega_{m_2}^2) h_{m_1} \hat{u}_{m_2} \right\} \\
+ b \sum_{m_1 + m_2 = m \in S_\delta} \left\{ (m_1 m_2 \alpha^2 + n_1 n_2 \beta^2 - \omega_{m_2}^2) h_{m_1} \hat{u}_{m_2} \right\} \\
+ 2p \sum_{m_1 + m_2 = m \in S_\delta} \left\{ u_{m_1} \hat{h}_{m_2} \right\} \right\}.
$$

(5.14)

This yields the cubic normal form (5.3).

Acknowledgments

The author would like to express his sincere appreciation to Professor Koichi Osaki (of the Kwansei Gakuin University) for a number of valuable discussions. The author would like to thank also referees for their careful reading and constructive comments.

References


