On Time-Dependent Bivariate Copulas

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We propose a new simple method of constructing bivariate copulas, which evolve according to the time variable. It is proved that for any copula there exists a time-parametrized family of copulas which realizes this copula as the time tends to the maturity. Applications are also discussed.

1. INTRODUCTION

A copula function, or simply a copula, is introduced as a tool for understanding the dependence structure among random variables. Copulas make a link between multivariate joint distributions and univariate marginal distributions. We first recall the definition of copula in the bivariate case.

Definition. A function $C$ defined on $I^2 := [0,1] 	imes [0,1]$ and valued in $I$ is called a copula if the following conditions are fulfilled.

(i) For every $(u, v) \in I^2$,

$$C(u, 0) = C(0, v) = 0, \quad C(u, 1) = u \quad \text{and} \quad C(1, v) = v. \quad (1)$$

(ii) For every $(u_i, v_i) \in I^2$ ($i = 1, 2$) with $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0. \quad (2)$$

The requirement (2) is referred to as the 2-increasing condition. We also note that a copula is continuous by its definition.

The well-know result due to A. Sklar5, who employed the term "copula" almost for the first time, describes the above mentioned dependence relation. For completeness of our presentation, we here recall Sklar's theorem in the case of a bivariate joint distribution.

Theorem 1. (Sklar's theorem) Let $H$ be a bivariate joint distribution function with marginal distribution functions $F$ and $G$; that is,

$$\lim_{x \to \infty} H(x, y) = G(y), \quad \lim_{y \to \infty} H(x, y) = F(x).$$

Then there exists a copula, which is uniquely determined on $\text{Ran}F \times \text{Ran}G$, such that

$$H(x, y) = C(F(x), G(y)). \quad (3)$$

Conversely, if $C$ is a copula and $F$ and $G$ are distribution functions, then the function $H$ defined by (3) is a bivariate joint distribution function with margins $F$ and $G$.  

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The study of copulas has been developed extensively these days. Because of their flexible structure, copulas have been applied in many situations. We refer for instance to W.F. Darsow et al.\textsuperscript{1}, F. Durante et al.\textsuperscript{2}, and E.W. Frees and E.A. Valdez\textsuperscript{3}. The book of R.B. Nelsen\textsuperscript{4} is a well-summarized excellent monograph. For other materials, we refer to the nice review articles of H. Tsukahara\textsuperscript{5} and Y. Yoshizawa\textsuperscript{6}.\textsuperscript{7}

In this note, we propose a new simple method of constructing bivariate copulas; our family is parametrized by the time variable.

Various kinds of one-parameter families of copulas are already known. Just to mention a few of them, we recall the Clayton family, the Gumbel-Hougaard family, and the Frank family. However, it seems that these parameters do not have a direct connection to the time variable; copulas are not usually used for the modeling of the interaction between different stochastic processes. Observe an excellent work of W.F. Darsow, B. Nguyen, and E.T. Olsen\textsuperscript{1}, where the relation at different times of the single Markov process is discussed. On the other hand, random events in the real world normally occur according to time; the dependence structure should be changed as the time proceeds. We thus believe that the time-dependent family of copulas is worth investigation.

We therefore take this stance and proceed to consider evolution equations for copulas. As a first step, we are concerned with the heat equation: namely, for a time parameterized family of copulas \(\{C(u,v,t)\}_{t\geq 0}\), we impose

\[
\frac{\partial C}{\partial t}(u,v,t) = \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) C(u,v,t).
\] (4)

The stationary solution to (4), which is referred to as the harmonic copula (see for instance Nelsen\textsuperscript{4}), is uniquely determined to be \(\Pi(u,v) := uv\), in view of the boundary condition (1). We note that the copula \(\Pi\) represents the independent structure between two respective random variables.

In the rest of the paper, we prove the existence for the evolution equation of copulas in terms of the initial value problem. For the applications, it would be better to deal with the maturity value condition, which is discussed in the final section.

2. EVOLUTION EQUATION OF COPULAS

First we treat the initial value problem and prove the next theorem.

**Theorem 2.** For any bivariate copula \(C_0(u,v)\), there exists a unique family of time-parametrized bivariate copulas \(\{C(u,v,t)\}_{t\geq 0}\) such that

\[
\frac{\partial C}{\partial t}(u,v,t) = \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) C(u,v,t) \quad \text{for} \quad (u,v,t) \in I^2 \times (0,\infty)
\]

\[
C(u,v,0) = C_0(u,v) \quad \text{on} \quad (u,v) \in I^2.
\] (5)

Moreover we have

\[
\lim_{t\to\infty} C(u,v,t) = \Pi(u,v) \quad \text{uniformly on} \quad (u,v) \in I^2.
\]

**Remark.** By the definition of copula, we understand that \(C(\cdot,\cdot,t)\) fulfills (1)(2); explicitly stated, (i) for every \((u,v,t) \in I^2 \times (0,\infty)\),

\[
C(u,0,t) = C(0,v,t) = 0, \quad \text{and}
\]

\[
C(u,1,t) = u \quad \text{and} \quad C(1,v,t) = v.
\] (6)
(ii) for every \((u_i, v_i, t) \in I^2 \times (0, \infty)\) \((i = 1, 2)\) with \(u_1 \leq u_2\) and \(v_1 \leq v_2\),
\[
C(u_1, v_1, t) - C(u_2, v_1, t) - C(u_2, v_2, t) + C(u_2, v_2, t) \geq 0.
\] (7)

Proof. Taking into account of the boundary conditions (6), we wish to find a solution \(C(u, v, t)\) of (5) with the form
\[
C(u, v, t) = \hat{C}(u, v, t) + uv,
\]
where \(\hat{C}(u, v, t)\) satisfies
\[
\frac{\partial \hat{C}}{\partial t}(u, v, t) = \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \hat{C}(u, v, t) \quad \text{for } (u, v, t) \in I^2 \times (0, \infty)
\]
\[
\hat{C}(u, v, t) = 0 \quad \text{on } (u, v, t) \in \partial I^2 \times (0, \infty)
\]
\[
\hat{C}(u, v, 0) = C_0(u, v) - uv \quad \text{on } (u, v) \in I^2.
\] (8)

It is easy to see that the solutions \(\hat{C}\) to (8) are expressed by use of the well-known formula involving the kernel; we thus infer that
\[
C(u, v, t) = \left( \frac{1}{2\pi} \right)^{n^2} e^{-\tau(u^2 + v^2)/2} \sin \pi u \sin \pi v \int_{I^2} \sin \pi u \sin n \pi u (C_0(\xi, \eta) - \xi \eta) d\xi d\eta.
\] (9)

The asymptotic profile of \(C\) is deduced immediately from (9); we learn that
\[
C(u, v, t) \to \Pi(u, v) \quad \text{exponentially as } t \to \infty \text{ uniformly on } (u, v) \in I^2.
\]

It remains to show that \(C(\cdot, \cdot, t)\) verifies the 2-increasing condition. To accomplish this, we first assume that \(C_0\) is of \(C^2\)-class. For a \(C^2\)-class copula, it is clear that the 2-increasing condition is equivalent to
\[
\frac{\partial^2 C}{\partial u \partial v}(u, v, t) \geq 0 \quad \text{for } (u, v) \in I^2.
\]

Invoking the formula (9), we compute
\[
\frac{\partial^2 C}{\partial u \partial v}(u, v, t)
\]
\[
= 1 + 4 \sum_{m,n=1}^{\infty} e^{-\pi^2(t^2+n^2)} \sin \pi u \sin n \pi v \int_{I^2} \sin \pi u \sin n \pi u (C_0(\xi, \eta) - \xi \eta) d\xi d\eta
\]
\[
= 1 + 4 \sum_{m,n=1}^{\infty} e^{-\pi^2(t^2+n^2)} \sin \pi u \sin n \pi v \int_{I^2} \cos \pi u \cos n \pi v \left( \frac{\partial^2 C_0}{\partial \xi \partial \eta}(\xi, \eta) - 1 \right) d\xi d\eta.
\]

where the use of the boundary condition (1) was made. We write the second term of the above right hand side as \(p(u, v, t)\):
\[
p(u, v, t) := 4 \sum_{m,n=1}^{\infty} e^{-\pi^2(t^2+n^2)} \sin \pi u \sin n \pi v \int_{I^2} \cos \pi u \cos n \pi v \left( \frac{\partial^2 C_0}{\partial \xi \partial \eta}(\xi, \eta) - 1 \right) d\xi d\eta.
\]
Since $p$ fulfills
$$
\frac{\partial p}{\partial t}(u,v,t) = \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) p(u,v,t) \quad \text{for} \quad (u,v,t) \in I^2 \times (0,\infty)
$$
$$
\frac{\partial p}{\partial v}(u,v,t) = 0 \quad \text{on} \quad (u,v,t) \in \partial I^2 \times (0,\infty) \quad (\nu: \text{the outer normal to } \partial I^2)
$$
$$
\int_I p(u, \cdot, t) du = \int_I p(\cdot, v, t) dv = 0 \quad \text{for} \quad t \in (0,\infty)
$$
$$
p(u,v,0) = \frac{\partial^2 C_0}{\partial u \partial v}(u,v) - 1 \quad \text{on} \quad (u,v) \in I^2,
$$
the maximum principle implies that $p(u,v,t) \geq -1$ for $(u,v,t) \in I^2 \times (0,\infty)$ and thus it follows that $\frac{\partial^2 C}{\partial u \partial v} \geq 0$ on $I^2 \times (0,\infty)$. This proves the 2-increasing condition in the case that the initial copula $C_0$ is of $C^2$-class.

Next we turn our attention to the case of continuous initial copula $C_0$. We approximate $C_0$ by a sequence of $C^2$-copulas $\{C^\varepsilon(u,v)\}_{\varepsilon > 0}$ such that $\lim_{\varepsilon \to 0} C^\varepsilon(u,v) = C_0(u,v)$ uniformly on $(u,v) \in I^2$. The result proved above implies that for every $\varepsilon > 0$ there corresponds a family of copulas $\{C^\varepsilon(u,v,t)\}_{\varepsilon > 0}$ which satisfies (5) with the initial value $C^\varepsilon_0$. Moreover, $C^\varepsilon(u,v,t)$ verifies the 2-increasing condition; that is, for every $(u_i,v_i,t) \in I^2 \times (0,\infty)$ $(i = 1,2)$ with $u_1 \leq u_2, v_1 \leq v_2$, and for every $\varepsilon > 0, t > 0$,
$$
C^\varepsilon(u_1,v_1,t) - C^\varepsilon(u_1,v_2,t) - C^\varepsilon(u_2,v_1,t) + C^\varepsilon(u_2,v_2,t) \geq 0. \quad (10)
$$
Now we wish to send $\varepsilon \to 0$. The convergence of $C^\varepsilon(u,v,t)$ as $\varepsilon \to 0$ is assured by the standard argument and we infer that $C(u,v,t) = \lim_{\varepsilon \to 0} C^\varepsilon(u,v,t)$ is defined for all $(u,v) \in I^2, t > 0$, where $C(u,v,t)$ fulfills (5) with the initial value $C_0$. In particular, letting $\varepsilon \to 0$ in (10), we have the 2-increasing condition; for every $(u_i,v_i,t) \in I^2 \times (0,\infty)$ $(i = 1,2)$ with $u_1 \leq u_2$ and $v_1 \leq v_2$,
$$
C(u_1,v_1,t) - C(u_1,v_2,t) - C(u_2,v_1,t) + C(u_2,v_2,t) \geq 0,
$$
from where we conclude that the proof of theorem is finally completed.

3. DISCUSSIONS

We have introduced a new simple method for constructing a family of bivariate copulas, which are parametrized by the time variable. Here we have considered the case of copulas satisfying the linear heat equation and we proved a theorem on the existence and convergence of solutions. Our concept is that a family of copulas is transformed along with a solution to a certain evolution equation. We believe that such a procedure is important and interesting enough to be pursued further.

Finally, as an application of our methodology, we include the next observation. This is just a time reversal version of the initial value problem (5). However, real financial products customarily prescribe the maturity condition; the present form below of establishing time-dependent copulas is natural in financial mathematics.

**Theorem 3.** For any bivariate copula $C_T(u,v)$, where $T$ ($> 0$) denotes the maturity, there exists a unique family of time-parametrized bivariate copulas $\{C(u,v,t)\}_{0 \leq t \leq T}$ such that
$$
\frac{\partial C}{\partial t}(u,v,t) + \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) C(u,v,t) = 0 \quad \text{for} \quad (u,v,t) \in I^2 \times \{t \leq T\},
$$
$$
C(u,v,T) = C_T(u,v) \quad \text{on} \quad (u,v) \in I^2.
$$
The above theorem tells us that for any intended dependence relation represented by copula at maturity, there corresponds certain time-parametrized family of copulas which realize this prescribed structure as time tends the maturity. Our next project of research is to extend this theorem to real world applications.

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