The Parameterization of All Robust Stabilizing Simple Multi-Period Repetitive Controllers

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Multi-period repetitive (MPR) controllers were proposed by Gotou et al., in order to improve the disturbance attenuation characteristic of modified repetitive control systems. Using MPR controllers, transfer functions from the periodic reference input to the output and from the disturbance to the output of the control system generally have infinite numbers of poles. To specify the input-output characteristic and the disturbance attenuation characteristic easily, Yamada et al. proposed MPR control systems, named simple MPR control systems, where these transfer functions have finite numbers of poles. However, the parameterization of all robust stabilizing simple MPR controllers for the plant with uncertainties has not been considered yet. In this paper, we propose the parameterization of all robust stabilizing simple MPR controllers.

1 Introduction

In this paper, we investigate the parameterization of all robust stabilizing simple multi-period repetitive (simple MPR) controllers for single-input/single-output plants. Multi-period repetitive (MPR) controllers improve the disturbance attenuation characteristic of the modified repetitive control system that follows the periodic reference input with small steady state error $^1,^2$. Yamada et al. pointed out that the disturbance attenuation characteristic for frequency component, which is same to that of the periodic reference input, of MPR control systems are not very enough and proposed a design method for MPR controllers to attenuate wide-frequency disturbance which is the same as the frequency of periodic reference input based on the idea of changing time-delay $^3$. In this way, several design methods of MPR controllers to improve the disturbance attenuation characteristic of conventional repetitive control systems have been considered.

On the other hand, there exists important control problem to find all stabilizing controllers named the parameterization problem. The parameterization of all stabilizing MPR controllers was studied by Yamada et al.$^4$. Using the parameterization of all stabilizing MPR controllers in $^3$, we can design the MPR controller, which guarantee the stability of the MPR control system. When the MPR control design method in $^3$ is applied to real systems, the influence of uncertainties in the plant must be considered. Because, in some cases, uncertainties in the plant make the MPR control system unstable. However, MPR controllers in $^3$ cannot guarantee the stability of control system for plants with uncertainties. Satoh et al. clarified the parameterization of all robust stabilizing MPR controllers for plants with uncertainties $^4$. However, using the method in $^4$, transfer functions from the periodic reference input to the output and from the disturbance to the output have infinite numbers of poles, even if the uncertainty does not exist. When transfer functions from the periodic reference input to the output and from the disturbance to the output have infinite numbers of poles, it is difficult to specify the input-output characteristic and the disturbance attenuation characteristic. From the practical point of view, it is desirable that the input-output characteristic and the disturbance attenuation characteristic are easily specified. In order to specify the input-output characteristic and the disturbance attenuation characteristic easily, transfer functions from the periodic reference input to the output and from the disturbance to the output are desirable to have finite numbers of poles.
In this paper, we propose the concept of robust stabilizing simple MPR controllers for the plant with uncertainty and clarify the parameterization of all robust stabilizing simple MPR controllers such that the controller works as a robust stabilizing MPR controller and transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles, when the uncertainty does not exist. This paper is organized as follows. In Section 2, the concept of the robust stabilizing simple MPR controller is presented. In addition, in Section 2, the problem considered in this paper is described. In Section 3, the parameterization of all robust stabilizing simple MPR controllers is clarified. In Section 4, control characteristics of a robust stabilizing simple MPR control system are described. In Section 5, we present a design procedure of robust stabilizing simple MPR control system. In Section 6, we show a numerical example to illustrate the effectiveness of the proposed method. Section 7 gives concluding remarks.

**Notations**

- $\mathbb{R}$ the set of real numbers.
- $\mathbb{R}_+$ $\mathbb{R} \cup \{\infty\}$
- $\mathbb{R}(s)$ the set of real rational functions with $s$.
- $\mathbb{RH}_\infty$ the set of stable proper real rational functions.
- $\mathbb{H}_\infty$ the set of stable causal functions.
- $D^\perp$ orthogonal complement of $D$, i.e., $\begin{bmatrix} D & D^\perp \end{bmatrix}$ or $\begin{bmatrix} D \\ D^\perp \end{bmatrix}$ is unitary.
- $A^T$ transpose of $A$.
- $A^+$ pseudo inverse of $A$.
- $\rho(\cdot)$ spectral radius of $\cdot$.
- $\|\cdot\|_\infty$ $\mathbb{H}_\infty$ norm of $\cdot$.
- $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ represents the state space description $C(sI - A)^{-1}B + D$.

## 2 Problem formulation

Consider the unity feedback control system shown in

$$
\begin{align*}
    y &= G(s)u + d \\
    u &= C(s)(r - y)
\end{align*}
$$

where $G(s) \in \mathbb{R}(s)$ is the single-input/single-output plant, $C(s)$ is the controller, $u \in \mathbb{R}$ is the control input, $d \in \mathbb{R}$ is the disturbance, $y \in \mathbb{R}$ is the output and $r \in \mathbb{R}$ is the periodic reference input with period $T > 0$ satisfying

$$
r(t + T) = r(t) \quad (\forall t \geq 0).
$$

The nominal plant of $G(s)$ is denoted by $G_m(s) \in \mathbb{R}(s)$. Both $G(s)$ and $G_m(s)$ are assumed to have no pole on the imaginary axis. In addition, it is assumed that the number of poles of $G(s)$ in the closed right half plane is equal to that of $G_m(s)$. The relation between the plant $G(s)$ and the nominal plant $G_m(s)$ is written as

$$
G(s) = G_m(s)(1 + \Delta(s)),
$$

where $\Delta(s)$ is an uncertainty. The set of $\Delta(s)$ is all functions satisfying

$$
|\Delta(j\omega)| < |W_T(j\omega)| \quad (\forall \omega \in \mathbb{R}_+),
$$

where $W_T(s)$ is a stable rational function.

Under these assumptions, the robust stability condition for the plant $G(s)$ with uncertainty $\Delta(s)$ satisfying (4) is given by

$$
\|T_s(s)W_T(s)\|_\infty < 1,
$$

where $T_s(s)$ is the complementary sensitivity function given by

$$
T_s(s) = \frac{G_m(s)C(s)}{1 + G_m(s)C(s)}.
$$

According to $^1,^3$, the MPR controller $C(s)$ in (1) is written by the form in

$$
C(s) = C_0(s) + \left( \sum_{i=1}^{N} C_i(s)e^{-\tau_i} \right) C_r(s),
$$
where \( C_0(s) \in R(s), C_i(s) \in R(s) (i = 1, \ldots, N) \) and \( N \) is an arbitrary positive integer. \( C_i(s) \) is an internal model for the periodic reference input with period \( T \) written by

\[
C_r(s) = \frac{1}{1 - \sum_{i=1}^{N} q_i(s)e^{-sT_i}},
\]

(8)

where \( q_i(s) \in R(s) (i = 1, \ldots, N) \) are low-pass filters satisfying \( \sum_{i=1}^{N} q_i(0) = 1 \) and \( T_i \in R > 0 \ (i = 1, \ldots, N) \). Without loss of generality, it is assumed to hold \( C_i(s) \neq 0 \ (\forall i = 1, \ldots, N) \). Gotou et al. 1 proposed the design method for MPR controller as

\[
T_i = T \cdot i \quad (i = 1, \ldots, N).
\]

(9)

On the other hand, Yamada et al. 3 proposed the design method for MPR controller such that \( T_i (i = 1, \ldots, N) \) do not necessarily satisfy (9). Therefore, in this paper, we attach importance to the generality and assume that \( T_i (i = 1, \ldots, N) \) do not necessarily satisfy (9). If low-pass filters \( q_i(s) (i = 1, \ldots, N) \) and \( T_i (i = 1, \ldots, N) \) satisfy

\[
1 - \sum_{i=1}^{N} q_i(j\omega_k)e^{-j\omega_k T_i} \approx 0 \quad (\forall k = 0, \ldots, N_{\text{max}}),
\]

(10)

where \( \omega_k(k = 0, \ldots, N_{\text{max}}) \) are frequency components of the periodic reference input \( r \) written by

\[
\omega_k = \frac{2\pi k}{T} \quad (k = 0, \ldots, N_{\text{max}})
\]

(11)

and \( \omega_{N_{\text{max}}} \) is the maximum frequency component of the periodic reference input \( r \), then the output \( y \) in (1) follows periodic reference input \( r \) with small steady state error.

Using the MPR controller \( C(s) \) in (7), transfer functions from the periodic reference input \( r \) to the output \( y \) and from the disturbance \( d \) to the output \( y \) in (1) are written as

\[
\frac{y}{r} = \frac{C(s)G(s)}{1 + C(s)G(s)}
\]

\[
= \frac{C_0(s)G_m(s) (1 + \Delta(s)) - \sum_{i=1}^{N} (C_0(s)q_i(s) - C_i(s)) e^{-sT_i} G_m(s) (1 + \Delta(s))}{1 + C_0(s)G_m(s) (1 + \Delta(s)) - \sum_{i=1}^{N} [q_i(s) (1 + C_0(s)G_m(s) (1 + \Delta(s))) - C_i(s)G_m(s) (1 + \Delta(s))] e^{-sT_i}},
\]

(12)

and

\[
\frac{y}{d} = \frac{1}{1 + C(s)G(s)}
\]

\[
= \frac{1 - \sum_{i=1}^{N} q_i(s)e^{-sT_i}}{1 + C_0(s)G_m(s) (1 + \Delta(s)) - \sum_{i=1}^{N} [q_i(s) (1 + C_0(s)G_m(s) (1 + \Delta(s))) - C_i(s)G_m(s) (1 + \Delta(s))] e^{-sT_i}},
\]

(13)

respectively. Generally, transfer functions from the periodic reference input \( r \) to the output \( y \) in (12) and from the disturbance \( d \) to the output \( y \) in (13) have infinite numbers of poles, even if \( \Delta(s) = 0 \). When transfer functions from the periodic reference input \( r \) to the output \( y \) and from the disturbance \( d \) to the output \( y \) have infinite numbers of poles, it is difficult to specify the input-output characteristic and the disturbance attenuation characteristic. From the practical point of view, it is desirable that the input-output characteristic and the disturbance attenuation characteristic are easily specified. In order to specify the input-output characteristic and the disturbance attenuation characteristic easily, transfer functions from the periodic reference input \( r \) to the output \( y \) and from the disturbance \( d \) to the output \( y \) are desirable to have finite numbers of poles.

From above practical requirement, we propose the concept of a robust stabilizing simple MPR controller defined as follows:
Definition 1 (robust stabilizing simple MPR controller)
We call the controller $C(s)$ a “robust stabilizing simple MPR controller”, if following expressions hold true:

1. The controller $C(s)$ works as a MPR controller. That is, the controller $C(s)$ is written by (7), where $C_0(s) \in R(s), C_1(s) \neq 0 \in R(s) (i = 1, \ldots, N)$ and $q_i(s) \in R(s) (i = 1, \ldots, N)$ satisfy $\sum_{i=1}^{N} q_i(0) = 1$.

2. When $\Delta(s) = 0$, the controller $C(s)$ makes transfer functions from the periodic reference input $r$ to the output $y$ in (1) and from the disturbance $d$ to the output $y$ in (1) have finite numbers of poles.

3. The controller $C(s)$ satisfies the robust stability condition in (5).

The problem considered in this paper is to clarify the parameterization of all robust stabilizing simple MPR controllers.

3 Parameterization

In this section, we propose the parameterization of all robust stabilizing simple MPR controllers defined in Definition 1.

In order to obtain the parameterization of all robust stabilizing simple MPR controllers, we must see that the robust stabilizing controllers hold (5). The problem of obtaining the controller $C(s)$ satisfying (5), which is not necessarily a simple MPR controller, is equivalent to the following $H_{\infty}$ control problem. In order to obtain the controller $C(s)$ satisfying (5), we consider the control system shown in Fig. 1. Here, $P(s)$ is selected such that the transfer function from $w$ to $z$ in Fig. 1 is equal to $T_4(s)W_4^T(s)$. The state space description of $P(s)$ is, in general,

\[
\begin{bmatrix}
\dot{x}(t) \\
z(t) \\
y(t)
\end{bmatrix} = \begin{bmatrix}
Ax(t) + B_1w(t) + B_2u(t) \\
C_1x(t) + D_{12}u(t) \\
C_2x(t) + D_{21}w(t)
\end{bmatrix},
\]

where $A \in R^{n\times n}, B_1 \in R^n, B_2 \in R^n, C_1 \in R^{1\times n}, C_2 \in R^{1\times n}, D_{12} \in R, D_{21} \in R$. $P(s)$ is called the generalized plant. $P(s)$ is assumed to satisfy following assumptions:

1) $(A, B_3)$ is stabilizable and $(C_2, A)$ is detectable;

2) $D_{12} \neq 0$ and $D_{21} \neq 0$;

3) \[
\begin{bmatrix}
A - j\omega I & B_2 \\
C_1 & D_{12}
\end{bmatrix}
\]
has full column rank for all $\omega \in R_4$ and \[
\begin{bmatrix}
A - j\omega I & B_1 \\
C_2 & D_{21}
\end{bmatrix}
\]
has full row rank for all $\omega \in R_4$.

Under these assumptions, according to (5), the following lemma holds true.

Lemma 1 If controllers $C(s)$ satisfying (5) exist, both

\[
X \left( A - B_2D_{12}^{-1}C_1 \right) + \left( A - B_2D_{12}^{-1}C_1 \right)^T X + X \left\{ B_1B_1^T - B_2 \left( D_{12}^{-1}D_{12}^T \right)^{-1} B_2^T \right\} X + \left( D_{12}^{-1}C_1 \right)^T D_{12}^{-1}C_1 = 0
\]

and

\[
Y \left( A - B_1D_{21}^{-1}C_2 \right)^T + \left( A - B_1D_{21}^{-1}C_2 \right) Y + Y \left\{ C_1^T C_1 - C_2^T \left( D_{21}D_{21}^T \right)^{-1} C_2 \right\} Y + B_1D_{21}^{-1} \left( B_1D_{21}^{-1} \right)^T = 0
\]
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have solutions $X \geq 0$ and $Y \geq 0$ such that

$$\rho(XY) < 1$$

(17)

and neither $A - B_2D_{12}^T C_1 + \{B_1B_1^T - B_2(D_{12}^T D_{23})^{-1} B_2^T\} X$ nor $A - B_1D_{12}^T C_2 + Y\{C_1^T C_1 - C_2^T (D_{21} D_{21}^T)^{-1} C_2\}$ has an eigenvalue in the closed right half plane. Using $X$ and $Y$, the parameterization of all controllers satisfying (5) is given by

$$C(s) = C_{11}(s) + C_{12}(s)Q(s)(I - C_{22}(s)Q(s))^{-1} C_{21}(s),$$

(18)

where

$$\begin{bmatrix}
    C_{11}(s) & C_{12}(s) \\
    C_{21}(s) & C_{22}(s)
\end{bmatrix} = \begin{bmatrix}
    A_e & B_{c1} & B_{c2} \\
    C_{c1} & D_{c11} & D_{c12} \\
    C_{c2} & D_{c21} & D_{c22}
\end{bmatrix},$$

(19)

$$A_e = A + B_1B_1^T X - B_2 \left( D_{12}^T C_1 + E_{12}^{-1} B_2^T X \right) - (I - YX)^{-1} \left( B_1D_{21} + YC_2^T E_{21}^{-1} \right) (C_2 + D_2B_1^T X),$$

$$B_{c1} = (I - YX)^{-1} \left( B_1D_{21} + YC_2^T E_{21}^{-1} \right), \quad B_{c2} = (I - YX)^{-1} \left( B_2 + YC_1^T D_{12} \right) E_{12}^{-1/2},$$

$$C_{c1} = -D_{12}^T C_1 - E_{12}^{-1/2} B_2^T X, \quad C_{c2} = -E_{21}^{-1/2} (C_2 + D_2B_1^T X),$$

$$D_{c11} = 0, \quad D_{c12} = E_{12}^{-1/2}, \quad D_{c21} = E_{21}^{-1/2}, \quad D_{c22} = 0, \quad E_{12} = D_{12}^T D_{12}, \quad E_{21} = D_{21} D_{21}^T$$

and $Q(s) \in H_\infty$ is any function satisfying $\|Q(s)\|_\infty < 1$.

Using Lemma 1, the parameterization of all robust stabilizing simple MPR controllers is summarized in Theorem 1.

**Theorem 1** If simple MPR controllers satisfying (5) exist, both (15) and (16) have solutions $X \geq 0$ and $Y \geq 0$ satisfying (17) and both $A - B_2D_{12}^T C_1 + \{B_1B_1^T - B_2(D_{12}^T D_{23})^{-1} B_2^T\} X$ and $A - B_1D_{12}^T C_2 + Y\{C_1^T C_1 - C_2^T (D_{21} D_{21}^T)^{-1} C_2\}$ have no eigenvalue in the closed right half plane. Using $X$ and $Y$, the controller $C(s)$ is a robust stabilizing simple MPR controller satisfying (5) if and only if $C(s)$ is written by

$$C(s) = C_{11}(s) + C_{12}(s)Q(s)(I - C_{22}(s)Q(s))^{-1} C_{21}(s),$$

(20)

where $C_{ij}(s) (i = 1, 2; j = 1, 2)$ are given by (19) and $Q(s) \in H_\infty$ is any function satisfying $\|Q(s)\|_\infty < 1$ and written by

$$Q(s) = \frac{Q_{s0}(s) + \sum_{i=1}^{N} Q_{si}(s)e^{-sT_i}}{Q_{d0}(s) + \sum_{i=1}^{N} Q_{di}(s)e^{-sT_i}},$$

(21)

where

$$Q_{ni}(s) = G_{d}(s)Q_{i}(s) (i = 1, \ldots, N),$$

(22)

$$Q_{ds}(s) = \frac{1}{1 + C_{11}(s)G_{m}(s)}G_{m}(s)Q_{i}(s) (i = 1, \ldots, N),$$

(23)

$Q_{s0}(s) \in RH_\infty$, $Q_{d0}(s) \in RH_\infty$, $G_{n}(s) \in RH_\infty$ and $G_{d}(s) \in RH_\infty$ are coprime factors of $-C_{22}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s))G_{m}(s) on RH_\infty$ satisfying

$$\frac{G_{m}(s)}{G_{d}(s)} = -C_{22}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s))G_{m}(s),$$

(24)
\( \hat{Q}_i(s) \neq 0 \in RH_\infty (i = 1, \ldots, N) \) are any functions and

\[
\sum_{i=1}^{N} (Q_{d0}(0) + Q_{d1}(0)) - (Q_{n0}(0) + Q_{n1}(0)) C_{22}(0) = 0
\]

and

\[
C_{12}(s)C_{21}(s) (Q_{n1}(s)Q_{d0}(s) - Q_{n0}(s)Q_{d1}(s)) \neq 0 \quad (\forall i = 1, \ldots, N).
\]

(Proof) First, the necessity is shown. That is, we show that if the MPR controller written by (7) stabilizes the control system in (1) robustly and makes transfer functions from the periodic reference input \( r \) to the output \( y \) in (12) and from the disturbance \( d \) to the output \( y \) in (13) have finite numbers of poles when \( \Delta(s) = 0 \), then \( C(s) \) and \( Q(s) \) are written by (20) and (21), respectively. From Lemma 1, the parameterization of all robust stabilizing controllers \( C(s) \) for \( G(s) \) is written by (20), where \( \|Q(s)\|_\infty < 1 \). In order to prove the necessity, we will show that if \( C(s) \) written by (7) stabilizes the control system in (1) robustly and makes transfer functions from the periodic reference input \( r \) to the output \( y \) in (12) and from the disturbance \( d \) to the output \( y \) in (13) have finite numbers of poles when \( \Delta(s) = 0 \), then \( Q(s) \) in (20) is written by (21). Substituting \( C(s) \) in (7) into (20), we have (21) where

\[
Q_{n0} = \hat{C}_d(s)C_{12d}(s)C_{21d}(s)C_{22d}(s)\hat{q}_d(s) (\hat{C}_{in}(s)C_{11d}(s) - C_{0d}(s)C_{11n}(s)),
\]

\[
Q_{nd}(s) = (-\tilde{q}_{in}(s)C_{in}(s)\hat{C}_d(s)C_{11d}(s) + \tilde{q}_d(s)C_{0d}(s)\hat{C}_d(s)C_{11d}(s) + \tilde{q}_{in}(s)C_{0d}(s)\hat{C}_d(s)C_{11n}(s)) \hat{C}_{12d}(s)C_{21d}(s)C_{22d}(s) (i = 1, \ldots, N),
\]

\[
Q_{d0}(s) = \hat{C}_d(s)\hat{q}_d(s) (\hat{C}_{in}(s)C_{11d}(s)C_{21d}(s)C_{22d}(s) - C_{0d}(s)C_{11n}(s)C_{22n}(s)C_{12d}(s)C_{21d}(s) + C_{0d}(s)C_{11d}(s)C_{22d}(s)C_{12n}(s)C_{21n}(s))
\]

and

\[
Q_{d1}(s) = C_{11d}(s)C_{22n}(s)C_{12d}(s)C_{21d}(s)\tilde{q}_d(s)C_{0d}(s)C_{11n}(s) - \tilde{q}_{in}(s)C_{0n}(s)\hat{C}_d(s)
\]

\[
+ \tilde{q}_{in}(s)C_{0d}(s)\hat{C}_d(s) (\hat{C}_{11n}(s)C_{22n}(s)C_{12d}(s)C_{21d}(s) - C_{11d}(s)C_{22d}(s)C_{12n}(s)C_{21n}(s))
\]

\( (i = 1, \ldots, N). \)

Here, \( C_{0n}(s) \in RH_\infty, C_{0d}(s) \in RH_\infty, C_{1jn}(s) \in RH_\infty (i = 1, 2; j = 1, 2) \) and \( C_{ijd}(s) \in RH_\infty (i = 1, 2; j = 1, 2) \) are coprime factors satisfying

\[
C_0(s) = \frac{C_{0n}(s)}{C_{0d}(s)}
\]

and

\[
C_{ij}(s) = \frac{C_{ijn}(s)}{C_{ijd}(s)} (i = 1, 2; j = 1, 2).
\]

\( \tilde{q}_{in}(s) \in RH_\infty (i = 1, \ldots, N), \tilde{q}_d(s) \in RH_\infty, \tilde{C}_{in}(s) \in RH_\infty (i = 1, \ldots, N) \) and \( \tilde{C}_d(s) \in RH_\infty \) are defined by

\[
\tilde{q}_{in}(s) = q_{in}(s) \prod_{i=1}^{j-1} q_{d}(s) \prod_{i=j+1}^{N} q_{ud}(s) (i, j = 1, \ldots, N),
\]

\[
\tilde{q}_d(s) = \prod_{i=1}^{N} q_{d}(s),
\]

\[
\tilde{C}_{in}(s) = C_{in}(s) \prod_{i=1}^{j-1} C_{id}(s) \prod_{i=j+1}^{N} C_{ud}(s) (i, j = 1, \ldots, N)
\]
\[
\hat{C}_d(s) = \prod_{i=1}^{N} C_{id}(s),
\]

(36)

respectively. Here, \(q_m(s) \in RH_{\infty}(i = 1, \ldots, N)\), \(q_{id}(s) \in RH_{\infty}(i = 1, \ldots, N)\), \(C_m(s) \in RH_{\infty}(i = 1, \ldots, N)\) and \(C_{id}(s) \in RH_{\infty}(i = 1, \ldots, N)\) are coprime factors satisfying

\[
q_i(s) = \frac{q_{im}(s)}{q_{id}(s)} (i = 1, \ldots, N),
\]

(37)

and

\[
C_i(s) = \frac{C_{im}(s)}{C_{id}(s)} (i = 1, \ldots, N)
\]

(38)

From (27)~(38), all of \(Q_m(s)\), \(Q_{ni}(s)(i = 1, \ldots, N)\), \(Q_{d0}(s)\) and \(Q_{di}(s)(i = 1, \ldots, N)\) are included in \(RH_{\infty}\). Thus, we have shown that if \(C(s)\) written by (7) stabilizes the control system in (1), \(Q(s)\) in (20) is written by (21). Since \(\sum_{i=1}^{N} q_i(0) = 1\) and from (27), (28), (29) and (23), (25) holds true. In addition, from the assumption of \(\hat{C}_i(s) \neq 0 (i = 1, \ldots, N)\), (26) is satisfied.

The rest to prove the necessity is to show that when \(\Delta(s) = 0\), if \(C(s)\) in (7) make transfer functions from the periodic reference input \(r\) to the output \(y\) and from the disturbance \(d\) to the output \(y\) have finite numbers of poles, then \(Q_{ni}(s)(i = 1, \ldots, N)\) and \(Q_{di}(s)(i = 1, \ldots, N)\) are written by (22) and (23), respectively. From (20) and (21), when \(\Delta(s) = 0\), transfer functions from the periodic reference input \(r\) to the output \(y\) and from the disturbance \(d\) to the output \(y\) are written by

\[
\frac{y}{r} = \frac{G_{dyn}(s)}{G_{d}\hat{y}(s)}
\]

(39)

and

\[
\frac{y}{d} = \frac{G_{dyn}(s)}{G_{d}\hat{y}(s)}
\]

(40)

respectively, where

\[
G_{dyn}(s) = \left\{ \left[ C_{11}(s)Q_{d0}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s)) Q_{n0}(s) \right] \right.
\]

\[
+ \sum_{i=1}^{N} \left[ C_{11}(s)Q_{di}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s)) Q_{ni}(s) \right] e^{-sT_i} \right\} G_m(s),
\]

(41)

\[
G_{d}\hat{y}(s) = \left\{ \left[ (Q_{d0}(s) - C_{22}(s)Q_{n0}(s)) + C_{11}(s)Q_{d0}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s)) Q_{n0}(s) \right] \right. G_m(s) \]

\[
+ \sum_{i=1}^{N} \left[ Q_{di}(s) - C_{22}(s)Q_{ni}(s) + C_{11}(s)Q_{di}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s)) Q_{ni}(s) \right] G_m(s) \right\} e^{-sT_i},
\]

(42)

\[
G_{dyn}(s) = Q_{d0}(s) - C_{22}(s)Q_{n0}(s) + \sum_{i=1}^{N} (Q_{di}(s) - C_{22}(s)Q_{ni}(s)) e^{-sT_i}
\]

(43)

and

\[
G_{d}\hat{y}(s) = \left\{ \left[ (Q_{d0}(s) - C_{22}(s)Q_{n0}(s)) + C_{11}(s)Q_{d0}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s)) Q_{n0}(s) \right] \right. G_m(s) \]

\[
+ \sum_{i=1}^{N} \left[ Q_{di}(s) - C_{22}(s)Q_{ni}(s) + C_{11}(s)Q_{di}(s) + (-C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s)) Q_{ni}(s) \right] G_m(s) \right\} e^{-sT_i}.
\]

(44)
From the assumption that transfer functions from the periodic reference input \( r \) to the output \( y \) in (39) and from the disturbance \( d \) to the output \( y \) in (40) have finite numbers of poles, (42) and (44),

\[
Q_{di}(s) - C_{22}(s)Q_{ni}(s) + \{ C_{11}(s)Q_{di}(s) + ( - C_{11}(s)C_{22}(s) + C_{12}(s)C_{21}(s) ) Q_{ni}(s) \} G_m(s) = 0 \quad (i = 1, \ldots, N)
\]

is satisfied. Using (24), this equation is rewritten by

\[
Q_{di}(s) = - \frac{1}{1 + C_{11}(s)G_m(s) G_d(s)} Q_{ni}(s) \quad (i = 1, \ldots, N).
\]

Since \( Q_{ni}(s) \in RH_\infty(\forall i = 1, \ldots, N) \) and \( Q_{di}(s) \in RH_\infty(\forall i = 1, \ldots, N) \), \( Q_{ni}(s) \in \mathbb{H}_\infty(\forall i = 1, \ldots, N) \) and \( Q_{di}(s) \in \mathbb{H}_\infty(\forall i = 1, \ldots, N) \) are written by (22) and (23), respectively, where \( Q_i(s) \in RH_\infty(\forall i = 1, \ldots, N) \). From the assumption that \( C_i(s) \neq 0 \) (\( \forall i = 1, \ldots, N \)) and from (28) and (30), \( Q_i(s) \neq 0 \) (\( \forall i = 1, \ldots, N \)) hold true. We thus have proved the necessity.

Next, the sufficiency is shown. That is, we show that if \( C(s) \) and \( Q(s) \in H_\infty \) are settled by (20) and (21), respectively, then the controller \( C(s) \) is written by the form in (7) and \( \sum_{i=1}^{N} q_i(0) = 1 \) holds true and transfer functions from the periodic reference input \( r \) to the output \( y \) and from the disturbance \( d \) to the output \( y \) have finite numbers of poles. Substituting (21) into (20), we have (7). From simple manipulation, \( C_0(s) \), \( C_i(s)(i = 1, \ldots, N) \) and \( q_i(s)(i = 1, \ldots, N) \) are denoted by

\[
C_0(s) = \frac{C_{11}(s)Q_{do}(s) + (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) Q_{no}(s)}{Q_{do}(s) - C_{22}(s)Q_{no}(s)}
\]

\[
C_i(s) = \frac{C_{11}(s)Q_{di}(s) + (C_{12}(s)C_{22}(s) + C_{12}(s)C_{21}(s)) Q_{ni}(s)}{Q_{do}(s) - C_{22}(s)Q_{no}(s)} + C_0(s)q_i(s) \quad (i = 1, \ldots, N)
\]

and

\[
q_i(s) = - \frac{Q_{di}(s) - C_{22}(s)Q_{ni}(s)}{Q_{do}(s) - C_{22}(s)Q_{no}(s)} \quad (i = 1, \ldots, N).
\]

We find that if \( C(s) \) and \( Q(s) \) are settled by (20) and (21), respectively, then the controller \( C(s) \) is written by the form in (7). From \( q_i(s) \neq 0 \) (\( \forall i = 1, \ldots, N \)), (26) and (48), \( C_i(s) \neq 0 \) (\( \forall i = 1, \ldots, N \)) hold true. Substituting (25) into (49), we have \( \sum_{i=1}^{N} q_i(0) = 1 \). In addition, from (22) and (23) and easy manipulation, we can confirm that when \( \Delta(s) = 0 \), transfer functions from the periodic reference input \( r \) to the output \( y \) and from the disturbance \( d \) to the output \( y \) have finite numbers of poles.

We have thus proved Theorem 1.

Note 1 Note that it is misunderstood that the controller \( C(s) \) in (20) represents limited case of the controller in (7). However, the controller \( C(s) \) in (20) represents all of stabilizing controller in (7). Using the controller \( C(s) \) in (20), \( C_0(s), C_i(s)(i = 1, \ldots, N) \) and \( q_i(s)(i = 1, \ldots, N) \) are written by

\[
C_0(s) = \frac{C_{11}(s)Q_{do}(s) + (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) Q_{no}(s)}{Q_{do}(s) - C_{22}(s)Q_{no}(s)}
\]

\[
C_i(s) = \frac{C_{12}(s)C_{21}(s)(Q_{mi}(s)Q_{do}(s) - Q_{no}(s)Q_{di}(s))}{(Q_{do}(s) - C_{22}(s)Q_{no}(s))^2} \quad (i = 1, \ldots, N)
\]

and

\[
q_i(s) = - \frac{Q_{di}(s) - C_{22}(s)Q_{ni}(s)}{Q_{do}(s) - C_{22}(s)Q_{no}(s)} \quad (i = 1, \ldots, N).
\]

4 Control characteristics

In this section, we describe control characteristics of control system in (1) using the parameterization of all robust stabilizing simple MPR controllers \( C(s) \) in (20).
First, we mention the input-output characteristic. The transfer function \( S(s) \) from the periodic reference input \( r \) to the error \( e = r - y \) is written by

\[
S(s) = \frac{1}{1 + C(s)G(s)} = \frac{S_n(s)}{S_d(s)},
\]

where

\[
S_n(s) = \left( 1 + \sum_{i=1}^{N} \frac{Q_{di}(s) - C_{22}(s)Q_{ni}(s)}{Q_{di}(s) - C_{22}(s)Q_{ni}(s)} e^{-sT_i} \right) (Q_{d0}(s) - C_{22}(s)Q_{n0}(s))
\]

and

\[
S_d(s) = \left( Q_{d0}(s) - C_{22}(s)Q_{n0}(s) + \{ C_{11}(s)Q_{d0}(s) + (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s))Q_{n0}(s) \} G(s) \right)
+ \sum_{i=1}^{N} \left( Q_{d1}(s) - C_{22}(s)Q_{n1}(s) + \{ C_{11}(s)Q_{d1}(s) + (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s))Q_{n1}(s) \} G(s) \right) e^{-sT_i}.
\]

From (50), for \( \omega_k(k = 0, \ldots, N_{\text{max}}) \) in (11), which are frequency components of the periodic reference input \( r \), if

\[
1 + \sum_{i=1}^{N} \frac{Q_{di}(j\omega_k) - C_{22}(j\omega_k)Q_{ni}(j\omega_k)}{Q_{d0}(j\omega_k) - C_{22}(j\omega_k)Q_{n0}(j\omega_k)} = 0,
\]

then the output \( y \) in (1) follows periodic reference input \( r \) without steady state error.

From Theorem 1, \( Q(s) \) in (21) must satisfy \( \| Q(s) \|_\infty < 1 \), that is,

\[
|Q(j\omega_k)| = \left| \frac{Q_{n0}(j\omega_k) + \sum_{i=1}^{N} Q_{ni}(j\omega_k)e^{-j\omega_k T_i}}{Q_{d0}(j\omega_k) + \sum_{i=1}^{N} Q_{di}(j\omega_k)e^{-j\omega_k T_i}} \right| \leq \frac{N}{N} \left| \frac{Q_{n0}(j\omega_k) + \sum_{i=1}^{N} Q_{ni}(j\omega_k)}{Q_{d0}(j\omega_k) + \sum_{i=1}^{N} Q_{di}(j\omega_k)} \right| < 1 \quad (\forall k = 0, 1, \ldots).
\]

Equation (54) does not hold for any \( \omega_k(k = 0, 1, \ldots) \). The maximum frequency component \( \omega_k \) satisfying (54) depends on \( W_T(s) \).

Next, the maximum frequency \( \omega_k \) that satisfies both (53) and (54), in other words the maximum \( k \) satisfying both (53) and (54), is clarified. For the maximum \( k \) satisfying both (53) and (54), next theorem holds true.

**Theorem 2** The maximum \( k \) satisfying both (53) and (54) is written by

\[
k = \max \left( n_m \left| \frac{1}{C_{22}(j\omega_k)} \right| < 1; \forall i = 0, 1, \ldots, n_m \right).
\]

**(Proof)** First, the necessity is shown. That is, we show that if (53) and (54) are satisfied, then

\[
\left| \frac{1}{C_{22}(j\omega_k)} \right| < 1
\]

holds true. From (53), we have

\[
Q_{d0}(j\omega_k) + \sum_{i=1}^{N} Q_{di}(j\omega_k) = C_{22}(j\omega_k) \left( Q_{n0}(j\omega_k) + \sum_{i=1}^{N} Q_{ni}(j\omega_k) \right).
\]

Substituting (57) into (54), we have

\[
|Q(j\omega_k)| = \left| \frac{Q_{n0}(j\omega_k) + \sum_{i=1}^{N} Q_{di}(j\omega_k)}{Q_{d0}(j\omega_k) + \sum_{i=1}^{N} Q_{di}(j\omega_k)} \right| = \left| \frac{1}{C_{22}(j\omega_k)} \right| < 1 \quad (\forall k = 0, 1, \ldots).
\]
We have thus proved the necessity.

Next, the sufficiency is shown. That is, we show that if

$$\left| \frac{1}{C_{22}(j\omega_k)} \right| < 1$$  \hspace{1cm} (59)$$

is satisfied, then there exist $Q_{n0}(j\omega_k), Q_{m}(j\omega_k)$ (i = 1, ..., N), $Q_{d0}(j\omega_k)$ and $Q_{di}(j\omega_k)$ (i = 1, ..., N) satisfying (53) and (54). Let $Q_{n0}(j\omega_k), Q_{m}(j\omega_k)$ (i = 1, ..., N), $Q_{d0}(j\omega_k)$ and $Q_{di}(j\omega_k)$ (i = 1, ..., N) be set satisfying

$$Q_{d0}(j\omega_k) + \sum_{i=1}^{N} Q_{di}(j\omega_k) = C_{22}(j\omega_k) \left( Q_{n0}(j\omega_k) + \sum_{i=1}^{N} Q_{mi}(j\omega_k) \right).$$ \hspace{1cm} (60)

Substitution of (60) for (59) gives

$$\left| \frac{1}{C_{22}(j\omega_k)} \right| = \frac{|Q_{n0}(j\omega_k) + \sum_{i=1}^{N} Q_{mi}(j\omega_k)|}{|Q_{d0}(j\omega_k) + \sum_{i=1}^{N} Q_{di}(j\omega_k)|} = |Q(j\omega_k)| < 1.$$ \hspace{1cm} (61)

In addition, from (60), (53) holds true. Thus, the sufficiency has been shown.

We have thus proved Theorem 2.

Next, we mention the disturbance attenuation characteristic. The transfer function $S(s)$ from the disturbance $d$ to the output $y$ is written by

$$S(s) = \frac{1}{1 + C(s)G(s)} = \frac{S_n(s)}{S_d(s)},$$ \hspace{1cm} (62)

where

$$S_n(s) = \left( 1 + \sum_{i=1}^{N} \frac{Q_{di}(s) - C_{22}(s)Q_{m}(s)}{Q_{d0}(s) - C_{22}(s)Q_{n0}(s)} e^{-s\tau_i} \right) (Q_{d0}(s) - C_{22}(s)Q_{n0}(s))$$ \hspace{1cm} (63)

and

$$S_d(s) = [Q_{d0}(s) - C_{22}(s)Q_{n0}(s) + (C_{11}(s)Q_{d0}(s) + (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) Q_{n0}(s))] G(s) + \sum_{i=1}^{N} [Q_{di}(s) - C_{22}(s)Q_{m}(s) + (C_{11}(s)Q_{d0}(s) + (C_{12}(s)C_{21}(s) - C_{11}(s)C_{22}(s)) Q_{m}(s))] G(s) e^{-s\tau_i}. \hspace{1cm} (64)$$

From (62), for frequency components $\omega_k (k = 0, ..., N_{max})$ in (11) of the disturbance $d$ those are same to those of the periodic reference input $r$, if

$$1 + \sum_{i=1}^{N} \left\{ \frac{Q_{di}(j\omega_k) - C_{22}(j\omega_k)Q_{m}(j\omega_k)}{Q_{d0}(j\omega_k) - C_{22}(j\omega_k)Q_{n0}(j\omega_k)} \right\} \simeq 0,$$ \hspace{1cm} (65)

then the disturbance $d$ is attenuated effectively. For the frequency component $\omega_d$ of the disturbance $d$ that is different from that of the periodic reference input $r$, that is $\omega_d \neq \omega_k$, even if

$$1 + \sum_{i=1}^{N} \left\{ \frac{Q_{di}(j\omega_d) - C_{22}(j\omega_d)Q_{m}(j\omega_d)}{Q_{d0}(j\omega_d) - C_{22}(j\omega_d)Q_{n0}(j\omega_d)} \right\} \simeq 0,$$ \hspace{1cm} (66)

the disturbance $d$ cannot be attenuated, because

$$e^{-j\omega_d\tau_i} \neq 1,$$ \hspace{1cm} (67)

and

$$1 + \sum_{i=1}^{N} \left\{ \frac{Q_{di}(j\omega_d) - C_{22}(j\omega_d)Q_{m}(j\omega_d)e^{-j\omega_d\tau_i}}{Q_{d0}(j\omega_d) - C_{22}(j\omega_d)Q_{n0}(j\omega_d)} \right\} \neq 0.$$ \hspace{1cm} (68)

In order to attenuate the frequency component $\omega_d$ of the disturbance $d$ that is different from that of the periodic reference input $r$, if

$$Q_{d0}(j\omega_d) - C_{22}(j\omega_d)Q_{n0}(j\omega_d) \simeq 0,$$ \hspace{1cm} (69)

then the disturbance $d$ is attenuated effectively.
5 Design procedure

In this section, a design procedure of robust stabilizing simple MPR controller $C(s)$ is presented.

A design procedure of robust stabilizing simple MPR controller $C(s)$ satisfying Theorem 1 is summarized as follows:

**Procedure**

Step 1) Obtain $C_{11}(s)$, $C_{12}(s)$, $C_{21}(s)$ and $C_{22}(s)$ by solving the robust stability problem using the Riccati equation based $H_{\infty}$ control.

Step 2) $Q_{n0}(s) \in RH_{\infty}$ is settled so that for the frequency component $\omega_d$ of the disturbance, $|Q_{d0}(j\omega_d) - C_{22}(j\omega_d)Q_{n0}(j\omega_d)|$ is effectively small. In order to design $Q_{n0}(s)$ to make $|Q_{d0}(j\omega_d) - C_{22}(j\omega_d)Q_{n0}(j\omega_d)|$ effectively small, $Q_{n0}(s)$ is settled by

$$Q_{n0}(s) = \frac{Q_{d0}(s)}{C_{22}(s)s^\alpha_d},$$

where $C_{22}(s) \in RH_{\infty}$ is an outer function of $C_{22}(s)$ satisfying

$$C_{22}(s) = C_{22}(s)C_{22}(s),$$

$C_{22}(s) \in RH_{\infty}$ is an inner function satisfying $C_{22}(0) = 1$ and $|C_{22}(j\omega)| = 1(\forall \omega \in R_+)$, $\overline{q_d}(s)$ is a low-pass filter satisfying $\overline{q_d}(0) = 1$, as

$$\overline{q_d}(s) = \frac{1}{(1 + s\tau_d)^{\alpha_d}},$$

is valid, $\alpha_d$ is an arbitrary positive integer to make $\overline{q_d}(s)/C_{22}(s)$ proper and $\tau_d \in R$ is any positive real number satisfying

$$1 - C_{22}(j\omega_d) \frac{1}{(1 + j\omega_d\tau_d)^{\alpha_d}} \simeq 0.$$  (73)

Step 3) $\overline{Q_i}(s) \in RH_{\infty}(i = 1, \ldots, N)$ are settled so that for frequency components $\omega_k(k = 0, \ldots, N_{max})$ of the periodic reference input $r$, $1 + \sum_{i=1}^{N}|Q_{d1}(j\omega_k) - C_{22}(j\omega_k)Q_{n1}(j\omega_k)|/(Q_{d0}(j\omega_k) - C_{22}(j\omega_k)Q_{n0}(j\omega_k)) \simeq 0$ is satisfied. In order to design $Q_{i}(s)(i = 1, \ldots, N)$ to hold

$$1 + \sum_{i=1}^{N} \frac{Q_{d1}(j\omega_k) - C_{22}(j\omega_k)Q_{n1}(j\omega_k)}{Q_{d0}(j\omega_k) - C_{22}(j\omega_k)Q_{n0}(j\omega_k)} = 1 - \frac{G_{n}(j\omega_k)}{1 + C_{11}(j\omega_k)G_m(j\omega_k)} + C_{22}(j\omega_k)G_{d}(j\omega_k) \sum_{i=1}^{N} \overline{Q_i}(j\omega_k) \simeq 0, $$

$\overline{Q_i}(s) \in RH_{\infty}(i = 1, \ldots, N)$ are settled by

$$\overline{Q_i}(s) = C_{22}(s)Q_{d0}(s) - C_{22}(s)Q_{n0}(s)$$

$$H_{d}(s) = C_{22}(s)G_{n}(s)$$

satisfying

$$H(s) = H_{i}(s)H_{o}(s),$$

$H_{i}(s) \in RH_{\infty}$ is an inner function satisfying $H_{i}(0) = 1$ and $|H_{i}(j\omega)| = 1(\forall \omega \in R_+)$, $\overline{q}_{ri}(s)(i = 1, \ldots, N)$ are low-pass filters satisfying $\sum_{i=1}^{N} \overline{q}_{ri}(0) = 1$, as

$$\overline{q}_{ri}(s) = \frac{1}{N(1 + s\tau_r)^{\alpha_r}} (i = 1, \ldots, N)$$

are valid, $\alpha_r$ is an arbitrary positive integer to make $\overline{q}_{ri}(s)/H_{o}(s)$ proper and $\tau_r \in R$ is any positive real number satisfying

$$1 - H_{i}(j\omega_k) \sum_{i=1}^{N} \frac{1}{N(1 + j\omega_k\tau_r)^{\alpha_r}} = 1 - H_{i}(j\omega_k) \frac{1}{(1 + j\omega_k\tau_r)^{\alpha_r}} \simeq 0.$$  (79)
Note 2 Note that even if the robust stabilizing simple MPR controller $C(s)$ is designed according to the above-mentioned procedure, $\|Q(s)\|_\infty < 1$ is not necessarily satisfied.

6 Numerical example

In this section, a numerical example is shown to illustrate the effectiveness of the proposed parameterization.

Consider the problem to obtain the parameterization of all robust stabilizing simple MPR controllers for plant $G(s)$ in (3), where the nominal plant $G_m(s)$ and the upper bound $W_T(s)$ of the set of $\Delta(s)$ are given by

$$G_m(s) = \frac{1}{(s-1)(s+2)(s+3)}$$

and

$$W_T(s) = \frac{(s+5)(s+75)(s+1000)}{7.5 \times 10^6}.$$  

The period $T$ of the periodic reference input $r$ in (2) is $T = 20\text{[sec]}$. Solving the robust stability problem using Riccati equation based $H_{\infty}$ control as Theorem 1, the parameterization of all robust stabilizing simple MPR controllers $C(s)$ is obtained as (20), where $N$ is selected as $N = 3$ and $T_i (i = 1, 2, 3)$ are set as $T_i = T \cdot i (i = 1, 2, 3)$. Here, $C_{ij}(s) (i = 1, 2; j = 1, 2)$ are

$$C_{11}(s) = \frac{9.163 \times 10^5(s+3)(s+2)}{(s+1000)(s+74.81)(s+7.199)}, C_{12}(s) = \frac{7.5 \times 10^6(s+4.644)(s^2+1.356s+6.056)}{(s+1000)(s+74.81)(s+7.199)},$$

$$C_{21}(s) = \frac{(s+75)(s+5)}{(s+74.81)(s+7.199)}, C_{22}(s) = \frac{0.1222(s^2+1081s+6.147 \times 10^7)}{(s+1000)(s+74.81)(s+7.199)}.$$  

In order for disturbances both

$$d = \sin(0.1\pi t) + \sin(0.2\pi t) + \sin(0.3\pi t)$$

and

$$d = \sin(0.025\pi t) + \sin(0.05\pi t) + \sin(0.075\pi t))$$

to be attenuated effectively and for the output $y$ to follow the periodic reference input

$$r = \sin(0.1\pi t) + \sin(0.2\pi t) + \sin(0.3\pi t)$$

with small steady state error, $Q_{u0}(s)$ and $\bar{Q}_i(s) (i = 1, 2, 3)$ are settled by (70) and (75), where

$$Q_{u0}(s) = 1,$$

$$\bar{Q}_d(s) = \frac{1}{(1+0.01s)^3},$$

$$Q_{u0}(s) = \frac{8.185 \times 10^6(s+1000)(s+74.81)(s+7.199)}{(s+100)^3(s^2+1081s+6.147 \times 10^7)}.$$  

The gain plot of $Q(s)$ is shown in Fig. 2. Figure 2 shows that the designed $Q(s)$ satisfies $\|Q(s)\|_\infty < 1$.

When $\Delta(s)$ is given by

$$\Delta(s) = \frac{s^2-240s+14400}{s+100000},$$

the gain plot of $1/\Delta(s)$ and $1/W_T(s)$ are shown in Fig. 3. Here, the dotted line shows the gain plot of $\Delta(s)$.
and the solid line shows that of $1/\Delta(s)$. Figure 3 shows that the uncertainty $\Delta(s)$ satisfies (4).

Using above-mentioned parameters, we have a robust stabilizing simple MPR controller. When the designed robust stabilizing simple MPR controller $C(s)$ is used, the response of the output $y$ in (1) for the periodic reference input $r$ in (85) is shown in Fig. 4. Here, the dotted line shows the response of the periodic reference input $r$ in (85) and the solid line shows that of the output $y$. Figure 4 shows that the output $y$ follows the periodic reference input $r$ in (85) with small steady state error, even if the plant has uncertainty $\Delta(s)$.

Next, using the designed robust stabilizing simple MPR controller $C(s)$, the disturbance attenuation characteristic is shown. The response of the output $y$ for the disturbance $d$ in (83) of which the frequency component is equivalent to that of the periodic reference input $r$ is shown in Fig. 5. Here, the dotted line shows the response of the disturbance $d$ in (83) and the solid line shows that of the output $y$. Figure 5 shows that the disturbance $d$ in (83) is attenuated effectively. Finally, the response of the output $y$ for the disturbance $d$ in (84) of which the frequency component is different from that of the periodic reference input $r$ is shown in Fig. 6. Here, the dotted line shows the response of the disturbance $d$ in (84) and the solid line shows that of the output $y$. Figure 6 shows that the disturbance $d$ in (84) is attenuated effectively.

In order to show the effectiveness of the proposed method, the difference was clarified by comparison with the response using the parameterization of all stabilizing simple multi-period repetitive controllers in $^9$. According
Figure 4: The response of the output $y$ for the periodic reference input $r = \sin(0.1\pi t) + \sin(0.2\pi t) + \sin(0.3\pi t)$

Figure 5: The response of the output $y$ for the disturbance $d = \sin(0.1\pi t) + \sin(0.2\pi t) + \sin(0.3\pi t)$

to \(^9\), the parameterization of all stabilizing simple MPR controllers is given by

$$C(s) = \frac{X(s) + D(s) \left( Q(s) + \sum_{i=1}^{N} Q_i(s)e^{-sT_i} \right)}{Y(s) - N(s) \left( Q(s) + \sum_{i=1}^{N} \tilde{Q}_i(s)e^{-sT_i} \right)},$$ \hfill (91)

where $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ are coprime factors of $G_m(s)$ on $RH_\infty$ satisfying

$$G_m(s) = \frac{N(s)}{D(s)}.$$ \hfill (92)

where $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ are functions satisfying

$$X(s)N(s) + Y(s)D(s) = 1,$$ \hfill (93)

and $Q(s) \in RH_\infty$ and $\tilde{Q}_i(s) \neq 0 \in RH_\infty (i = 1, \ldots, N)$ are any functions satisfying

$$\sum_{i=1}^{N} \frac{N(0)\tilde{Q}_i(0)}{Y(0) - N(0)\tilde{Q}_i(0)} = 1.$$ \hfill (94)
Using the parameterization in (91), we design a stabilizing simple MPR controller for the nominal plant in (80). \( N \) in (7) is \( N = 3 \). The period \( T \) of the reference input is \( T = 20 \) [sec] and \( T_i = T : i (i = 1, 2, 3) \). A pair of coprime factors \( N(s) \in RH_\infty \) and \( D(s) \in RH_\infty \) of \( G_m(s) \) in (80) satisfying (92) are given by

\[
N(s) = \frac{1}{(s + 1)(s + 5)(s + 6)} \in RH_\infty
\]  

and

\[
D(s) = \frac{(s - 1)(s + 2)(s + 3)}{(s + 1)(s + 5)(s + 6)} \in RH_\infty.
\]  

\( X(s) \in RH_\infty \) and \( Y(s) \in RH_\infty \) satisfying (93) are settled by

\[
X(s) = \frac{576(s + 2.5)^2}{(s + 1)(s + 5)(s + 6)} \in RH_\infty
\]  

and

\[
Y(s) = \frac{(s + 10)(s^2 + 10s + 45)}{(s + 1)(s + 5)(s + 6)} \in RH_\infty.
\]  

Using the method in 9), in order for disturbances both in (83) and (84) to be attenuated effectively and for the output \( y \) to follow the periodic reference input in (85) with small steady state error, \( Q(s) \in PH_\infty \) and \( Q_i(s) \in RH_\infty(i = 1, \ldots, N) \) in (91) are settled by

\[
Q(s) = \frac{Y(s)}{N(s)} q_d(s),
\]  

and

\[
Q_i(s) = \frac{Y(s) - N(s)Q(s)}{N(s)} q_{ri}(s) \quad (i = 1, 2, 3),
\]

where \( q_d(s) \) and \( q_{ri}(s) (i = 1, 2, 3) \) are given by (87) and (89). Using the above parameters, we have a stabilizing simple MPR controller \( C(s) \). Using this controller, when the uncertainty \( \Delta(s) = 0 \), the response of the output \( y \) in (1) for the periodic reference input \( r \) in (85) is shown in Fig. 7. Here, the dotted line shows the response of the periodic reference input \( r \) in (85) and the solid line shows that of the output \( y \). Figure 7 shows that the output \( y \) follows the periodic reference input \( r \) in (85) with small steady state error. Next, when \( \Delta(s) \) is given by (90), the response of the output \( y \) in (1) for the periodic reference input \( r \) in (85) is shown in Fig. 8. Here, the dotted line shows the response of the periodic reference input \( r \) in (85) and the solid line shows that of the output \( y \). Figure 8 shows that the control system in (1) is unstable. The comparison of Fig. 4 with Fig. 7 and Fig. 8 shows that both robust stabilizing simple MPR control system and simple MPR control system have similar input-output characteristic when \( \Delta(s) = 0 \). On the other hand, simple MPR controller cannot guarantee the stability of control system with the uncertainty, but our proposed robust stabilizing simple MPR controller can.

In this way, we find that we can easily design a robust stabilizing simple MPR controller using Theorem 1.
Figure 7: The response of the output $y$ for the periodic reference input $r = \sin(0.1\pi t) + \sin(0.2\pi t) + \sin(0.3\pi t)$ when $\Delta(s) = 0$ using the simple multi-period repetitive controller

Figure 8: The response of the output $y$ for the periodic reference input $r = \sin(0.1\pi t) + \sin(0.2\pi t) + \sin(0.3\pi t)$ when $\Delta(s)$ is given by (90) using the simple multi-period repetitive controller

7 Conclusions

In this paper, we proposed the parameterization of all robust stabilizing simple MPR controllers for plant with uncertainties. That is, we found out the parameterization of all robust stabilizing simple MPR controllers $C(s)$ written as the form in (7) such that the control system in (1) is robustly stable, the output $y$ follows the periodic reference input $r$ with small steady state error even in the presence of uncertainty $\Delta(s)$ and transfer functions from the periodic reference input to the output and from the disturbance to the output have finite numbers of poles when the uncertainty $\Delta(s) = 0$. Control characteristics of a robust stabilizing simple MPR control system are presented, as well as a design procedure for a robust stabilizing simple MPR controller. A numerical example was shown to illustrate the effectiveness of the proposed parameterization.

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References


