An Analysis of Propagation of Spherical Waves in an Elastic-Plastic-Viscoplastic Body

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An analysis of dynamic behavior is carried out in the case when dynamic load with a central symmetry is applied on the inner face of the cavity with a central symmetry in a thick-walled elastic-plastic-viscoplastic sphere. First, formulas are derived of propagation speeds of elastic-plastic-viscoplastic spherical waves. Second, the theoretical formula is analytically proved to be strain rate and stress rate dependent. Third, ordinary differential equations among physical quantities are derived along characteristic curves. Fourth, the propagation theory of spherical wave based on the elastic-plastic-viscoplastic constitutive equation is shown to contain one based on the elastic-plastic constitutive equation. Fifth, calculated examples are shown on the basis of the elastic-plastic-viscoplastic theory. Last, a comparison is done of calculated results based on the elastic-plastic-viscoplastic theory with those based on the elastic-plastic theory.

1. INTRODUCTION

Many investigations have been done on inelastic spherical wave propagation in solid materials\(^9\). However, their investigations are almost on strain rate independent materials. There exists a theoretical research on spherical wave propagations in strain rate dependent elastic-viscoplastic materials by P. A. Fouti and F. Ziegler\(^9\). But, as is commonly known, the elastic-viscoplastic theory of wave propagation can’t be accepted sufficiently because the theoretical propagation speed based on the equation is the same as that of the elastic wave in a 1-dimensional bar\(^9\).

In this paper, a theoretical analysis of spherical wave propagation is carried out in a strain rate dependent elastic-plastic-viscoplastic thick-walled sphere with the characteristic method\(^9\). In there, theoretical equations are shown of the propagation speed of the elastic-plastic-viscoplastic spherical waves, and equations are shown among particle velocity, stress and strain along characteristic curves. And, the calculated examples are shown. In the case, calculated results given by use of the existing strain rate independent potential type, elastic-plastic constitutive equation (plastic strain has been expressed in terms of the plastic potential function)\(^9\) are also compared with those based on the elastic-plastic-viscoplastic constitutive equation. Also, the theoretical formula of propagation speed of the elastic-plastic-viscoplastic spherical wave is analytically proved to be strain rate and stress rate dependent. Moreover, it is shown that the theory of the elastic-plastic-viscoplastic spherical wave propagation contains the theory of the under-stress type, elastic-plastic one (plastic strain has been expressed in terms of the under-stress function), and the theory of the under-stress type elastic-plastic spherical wave propagation is the same as that of the plastic potential type one. In addition, an infinitesimal and an isothermal deformation are treated.

2. BASIC EQUATIONS

The general equations of motion and continuity\(^9\) are in curvilinear coordinates, respectively,

\[
p\dot{v} = \ddot{\tau}^\eta |_{j},
\]

and

\[
\dot{\tau}_\eta = (v_{1} |_j + v_{j} |_1)^2,
\]

where \(p\) is the density, \(\ddot{v}^\eta\) and \(\dot{\tau}_\eta\) are components of covariant stress and contravariant strain tensors, respectively, \(v^1\) and \(v_1\) are components of covariant and contravariant particle velocity vectors, respectively, and \(j\) (vertical bar sub \(j\)) and \(\cdot\) (dot) denote covariant and time differentiations, respectively.

The incompressible elastic-plastic-viscoplastic constitutive equation\(^9\) is in the orthogonal coordinates.
\[ 
\begin{align*}
\varepsilon_y &= \begin{cases} 
\frac{1}{2\mu} + \Psi(H) s_y + \Phi(F) s_y + \frac{1}{9K} \delta_{yy} \sigma_{kk} & \text{when } F > 0 \\
\frac{1}{2\mu} s_y + \frac{1}{9K} \delta_{yy} \sigma_{kk} & \text{when } F \leq 0 
\end{cases} 
\quad (3a,b) \\
\text{or} \\
\varepsilon_y &= \begin{cases} 
\frac{1}{2\mu} + \Psi(H) s_y + \Phi(F) s_y - \frac{3}{2} \frac{\nu}{1+v} \delta_{yy} \sigma_{kk} & \text{when } F > 0 \\
\frac{1}{2\mu} s_y - \frac{\nu}{1+v} \delta_{yy} \sigma_{kk} & \text{when } F \leq 0 
\end{cases} 
\quad (4a,b)
\end{align*} 
\]

where \(\sigma_y, s_y\), and \(\varepsilon_y\) are components of stress, stress deviator and strain tensors, respectively. \(\delta_{yy}\) and \(v\) are Kronecker’s delta and Poisson’s ratio, respectively. \(\mu\) and \(K\) are the elastic modulus of shear and volume, respectively. \(\Psi(H)\) and \(\Phi(F)\) are the generalized under-stress function (or generalized plastic function) and the generalized over-stress function (or generalized viscoplastic function), respectively. And following equations hold:

\[ 
\Psi(H) = H^3 / \zeta, \quad \Phi(F) = F^3 / \eta, 
\]

\[ 
H = \sqrt{J_2} / \sqrt{J_3} - 1, \quad F = R_S \sqrt{J_2} / |r_S| - 1 = R_S \sqrt{J_2} / |\sigma_s| - 1, 
\]

\[ 
J_2 = \left( \frac{5}{3} J_1^2 \right) J_3, \quad J_3 = \frac{s_y \mu}{2}, \quad J_3 = \frac{s_y \mu k}{3}, 
\]

\[ 
R_S = \sqrt{5} |r_S| / |\sigma_s|, \quad r_S = r_S(r_S), \quad \sigma_S = \sigma_S(e_S), 
\]

\[ 
Q_1 := \{ \text{Components of elastic strain which become zero are eliminated in a concrete problem} \}, \]

\[ 
Q_2 := \{ \text{Only elastic strain components related to the direction of motion in the problem are kept using Cauchy law} \}, 
\]

\[ 
Q_3 := \{ \text{Components of elastic strain are replaced by those of total strain} \}, 
\]

\[ 
J_2 = e^{\kappa} e^{\kappa} / 2, \quad e^{\kappa} = e^{\kappa} - \delta_{yy} \varepsilon_{kk} / 3. 
\]

\(\zeta\) and \(k_3\) in Eq.(5) are the plastic coefficient and constant, respectively. \(\eta\) and \(k_5\) in Eq.(6) are the viscoplastic coefficient and constant, respectively. Eq.(7) and (8a,b), F are the generalized non-dimensional under-stress and over-stress, respectively. \(H\) is the relative ratio of the upper limit stress and the dynamic stress, and \(F\) is the relative ratio of the dynamic stress and the static stress. Eq.(9), \(J_3\) is the third invariant of non-dimensional stress tensor deviator, and Eq.(10), \(J_2\) and Eq.(11), \(J_1\) are the second and the third invariant of stress tensors, respectively. Eq.(12), \(R_S\) is the ratio of the stresses between static simple shear and tension. Since the over-stress is adopted in the elastic-plastic-viscoplastic constitutive equation, Eq.(14a) and (14b) must be used instead of Eqs.(13a) and (13b) for \(r_S\) and \(e_S\) in Eq.(12), respectively. Eqs.(13a) and (13b) indicate a static simple shear stress-strain curve and tensile one, respectively. Eqs.(14a) and (14b) can be obtained by replacing \(r_S\) in Eq.(13a) and \(e_S\) in Eq.(13b) by \(3^2 L\) and \(L\), respectively. L in Eq.(15) is current strain (equivalent strain), \(e_S\) in the right side of Eq.(15) is the component of a strain tensor deviator. Eq.(16), \(J_3^2\) is the generalized upper limit stress. \(J_3^2\) is also the secondary invariant of the strain tensor deviator \(J_3 = e^{\kappa} e^{\kappa} / 2. \quad s^{\kappa} = \text{the component of the upper limit stress tensor deviator} \quad s^{\kappa} = s^{\kappa} e^{\kappa} / 2. \quad \delta^{\kappa} = \text{the component of the upper limit stress tensor deviator} \quad \delta^{\kappa} = \delta^{\kappa} e^{\kappa} / 2.
\]

\[ 
(19a), (20) 
\]

\[ 
(19b) 
\]

\[ 
(19c) 
\]

\[ 
(20a) 
\]

\[ 
(20b) 
\]

Figure 1 shows the generalized stress-strain plane with a parameter of the generalized arc strain rate. Vertical and horizontal coordinates are the generalized stress \((J_3)^{12}\) and the generalized strain \((S_2)^{12}\), respectively. \(O_1\) is the generalized dynamic stress-strain curve, \(J_2^{12}, J_3^{12}\), curve in the case when a strain rate is a generalized strain rate \(L\). \(O_2\) is the generalized static stress-strain curve, \(J_2^{12}, J_3^{12}\), curve in the case when a strain rate is a generalized static stress rate \(I_0\). This static curve coincides with a static simple tensile curve, \(\sigma_{S0}, \varepsilon_{S0}\), curve with a static simple tensile strain rate \(e_0\) when a static simple tensile curve is adopted as the reference curve (or, reference stress) \(\sigma_0\), and with a static simple shear curve, \(\gamma_{S0} \gamma_{B0}\), curve with a static simple shear strain rate \(\gamma_{B0}\).
when a static simple shear curve is adopted as one. OL₁ is the extension line of the generalized initial elastic line and the gradient is 2µ. Point Y is the generalized static initial yield point and point D is the current one on the OL₁. Points Q, S and E indicate points on the horizontal coordinate axis, OL₂ and OL₃ corresponding to current strain (l²)¹/² of the point D, respectively. Subscripts S and Y of any variable indicate points with the same symbol. Minus signs are put in front of the symbols (µ)⁰¹ and (µ)¹² at negative sides on their coordinate axes.

Arc strain rate (or equivalent strain rate), equivalent stress and equivalent stress rate are, respectively

\[
\dot{\varepsilon} = \sqrt{3/2} J_2 = \sqrt{3/2} \sum_{ij} \varepsilon_{ij} \varepsilon_{ij}, \quad \sigma = \sqrt{3/2} J_2, \quad \dot{\sigma} = \sqrt{3/2} \sum_{ij} \sigma_{ij} \varepsilon_{ij}, \quad \rho / \sqrt{s_{kk}}, \quad \text{(21a,b), (22), (23a,b)}
\]

where \( J_2 \) is the second invariant of strain tensor deviator. Using equivalent stress, arc strain and arc strain rate enable us to discuss multi-axes stresses state in a single plane which is an equivalent stress-arc strain plane with a parameter of a arc strain rate.

Fig. 1 A schematic diagram of the generalized stress-strain relation for the elastic-plastic- viscoplastic constitutive equation. The \( \dot{J}_2 - \dot{\varepsilon}_2 - \dot{\varepsilon}_3 \) relation.

3. ANALYSIS OF A THICK-WALLED SPHERE

3.1 Basic equations

State of deformation exhibits a point symmetry about the center, since the case is considered when a point symmetrical dynamic force is loaded on the inner face of the spherical cavity with a central symmetry in a thick-walled sphere which has the inner radius \( r \) and the outer radius \( R \). Lagrange coordinates is used. And spherical coordinates (\( r, \varphi, \theta \)) is adopted. Any physical quantity in the direction of \( \varphi \) is the same as that of \( \theta \). Any physical quantity is a function of time \( t \) and radial distance \( r \).

The equation of motion in the radial direction is given, from Eq. (1), by using physical components\(^{(9,10)}\) as follows:

\[
\rho \frac{\partial \varepsilon_r}{\partial t} = \frac{\partial \sigma_r}{\partial r} + \frac{2}{r} (\sigma_r - \sigma_{\varphi \varphi}).
\]

(24)

where \( \varepsilon_r \) is the radial particle velocity, and \( \sigma_r \) and \( \sigma_{\varphi \varphi} = \sigma_{\theta \theta} \) are the radial and circumferential normal stresses, respectively.

The radial and circumferential equations of continuity are, from Eq. (2), by using physical components as follows:

\[
\frac{\partial \sigma_r}{\partial r} = \frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{\varphi \varphi}}{\partial \varphi} = \frac{\varepsilon_r}{r},
\]

(25), (26)

where \( \varepsilon_r \) and \( \varepsilon_{\varphi \varphi} = \varepsilon_{\theta \theta} \) are the radial and circumferential strains, respectively. Since inelastic deformation is assumed to be incompressible, \( \varepsilon_r + \varepsilon_{\varphi \varphi} + \varepsilon_{\theta \theta} = 0 \), then \( \varepsilon_r + \varepsilon_{\varphi \varphi} + 2(\varepsilon_{\varphi \varphi} + \varepsilon_{\theta \theta}) = 0 \) holds, where the superscripts \( i, p \) and \( vp \) denote inelastic, plastic and viscoplastic, respectively.
The radial and circumferential elastic-plastic-viscoplastic constitutive equations are, from Eq.(3a,b) or (4a,b), respectively

\[
\frac{\partial \sigma_{rr}}{\partial t} = \begin{cases} \frac{1}{E} \frac{\partial \sigma_{rr}}{\partial t} - \frac{2\nu}{E} \frac{\partial \sigma_{\theta\theta}}{\partial t} & \text{when } F \leq 0 \\ p - \frac{\partial \sigma_{rr}}{\partial t} - 2s \frac{\partial \sigma_{rr}}{\partial t} + 2f \Phi & \text{when } F > 0 \end{cases}, \quad \frac{\partial \sigma_{\theta\theta}}{\partial t} = \begin{cases} \frac{\nu}{E} \frac{\partial \sigma_{rr}}{\partial t} + \frac{1 - \nu}{E} \frac{\partial \sigma_{\theta\theta}}{\partial t} & \text{when } F \leq 0 \\ -s \frac{\partial \sigma_{rr}}{\partial t} + q \frac{\partial \sigma_{rr}}{\partial t} + f \Phi & \text{when } F > 0 \end{cases}. \tag{27a,b, 28a,b} \]

Equations (27a,b) and (28a,b) express the elastic and elastic-plastic-viscoplastic deformations when \( F \leq 0 \) and \( F > 0 \) hold, respectively. From Eqs.(25) and (27a,b), and Eqs.(26) and (28a,b), the following equations are obtained, respectively:

\[
\frac{\partial \sigma_{rr}}{\partial r} = \begin{cases} \frac{1}{E} \frac{\partial \sigma_{rr}}{\partial t} - \frac{2\nu}{E} \frac{\partial \sigma_{\theta\theta}}{\partial t} & \text{when } F \leq 0 \\ p - \frac{\partial \sigma_{rr}}{\partial t} - 2s \frac{\partial \sigma_{rr}}{\partial t} + 2f \Phi & \text{when } F > 0 \end{cases}, \quad \frac{\partial \sigma_{\theta\theta}}{\partial r} = \begin{cases} \frac{\nu}{E} \frac{\partial \sigma_{rr}}{\partial t} + \frac{1 - \nu}{E} \frac{\partial \sigma_{\theta\theta}}{\partial t} & \text{when } F \leq 0 \\ -s \frac{\partial \sigma_{rr}}{\partial t} + q \frac{\partial \sigma_{rr}}{\partial t} + f \Phi & \text{when } F > 0 \end{cases}. \tag{29a,b, 30a,b} \]

where

\[
\rho = \frac{1}{E + 2\nu / 3}, \quad q = \frac{1 - \nu}{E + \nu / 3}, \quad s = \frac{\nu}{E + \nu / 3}, \quad f = \frac{\sigma_{rr} - \sigma_{\theta\theta}}{3}. \tag{31, 32, 33, 34} \]

\( J_2, J_3 \) and \( \overline{J}_3 \) are obtained, from Eqs.(10), (11) and (9), respectively:

\[
J_2 = \left( \sigma_{rr} - \sigma_{\theta\theta} \right)^2 / 3, \quad J_3 = \left( 2 / 27 \right) \left( \sigma_{rr} - \sigma_{\theta\theta} \right)^3, \quad \overline{J}_3 = \begin{cases} 1 & \text{when } \left( \sigma_{rr} - \sigma_{\theta\theta} \right) \geq 0 \\ -1 & \text{when } \left( \sigma_{rr} - \sigma_{\theta\theta} \right) < 0 \end{cases}. \tag{35, 36, 37a,b} \]

\( H \) and \( F \) are, from Eqs.(7), (10), (16)–(20) and (35), and from Eqs.(8a,b)–(12), (14a), (14b) and (35)–(37a,b), respectively:

\[
H = \frac{2 \mu \left( \sigma_{rr} - \sigma_{\theta\theta} \right)}{\left( \sigma_{rr} - \sigma_{\theta\theta} \right) - 1}, \quad F = \frac{R S J_2 \left( \sigma_{rr} - \sigma_{\theta\theta} \right)}{3 |F_S|} - 1 = \frac{R S J_2 - 1}{|F_S|}, \tag{38, 39a,b} \]

\( \Psi \) and \( \Phi \) are obtained, from Eqs.(5) and (38), and Eqs.(6) and (39a,b), respectively:

\[
\Psi = \frac{1}{\zeta} \left( \frac{2 \mu \left( \sigma_{rr} - \sigma_{\theta\theta} \right)}{\left( \sigma_{rr} - \sigma_{\theta\theta} \right) - 1} \right)^{\frac{1}{2}}, \quad \Phi = \frac{1}{\eta} \left( \frac{R S J_2 \left( \sigma_{rr} - \sigma_{\theta\theta} \right)}{3 |F_S|} - 1 \right)^{\frac{1}{2}}. \tag{40, 41a,b} \]

Equations (14a), (15) and (14b), \( \sigma_S \) are employed as the following forms, respectively:

\[
\sqrt{3} L_x = \frac{\tau_S}{\mu} + a \left( \frac{F_S}{\tau_S} \right)^{\frac{a}{b}}, \quad L_y = \frac{\sigma_S}{E} + a \left( \frac{F_S}{\sigma_S} \right)^{\frac{a}{b}}, \tag{12, 13} \]

where \( a, b, a \) and \( b \) are material constants from static simple tests. \( \tau_S \) and \( \sigma_S \) are the initial yield stresses in the static simple shear and tensile tests, respectively. The stress-strain curve for the material in static simple tension is assumed to be a smooth curve, concave toward the strain axis. Are strain, are strain rate, equivalent stress and equivalent stress rate are, from Eqs.(15), (21b), (22) and (23b), respectively:

\[
L = \frac{2}{3} \left[ \left( \sigma_{rr} - \sigma_{\theta\theta} \right) \right]^{\frac{1}{3}}, \quad L = \frac{2}{3} \left( \frac{\sigma_{rr} - \sigma_{\theta\theta}}{\sigma_{rr} - \sigma_{\theta\theta}} \right) \left[ \sigma_{rr} - \sigma_{\theta\theta} \right] = \frac{\sigma_{rr} - \sigma_{\theta\theta}}{E} \left[ \frac{\sigma_{rr} - \sigma_{\theta\theta}}{E} \right], \tag{44a,b, 45a,b,c,d} \]

\[
\overline{\sigma} = \left( \sigma_{rr} - \sigma_{\theta\theta} \right), \quad \overline{\sigma} = \left( \sigma_{rr} - \sigma_{\theta\theta} \right) \left( \sigma_{rr} - \sigma_{\theta\theta} \right) = \left( \sigma_{rr} - \sigma_{\theta\theta} \right)^2, \tag{46, 47a,b,c} \]

The radial displacement \( w_r \) at any radial distance \( r \) is obtained by using the following equation:

\[
\frac{\partial w_r}{\partial t} = \frac{\partial \sigma_{rr}}{\partial r}. \tag{48} \]

3.2 Analysis by use of the characteristic method

Equations (24), (29b) and (30b) can now be written in the form of the matrix partial differential equation:
\[ L(W) = AW_i + BW_i - \frac{1}{r} DW = 0, \]  

(49)

where

\[
A = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho & -2s \\ -s & q & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 2 & -2 \\ 0 & -2g & 2g \\ 1 & g & -g \end{bmatrix}, \quad W = \begin{bmatrix} \frac{\sigma_{rr}}{\sigma_{pp}} \\ \frac{\sigma_{rr}}{\sigma_{pp}} \end{bmatrix},
\]

(50a), (50b), (50c), (50d)

and \( g \) is

\[ g = (\Phi/3)r. \]

(51)

The system of equations, Eqs.(49), is a hyperbolic, partial differential equations of first order, and Eqs.(49) can be solved by the characteristic method. Equation (49) expresses dynamic elastic behavior when \( \Psi \) and \( \Phi \) are replaced by \( \Psi = 0 \) and \( \Phi = 0 \), respectively in case of \( F \leq 0 \).

The propagation velocity (i.e. the characteristic velocity) \( \lambda_{03} \) for Eqs.(49) are the roots of the characteristic equation \( |A_0 \mathbf{A} - \mathbf{B}| = 0 \):

\[ \lambda_{03} \left( \frac{\sigma_{rr}}{\sigma_{pp}} \right)^2 (\rho + s) = 0. \]

(52)

Consequently, from Eq.(52), we obtain \( \lambda_{03} = c, \) \( \lambda_{03} = c \) and \( \lambda_{03} = 0 \), where \( c \) is

\[ c = \sqrt{\frac{1 - v}{E} + \frac{\Psi}{E} + \frac{1}{2}}, \]

(53)

Equation (53), \( c \) is the propagation speed of the elastic-plastic-viscoplastic spherical wave. \( c \) includes the plastic function \( \Psi(\mathbf{H}) \) but doesn’t the viscoplastic function \( \Phi(\mathbf{F}) \) explicitly. Since \( \Psi(\mathbf{H}) = 0 \) and \( \Phi(\mathbf{F}) = 0 \) are realized, i.e. deformation is in elastic when \( F \geq 0 \) holds, Eq.(53) turns into the speed of the spherical elastic wave (i.e. plane elastic wave), \( c_1 \):

\[ c_1 = \sqrt{\frac{E(1 - v)}{\rho(1 - 2v)(1 + v)}} = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c \leq c_1, \]

(54a), (54b), (55)

where \( \lambda \) is Lamé’s constant.

Next, ordinary differential equations are derived among the physical quantities along the characteristic curves. There exists the eigenvector \( \mathbf{I}_{03} \) corresponding to the eigenvalue \( \lambda_{03} \), consequently we obtain the following equations from \( \mathbf{I}_{03} \mathbf{A} = \mathbf{B} \mathbf{I}_{03} = 0 \):

\[ I_{(1)} = \begin{bmatrix} 1 \\ -\rho c \\ -\rho cz \end{bmatrix}, \quad I_{(2)} = \begin{bmatrix} 1 \\ \rho c \\ \rho cz \end{bmatrix}, \quad I_{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \]

(56a), (56b), (56c)

where \( z = 2s / q \).

Accordingly, we obtain ordinary differential equations along the characteristic curve \( d\mathbf{r}/dt = \lambda_{03} \mathbf{I} \) from the linear combination \( \mathbf{I}^T \mathbf{L} = 0 \):

\[ \frac{d\sigma_{rr}}{dt} + \rho c \frac{d\nu_{rr}}{dt} = \frac{\lambda}{r} \left( 6f + \rho c v_z \right), \quad \frac{d\nu_{rr}}{dt} = 0 \]  

(58a), (58b)

\[ \frac{2\sigma_{rr} - \sigma_{pp}}{dt} + \frac{\nu_{rr}}{r} \frac{\sigma_{pp}}{dt} = 0 \]  

(58c)

The physical quantities produced in a thick-walled sphere can be obtained by making grid points passing through characteristic curves in a radial distance-time plane and by integrating Eqs.(58a,b,c) numerically along characteristic curves.

### 3.3 Alternative forms of \( c \)

Rearranging Eq.(53), we obtain alternative forms of the propagation speed of the elastic-plastic-viscoplastic spherical wave, \( c \):

\[ c = c_3 \left[ 1 + \frac{2 \cdot \frac{\sigma_{rr} - \sigma_{pp}}{dt}}{3K \cdot \frac{\sigma_{rr} - \sigma_{pp}}{dt} - \left( \frac{\sigma_{rr} - \sigma_{pp}}{dt} \right)^2} \right], \quad c = c_3 \left[ 1 + \frac{\pm 4\sigma}{3K \cdot 3L \mp 2\Phi \sigma} \right], \]

(59a), (59b)

where \( c_3 \) is the propagation speed of spherical bulk elastic wave (i.e. plane bulk elastic wave), and \( K \) is the bulk modulus of elasticity.
From Eqs. (53), (59a) and (59b), the propagation speed of the spherical elastic-plastic-viscoplastic wave, \( c \), is proved to be stress rate and strain rate dependent.

### 3.4 Case of an elastic-plastic body

There are two kinds of the theoretical propagation speed of the elastic-plastic spherical wave \( c_{ep} \) in form. The subscript \( e-p \) denotes the case of the elastic-plastic body. They are the under-stress type, elastic-plastic spherical wave \( c_{ep}\) and the plastic potential type, one \( c_{ep\psi} \). The subscripts \( \psi \) and \( U \) are the under-stress function and the gradient of equivalent stress with respect to equivalent plastic strain \( U=d\sigma/d\varepsilon^p \), respectively. In addition, \( c_{ep\psi} \) is equal to \( c_{ep\psi} \) and \( c_{p} \) is smaller than \( c \):

\[
    c \geq c_{e-p} = c_{e-p}\psi = c_{e-p}\psi = c_{e-p}U.
\]

#### 3.4.1 Case of the under-stress type elastic-plastic body

By setting viscoplastic coefficient \( \eta = \infty \) in Eq. (5) or (40a,b), therefore the over-stress is \( \Phi = 0 \) in Eq. (53), \( c \) becomes the propagation speed \( c_{ep} \) of the under-stress type, elastic-plastic spherical wave. Theoretical formulas of \( c \) and \( c_{ep} \) have the same form, Eq. (53), but \( \Phi = 0 \) holds in the former and \( \Phi = 0 \) in the latter:

\[
    c_{e-p} = c_{e-p}\psi = (e)_{p=0} = \left( \begin{array}{c}
    \frac{1-\nu}{E}\frac{\psi}{3} \\
    \frac{1-2\nu}{E}\frac{1+\nu}{E} + \psi
    \end{array} \right) = \sqrt[\psi]{} \frac{2}{3K} \left( \frac{\sigma_{pp} - \sigma_{pp}}{\sigma_{pp}} \right) = c_{3} = \frac{4}{9K} c_{3} = \frac{4}{9K} d\sigma/d\varepsilon^p.
\]

#### 3.4.2 Case of the plastic potential type elastic-plastic body

The propagation theory of the plastic potential type, elastic-plastic spherical wave has been described in the paper, but the theory can be derived by another method. It can be obtained by replacing \( \psi \) by \( 3(2U) \) in the propagation theory of the plastic potential type, elastic-plastic spherical wave (note \( \Phi = 0 \) in this case). For example, the propagation speed of the plastic potential type, elastic-plastic spherical wave \( c_{ep\psi} \) is

\[
    c_{e-p} = c_{e-p}\psi = \left( \begin{array}{c}
    \frac{1-\nu}{E}\frac{1}{2U} \\
    \frac{1-2\nu}{E}\frac{1+\nu}{2U} + \psi
    \end{array} \right) = \sqrt[\psi]{} \frac{2}{3K} \left( \frac{\sigma_{pp} - \sigma_{pp}}{\sigma_{pp}} \right) = c_{3} = \frac{4}{9K} c_{3} = \frac{4}{9K} d\sigma/d\varepsilon^p.
\]

In addition, Eqs. (62a,b) are equal to Eqs. (61a)-(61g).

#### 3.5 Case of the elastic-viscoplastic body

The theoretical propagation speed of the elastic-viscoplastic spherical wave, \( c_{ep} \) is the same formula as that of the plane elastic wave, Eq. (54a,b) since the theoretical propagation speed of the elastic-viscoplastic spherical wave is derived by setting \( \psi = 0 \) and \( \Phi = 0 \) in the propagation theory of the elastic-viscoplastic spherical wave:

\[
    c_{e-p} = c_{e-p}\psi = \left( \begin{array}{c}
    \frac{1-\nu}{E}\frac{1}{2U} \\
    \frac{1-2\nu}{E}\frac{1+\nu}{2U} + \psi
    \end{array} \right) = \sqrt[\psi]{} \frac{2}{3K} \left( \frac{\sigma_{pp} - \sigma_{pp}}{\sigma_{pp}} \right) = c_{3} = \frac{4}{9K} c_{3} = \frac{4}{9K} d\sigma/d\varepsilon^p.
\]

where the subscript \( e-p \) of \( c_{ep} \) denotes the elastic-viscoplastic body. The theoretical propagation speed of the elastic-viscoplastic spherical wave is independent on strain, stress rate and stress rate, and agrees with the propagation speed of the plane elastic wave.

### 4. CALCULATED EXAMPLES

Any physical quantities produced in a thick-walled sphere can be obtained by making grids in a radial distance-time plane and by integrating numerically ordinary differential equations (58a,b,c) along characteristic curves. Dimensions of a thick-walled sphere (Fig. 2) are \( d = 2d, a = 50[mm] \) in the inner diameter and \( d = 2d, a = 250[mm] \) in the outer one. An impulsive loading condition is a particle velocity one with a central symmetry at the inner face of the cavity, \( v = v_t = 125[v] \) in time, and \( c_1 \) is \( c_1 = 3.80 \) [km/s]. The radial particle velocity increases parabolic, after that, it keeps constant particle velocity \( v = 25[m/s] \) after the interval of the rising time \( t = 5[\mu s] \).

Material constants used in calculations are shown in Table 1. Experimental values of \( a, b, c, b_2 \) of material constants of the static simple tensile and shear stress-strain curves required to use the elastic-viscoplastic constitutive equation were employed from Ref.11 for Brass. But experimental values \( c, \eta, k_1 \) and \( k_2 \) of the constants used in the under-stress and the over-stress functions are decided by assuming the strain rate dependency suitably since there are no dynamic experiments for Brass. A dynamic simple tensile stress-strain
relation (i.e. the simple tensile elastic-plastic-viscoplastic constitutive equation) obtained by assuming material constants is, from Eq.(3a) or (4a)

\[ \dot{\varepsilon} = \frac{1}{E} \left[ \frac{2}{\sigma} \left( \dot{\varepsilon} - 1 \right) \right] \left[ \frac{2}{\sigma} \left( \frac{\sigma}{\sigma_p} - 1 \right) \right] \sigma \quad \text{when} \quad |\sigma| > \sigma_p. \]  

(64)

Table 1: Values of material coefficients

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
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<tr>
<td>E [GPa]</td>
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<tr>
<td>ρ [g/cm³]</td>
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<tr>
<td>ν</td>
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</tr>
<tr>
<td>σ_f [MPa]</td>
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<tr>
<td>k</td>
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<tr>
<td>k</td>
<td>2.16</td>
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<tr>
<td>g</td>
<td>0.39 × 10^5 [kg/(mm²)*s]</td>
</tr>
<tr>
<td>h</td>
<td>11.8 [GPa]</td>
</tr>
<tr>
<td>k</td>
<td>0.118 [MPa*sec]</td>
</tr>
<tr>
<td>g</td>
<td>0.85</td>
</tr>
<tr>
<td>k</td>
<td>0.0</td>
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</tbody>
</table>

Fig. 2: Thick-walled sphere. Particle velocity loading condition.

Fig. 3: Dynamic stress-strain curves in tension.

Fig. 4: Distances and times used in calculation in (t, r) plane.

Fig. 5: Radial particle velocity in (r, v_r) plane.

Fig. 6: Radial particle velocity in (r, v_r) plane.

Fig. 7: Radial stress in (r, v_r) plane.

Fig. 8: Radial stress in (r, v_r) plane.

Fig. 9: Circumferential stress in (r, v_r) plane.

Fig. 10: Circumferential stress in (r, v_r) plane.

Calculated results obtained by using Eq.(64) is shown in broken line in Fig.3. An examination of accuracy of the difference calculation is done by calculating the relative errors \( E_r \) between the input energy \( E_i \) and the output one \( E_o \). \( E_r \) is the sum of the kinetic energy \( E_k \) and the deformation energy \( E_d \) in the body. Positions and times in calculation range in a radial distance-time plane are shown in Fig.4. Time taken is \( 2(r_0, r_0)c = 52.7 \mu s \) for a precursor wave to go and return back in the radial direction of the thick-walled sphere. The boundary condition at the outer face of the sphere is a free end face. Calculations at the calculating points are done within the range of time before the arrival of the reflected waves from the outer surface of the thick-walled sphere.

Figures 5–33 correspond to one another. In any figure, quantities obtained by using the elastic-plastic-viscoplastic and the elastic-plastic constitutive equations are shown in solid line (we call E-P-VP theory \(^7\)) and broken line (E-P theory \(^8\)) respectively.
Figures 5 and 6 show the distribution of the radial particle velocity with parameters of t and r, respectively. In Fig. 5, v, (t) is an impact face particle velocity at the inner face of the thick-walled sphere. v, of both theories decrease with increasing radial distance from the impact face. v, of the E-P-VP theory becomes larger than that of the E-P one immediately after the initial yield point, after that, v, of the E-P-VP theory becomes smaller than that of the E-P one, and the difference decreases with increasing time.

Figures 7, 8 and 9–11 show the distribution of the radial and circumferential stresses. From Fig. 7, σr by both theories decrease extremely with increasing radial distance from the impact face. And σr by both theories decrease immediately after the elapse of the rising time, after that, they increase. σr by the E-P-VP theory almost becomes larger than that by the E-P one.

From Fig. 9, σθ by both theories decrease extremely with increasing radial distance from the impact face. σθ by both theories decrease immediately after the elapse of the rising time, after that, they increase. σθ by the E-P-VP theory becomes smaller than that by the E-P one until the time for a while after the elapse of the rising time, after that, σθ by the E-P-VP theory becomes larger than that by the E-P one (r=r1–r8). σθ is in tensile state in the vicinity of the center of the thickness (r=r8–r9) under the condition of calculation. Figure 10C shows the magnified figure from Fig. 9.

Figures 12 and 13 show the distribution of equivalent stress with parameters t and r, respectively. From Fig. 12,  by the E-P-VP theory becomes larger than that by the E-P one.  by both theories decrease remarkably with increasing radial distance from the impact face. Although the distributions of σr (Figs. 7 and 8) and σθ (Figs. 9–11) are somewhat complicated, the change in  (Figs. 12 and 13) combined with them is extremely simple and is easy to be understood.

Figures 14 and 15 show relations between the radial and circumferential stresses. A part of diagrams in Fig. 14 is shown in Fig. 15 to be understood easily. From Fig. 15, a typical relation between both stresses is on the whole such that both σr corresponding to σθ increase at the beginning, after that, decrease rapidly, moreover, after that, increase again. From Fig. 14, relations between σr and σθ by both theories decrease extremely with increasing radial distance from the impact face. σr corresponding to a certain σθ by the E-P-VP theory becomes larger than that by the E-P one (r=r1–r8). The deformation is nearly in elastic range away from the vicinity of the center of the thickness (r=r8–r9) under the conditions of calculation, and the circumferential stress becomes in tension at r=r8, r9.
Figures 16 and 17, and Figs.18 and 19 show distributions of radial and circumferential strains, respectively. There is almost no difference between both theories about \( \varepsilon_r \) and \( \varepsilon_{\theta\theta} \), respectively. \( \varepsilon_r \) by both theories and \( \varepsilon_{\theta\theta} \) by ones decrease extremely with increasing radial distance from the impact face, respectively.

Figures 20 and 21 show distributions of arc strain, \( L \) by both theories decreases extremely with increasing radial distance from the impact face. \( L \) is what \( \varepsilon_r \) (Figs.16 and 17) and \( \varepsilon_{\theta\theta} \) (Figs.18 and 19) have been combined. There is no difference between both theories about \( L \). Maximum values of both logarithmic arc strain rates are \( (\log L)_{\text{max}} \approx 3.5 \).

Figures 22 and 23 show relations between radial and circumferential strains. Figure 23 shows magnified diagram from Fig.22. \( \varepsilon_r \) corresponding to a certain \( \varepsilon_{\theta\theta} \) by both theories decrease slightly with increasing radial distance from the impact face, but the degree is small. Both theories equal on the whole, however, in detail, \( \varepsilon_r \) corresponding to a certain \( \varepsilon_{\theta\theta} \) by the E-P-VP theory becomes smaller than that by the E-P one at the beginning, after that, vice versa (Fig.23). \( \varepsilon_r=\varepsilon_{\theta\theta} \) relations change remarkably (Figs.14 and 13), but \( \varepsilon_r=\varepsilon_{\theta\theta} \) relations are simple on the whole.

Figures 24 and 25 show distributions of radial displacement, \( w_r \), by both theories decreases extremely with increasing radial distance from the impact face. \( w_r \) by both theories are almost equal.
Figures 26–28 show radial stress-strain relation, circumferential one and equivalent stress-arc strain relation, respectively. In Fig.26, \( \sigma_r \) corresponding to a certain \( \ell_0 \) by both theories becomes small with increasing radial distance. \( \sigma_r \) by the E-P-VP theory becomes larger than that by the E-P one. The difference in \( \sigma_r \) by the E-P-VP theory among radial positions becomes small with increasing \( \ell_0 \), and the curves have a tendency to lie in a curve. The same phenomenon exists in E-P theory.

In Fig.27, \( \sigma_{\phi\phi} \) corresponding to a certain \( \ell_0 \) by both theories becomes small with increasing radial distance. \( \sigma_{\phi\phi} \) by the E-P-VP theory becomes smaller than that by the E-P one until immediately after the rising time (Fig.5), after that, vice versa. The curve by E-P-VP theory at any distance lies in a curve with increasing strains. The same phenomenon exists in E-P theory.

In Fig.28, \( \sigma \) corresponding to a certain \( \ell_0 \) at any radial distance by the E-P theory is on the static simple tensile curve regardless of increasing radial distance. This means that the equivalent stress by the E-P theory is strain rate independent. \( \sigma \) corresponding to \( \ell \) certain \( \ell_0 \) at any radial distance by the E-P-VP theory becomes small with increasing radial distance from the impact face, and approaches one by the E-P theory. This means that the equivalent stress by the E-P-VP theory is strain rate dependent. \( \sigma \) corresponding to a certain \( \ell_0 \) by the E-P-VP theory becomes larger than that by the E-P one. The relations of components, \( \sigma_r-\ell_0 \) curve (Fig.26) and \( \sigma_{\phi\phi}-\ell_0 \) curve (Fig.27) are a little complicated, but \( \sigma \) vs. \( \ell_0 \) curve (Fig.28) in which \( \sigma \) and \( \ell_0 \) are derived by using \( \sigma_r \) and \( \sigma_{\phi\phi} \), and \( \ell_0 \) and \( \ell_{\phi\phi} \), respectively is extremely simple and is easy to be understood.

Figures 29 and 30 show distributions of the under-stress function (i.e. the plastic function) and the over-stress function (i.e. the viscoplastic function) in the propagation theory of the elastic-plastic-viscoplastic spherical wave, respectively. From Fig.29, \( \Psi \) increases monotonously with increasing time at any radial distance. \( \Psi \) decreases remarkably with increasing radial distance.

From Fig.30, \( \Phi \) increases until the rising time, and after that, decreases at any radial distance. \( \Phi \) decreases extremely with increasing radial distance.

From Figs.29 and 30, \( \Psi \) and \( \Phi \) are nearly zero in the middle of the wall thickness (\( r=r_0 \)) under the condition of calculation, and these correspond to the elastic deformations.

Figures 31 and 32 show propagation speeds of spherical waves corresponding to arc strain and equivalent stress, respectively. \( c \) corresponding to a certain \( \ell_0 \) or \( \sigma \) by the E-P-VP theory becomes larger than \( c_{\phi\phi} \) corresponding to those by the E-P one.

In Fig.31, \( c_{\phi\phi} \) corresponding to a certain \( \ell_0 \) at any radial distance by the E-P theory is on a certain curve regardless of increasing radial distance. This means that the propagation speed by the E-P theory is strain rate independent. \( c \) corresponding to a certain \( \ell_0 \) at any radial distance by the E-P-VP theory becomes small with increasing radial distance from the impact face, and approaches \( c_{\phi\phi} \) by the E-P theory. This means that the propagation speed by the E-P-VP theory is strain rate dependent. Under the condition of calculation, the propagation
5. CONCLUSIONS

In this paper, theoretical analyses and numerical calculations were carried out on the propagation of inelastic spherical wave in a thick-walled sphere. As a result, the elastic-plastic-viscoplastic constitutive equation of a spherical body was derived from the general incompressible elastic-plastic-viscoplastic one. A theoretical equation of the propagation speed of the elastic-plastic-viscoplastic spherical wave was derived by using the equation of motion, the equation of continuity and the elastic-plastic-viscoplastic constitutive equation, and the ordinal differential equations were derived among particle velocity, stress and strain along characteristic curves. The derived theoretical equation of the propagation speed of the elastic-plastic-viscoplastic spherical wave was analytically proved to be strain rate and stress rate dependent. The equation of the propagation speed of the under-stress type, elastic-plastic spherical wave was derived from the theoretical equation of the propagation speed of the elastic-plastic-viscoplastic spherical wave, and it was shown that this equation is the same as that of the propagation speed of the plastic potential type, elastic-plastic spherical wave. Consequently, it was shown that the propagation theory of the elastic-plastic-viscoplastic spherical wave contains that of the elastic-plastic one.

Numerical calculations on the propagation of the elastic-plastic-viscoplastic spherical wave were carried out by the difference method when radial particle velocity with central symmetry was loaded at the inner face of the cavity of a thick-walled sphere. As a result, physical quantities such as radial particle velocity, radial and circumferential stresses, equivalent stress, radial and circumferential strains, are strain and radial displacement were extremely decreased with increasing radial distance from the impact face. Equivalent stress based on the propagation theory of the elastic-plastic-viscoplastic spherical wave decreases and approaches that of the elastic-plastic one with increasing radial distance from the impact face in the equivalent stress-arc strain plane (i.e. strain rate dependency of stress). The propagation speed of the elastic-plastic-viscoplastic spherical wave decreases and approaches that of the elastic-plastic one with increasing radial distance from the impact face in the propagation speed-arc strain plane (i.e. strain rate dependency of propagation speed).

REFERENCES


