Numerical Conformal Mappings onto the Canonical Slit Domains

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Conformal mappings are familiar in science and engineering. Recently, the problem of multiply connected domains has attracted special interest. We here present a unified scheme for numerical conformal mappings of unbounded multiply connected domains onto the parallel, circular and radial slit domains under two different conditions. These are canonical slit domains important in potential flow problems. We express the mapping functions in terms of an analytic function so that its real part is subject to a Dirichlet boundary condition, and have approximate mapping functions of simple form and high accuracy using the charge simulation method. Numerical examples show the effectiveness of our method. We also make a short survey of relevant works.

1. INTRODUCTION

Conformal mappings are familiar in science and engineering\(^1\),\(^2\). However, their exact mapping functions are unknown in many cases. Therefore, numerical conformal mappings have been studied for a long time\(^3\)–\(^7\).

In the conformal mapping of multiply connected domains, a standard domain such as the unit disk for simply connected domains does not exist. Two domains can be mapped conformally onto each other if, and only if, they agree in connectivity \(n\), and moreover \(3(n – 2) \geq 3\) conformal invariants called moduli. Hence, canonical domains that specify geometric characters independent of moduli are introduced. They often have slits, and the following listed in Nehari\(^8\) (see Fig. 1) are well known: (a) the parallel slit domain, (b) the circular slit domain, (c) the radial slit domain, (d) the circle with concentric circular slits, and (e) the circular ring with concentric circular slits. Koebe\(^9\) formerly gave thirty nine canonical slit domains including these five domains. Also the

\[\theta\]

(a) (b) (c) (d) (e)

Fig. 1 Canonical slit domains listed in Nehari\(^8\).
circular domain, whose boundary curves are circles without slits, is an important canonical domain. Recently, conformal mappings of multiply connected domains have attracted special interest, and actively been studied. The charge simulation method, generally speaking the fundamental solution method, was originally a numerical method for Laplace's equation. Amano first applied it to the numerical conformal mapping. He approximated a pair of harmonic functions by a linear combination of complex logarithmic functions, and constructed a simple form of approximate mapping function with high accuracy. This method was easily applicable to the problems of not only simply and doubly connected domains but also multiply connected domains of arbitrary connectivities onto all the canonical slit domains shown in Fig. 1. In this paper, we refine some our recent works into a unified scheme for numerical conformal mappings of unbounded multiply connected domains onto the parallel, circular and radial slit domains under two different conditions. In Section 2, we reduce a scheme for these numerical conformal mappings under the condition \( f(\infty) = \infty \), where \( w = f(z) \) are mapping functions. They are applicable to a uniform, a vortex and a point source potential flows past obstacles. In Section 3, we reduce a scheme for the same conformal mappings under the condition \( f(v) = \infty, \, |v| < \infty \). They are applicable to a dipole source, a vortex pair, and a point source and sink pair flows past obstacles. In Section 4, we give concluding remarks in relation to some other relevant works.

2. NUMERICAL CONFORMAL MAPPINGS

Let \( D \) be an unbounded domain exterior to closed Jordan curves \( C_1, \ldots, C_n \) in the \( z \) plane as shown in Fig. 2. Consider the conformal mappings \( w = f(z) \) of \( D \) onto (a) the parallel slit domain (b) the circular slit domain and (c) the radial slit domain. They are the entire \( w \) plane with rectilinear parallel, circular or radial slits \( S_1, \ldots, S_n \). We suppose that both \( z \) and \( w \) planes include the point at infinity. Our aim is to obtain an approximation \( F(z) \) of the mapping function \( f(z) \) given by the subsequent theorems.

2.1 The parallel slit domain

Theorem 1 For an angle \( \theta \) arbitrarily assigned to the real axis, there exists a unique analytic function \( f_\theta(z) \) such that (i) conformally maps \( D \) onto the parallel slit domain and (ii) satisfies \( f_\theta(\infty) = \infty \) with the Laurent expansion near \( z = \infty \) of the form

\[
 f_\theta(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots .
\]  

(1)

We express the mapping function as

\[
 f_\theta(z) = z + i e^{i \theta} a(z),
\]  

(2)

Fig. 2 Conformal mappings onto the (a) parallel, (b) circular and (c) radial slit domains.
where \( a(z) \) is analytic in \( D \) and should satisfy the following conditions.

(i) Normalization condition. From the Laurent expansion (1),
\[
\lim_{z \to \infty} (f_0(z) - z) = 0, \quad \text{i.e.,} \quad a(\infty) = 0. \tag{3}
\]

(ii) Boundary condition. Since \( f_\theta(z) \) maps \( C_1, \ldots, C_n \) onto the parallel slits \( S_1, \ldots, S_n \) of the angle \( \theta \),
\[
\text{Im}(e^{-i\theta} f_\theta(z)) = p_m, \quad \text{i.e.,} \quad \text{Re} a(z) - p_m = -\text{Im}(e^{-i\theta} z), \quad z \in C_m, \ m = 1, \ldots, n, \tag{4}
\]
where \( p_m \) is the directed length of the perpendicular from \( w = 0 \) to the straight line containing \( S_m \).

The problem is now to find \( a(z) \) satisfying (3), (4), together with \( p_m \).

Using the charge simulation method, we approximate the analytic function \( a(z) \) by a linear combination of complex logarithmic functions,
\[
a(z) \simeq A(z) = Q_0 + \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log(z - \zeta_{lj}), \tag{5}
\]
where \( Q_0 \) is a complex constant and \( Q_{lj} \) are real constants called the charges. The singular points \( \zeta_{lj} \) called the charge points are placed inside \( C_l \), i.e., outside \( D \). We impose the following requirements on the approximate function (5).

(i) Single-valuedness condition. The approximate function (5) is single-valued in \( D \) if and only if
\[
\int_{C_l} dA(z) = i \int_{C_l} d\left( \sum_{m=1}^{n} \sum_{j=1}^{N_m} Q_{mj} \arg(z - \zeta_{mj}) \right) = 2\pi i \sum_{j=1}^{N_l} Q_{lj} = 0,
\]
so that
\[
\sum_{j=1}^{N_l} Q_{lj} = 0, \quad l = 1, \ldots, n. \tag{6}
\]

(ii) Normalization condition. From the normalization condition (3), we require
\[
A(\infty) = Q_0 + \lim_{z \to \infty} \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log(z - \zeta_{lj}) = 0,
\]
so that under the condition (6)
\[
Q_0 = 0, \quad A(z) = \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log(z - \zeta_{lj}). \tag{7}
\]

(iii) Collocation condition. We require \( A(z) \) to satisfy the boundary condition (4) at a finite number of boundary points \( z_{mk} \) called the collocation points, and have
\[
\sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log|z_{mk} - \zeta_{lj}| - P_m = -\text{Im}(e^{-i\theta} z_{mk}) \quad z_{mk} \in C_m, \ k = 1, \ldots, N_m, \ m = 1, \ldots, n, \tag{8}
\]
where \( P_m \) is the approximation of \( p_m \). Equation (8) is called the collocation condition.
Equations (6), (8) constitute a set of simultaneous linear equations in $N_1 + \cdots + N_n + n$ unknowns $Q_{ij}, P_m$. Once they are determined, we obtain $A(z)$ by (7), and an approximate mapping function $F_0(z)$ by (2).

2.2 The circular slit domain

**Theorem 2** There exists a unique analytic function $f_c(z)$ such that it (i) conformally maps $D$ onto the circular slit domain and (ii) satisfies $f_c(0) = 0, f_c(\infty) = \infty$ with the Laurent expansion near $z = \infty$ of the form

$$f_c(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots. \quad (9)$$

We express the mapping function as

$$f_c(z) = z \exp a(z), \quad (10)$$

where $a(z)$ is analytic in $D$ and should satisfy the following conditions.

(i) Normalization condition. From the Laurent expansion (9),

$$\lim_{z \to \infty} \frac{f_c(z)}{z} = 1, \quad \text{i.e.,} \quad a(\infty) = 0. \quad (11)$$

(ii) Boundary condition. Since $f_c(z)$ maps $C_1, \ldots, C_n$ onto the circular slits $S_1, \ldots, S_n$ of the radii $r_1, \ldots, r_n$,

$$|f_c(z)| = r_m, \quad \text{i.e.,} \quad \text{Re } a(z) - \log r_m = -\log |z|,$$

$$z \in C_m, \quad m = 1, \ldots, n. \quad (12)$$

The problem is now to find $a(z)$ satisfying (11), (12), together with $r_m$.

We approximate $a(z)$ by (5), and obtain the single-valuedness condition (6), the normalization condition (7), and the collocation condition

$$\sum_{i=1}^{N_1} \sum_{j=1}^{N_j} Q_{ij} \log |z_{mk} - \zeta_{ij}| - \log R_m = -\log |z_{mk}|,$$

$$z_{mk} \in C_m, \quad k = 1, \ldots, N_m, \quad m = 1, \ldots, n. \quad (13)$$

where $R_m$ is the approximation of $r_m$.

Equations (6), (13) constitute a set of simultaneous linear equations in $N_1 + \cdots + N_n + n$ unknowns $Q_{ij}, \log R_m$. Once they are determined, we obtain $A(z)$ by (7) and an approximate mapping function $F_c(z)$ by (10).

2.3 The radial slit domain

**Theorem 3** There exists a unique analytic function $f_r(z)$ such that it (i) conformally maps $D$ onto the radial slit domain and (ii) satisfies $f_r(0) = 0, f_r(\infty) = \infty$ with the Laurent expansion near $z = \infty$ of the form

$$f_r(z) = z + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots. \quad (14)$$

We express the mapping function as

$$f_r(z) = z \exp(ia(z)), \quad (15)$$

where $a(z)$ is analytic in $D$ and should satisfy the following conditions. Note that the imaginary unit $i$ in (15) is for describing the boundary condition in terms of Re $a(z)$.
(i) Normalization condition. From the Laurent expansion (14),
\[ \lim_{z \to \infty} \frac{f_r(z)}{z} = 1, \quad \text{i.e.,} \quad a(\infty) = 0. \quad (16) \]

(ii) Boundary condition. Since \( f_r(z) \) maps \( C_1, \ldots, C_n \) onto the radial slits \( S_1, \ldots, S_n \) of the arguments \( \theta_1, \ldots, \theta_n, \)
\[ \arg f_r(z) = \theta_m, \quad \text{i.e.,} \quad \text{Re} \ a(z) - \theta_m = - \arg z, \]
\[ z \in C_m, \quad m = 1, \ldots, n. \quad (17) \]

The problem is now to find \( a(z) \) satisfying (16), (17), together with \( \theta_m \).

We approximate \( a(z) \) by (5), and obtain the single-valuedness condition (6), the normalization condition (7), and the collocation condition
\[ \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log |z_{mk} - \zeta_{lj}| - \Theta_m = - \arg z_{mk}, \]
\[ z_{mk} \in C_m, \quad k = 1, \ldots, N_m, \quad m = 1, \ldots, n, \quad (18) \]
where \( \Theta_m \) is the approximation of \( \theta_m \).

Equations (6), (18) constitute a set of simultaneous linear equations in \( N_1 + \cdots + N_n + n \) unknowns \( Q_{lj}, \Theta_m \). Once they are determined, we obtain \( A(z) \) by (7) and an approximate mapping function \( F_r(z) \) by (15).

2.4 A unified scheme

We use in computation the principal value of logarithmic function. Consequently, the term \( \log(z - \zeta_{lj}) \) in (7) has the \( 2\pi i \) discontinuity on the half line \( \{ \zeta_{lj} - t \ | \ t > 0 \} \), which causes discontinuities of \( A(z) \) in \( D \). Therefore, we change the expression (7) into a form that is continuous in \( D \) when the principal value is used. We call an approximate mapping function using such an expression of \( A(z) \) the continuous scheme.

Here we assume, only for simplicity, that each boundary curve \( C_l \) is starlike with respect to its inside point \( \zeta_{l0} \). Using (6), we rewrite (7) into
\[ A(z) = \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log(z - \zeta_{lj}) - \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log(z - \zeta_{l0}) \]
\[ = \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log \frac{z - \zeta_{lj}}{z - \zeta_{l0}}. \]

The term \( \log[(z - \zeta_{lj})/(z - \zeta_{l0})] \) has the discontinuity on the line segment \( (\zeta_{lj}, \zeta_{l0}) \) inside \( C_l \), and is continuous in \( D \) when the principal value is used. We now have a unified scheme for the numerical conformal mappings onto parallel, circular and radial slit domains.

**Scheme 1** The approximate mapping functions are given by
\[ F_0(z) = z + i e^{i \theta} A(z), \]
\[ F_c(z) = z \exp A(z), \]
\[ F_r(z) = z \exp (i A(z)), \quad (19) \]
\[ A(z) = \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log \frac{z - \zeta_{lj}}{z - \zeta_{l0}}. \quad (20) \]
where the unknown coefficients $Q_{lj}$, together with the constants $P_m$, $R_m$ or $\Theta_m$ are determined by solving the linear equations

\begin{align}
\sum_{j=1}^{N_l} Q_{lj} &= 0, \quad l = 1, \ldots, n, \\
\sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log \left| \frac{z_{mk} - \zeta_0}{z_{mk} - \zeta_{lj}} \right| - S_m &= -t_{mk}, \\
& \quad z_{mk} \in C_m, \quad k = 1, \ldots, N_m, \quad m = 1, \ldots, n, \quad (22)
\end{align}

where

\begin{align}
S_m &= \begin{cases} 
P_m & \text{for } \theta \in [0, \pi], \\
\log R_m & \text{for } \theta \in [\pi, 2\pi), \\
\Theta_m & \text{for } \theta \in [2\pi, 3\pi),
\end{cases} \\
t_{mk} &= \begin{cases} 
\text{Im}(e^{-i\theta} z_{mk}) & \text{for } \theta \in [0, \pi), \\
\log |z_{mk}| & \text{for } \theta \in [\pi, 2\pi), \\
\arg z_{mk} & \text{for } \theta \in [2\pi, 3\pi),
\end{cases} \quad (23)
\end{align}

It should be noted that the coefficient matrix is the same for all the mapping functions.

### 2.5 An example

Computations were carried out on a dual Intel Xeon 3.06 GHz processor workstation with the Intel Fortran compiler in double precision working. The IMSL library was used for solving linear equations.

**Example 1** The problem domain $D$ is the exterior of three disks,

\begin{align*}
C_l : \quad |z - \zeta_0| &= \rho_l, \quad \rho_1 = 1, \quad \rho_2 = 0.5, \quad \rho_3 = 1.5, \\
\zeta_0 &= 2 \exp \frac{2(l-1)\pi i}{3}, \quad l = 1, 2, 3.
\end{align*}

Collocation points and charge points are placed by

\begin{align*}
z_{lj} = \zeta_0 + \rho_l \exp \frac{2(j-1)\pi i}{N}, \quad \zeta_{lj} = \zeta_0 + q \rho_l \exp \frac{2(j-1)\pi i}{N}, \\
j = 1, \ldots, N, \quad l = 1, 2, 3, \quad (24)
\end{align*}

where $0 < q < 1$ is a parameter for charge placement. Errors are estimated by

\begin{align*}
\epsilon_{P_{l1}} &= \max_{z \in C_l} |\text{Im}(e^{-i\theta} F_\theta(z)) - P_l|, \\
\epsilon_{R_{l1}} &= |P_l - P_{l1}^{(2N)}|, \quad (25) \\
\epsilon_{F_{l1}} &= \max_{z \in C_l} |F_c(z) - R_l|, \\
\epsilon_{R_{l1}} &= |R_l - R_{l1}^{(2N)}|, \quad (26) \\
\epsilon_{F_{l1}} &= \max_{z \in C_l} |\arg F_c(z) - \Theta_l|, \\
\epsilon_{\Theta_{l1}} &= |\Theta_l - \Theta_{l1}^{(2N)}|, \quad (27)
\end{align*}

where $P_{l1}^{(2N)}$, $R_{l1}^{(2N)}$ and $\Theta_{l1}^{(2N)}$ are the results for $2N$ charges. In practice $\epsilon_{F_{l1}}$, $\epsilon_{F_{l1}}$ and $\epsilon_{F_{l1}}$ are evaluated at $8N$ points uniformly placed on $C_l$.

Fig. 3 illustrates by square meshes the numerical conformal mapping of $D$ onto the (a) parallel, (b) circular and (c) radial slit domains. Small dots inside the boundary circles show the charge points.

Table 1 shows numerical results of the conformal mapping, where $\kappa$ is the $L_1$ condition number of the coefficient matrix of the linear equations solved. The values of $P_l$, $R_l$ and $\Theta_l$ are shown until a nonzero digit appears respectively in the right-hand side of (25), (26) and (27), so that they agree except for the last digit with those of $P_{l1}^{(2N)}$, $R_{l1}^{(2N)}$ and $\Theta_{l1}^{(2N)}$ that are expected to be more accurate.
Fig. 3 Numerical conformal mappings onto the (a) parallel \((\theta = \pi/3)\), (b) circular and (c) radial slit domains.

We can see that high accuracy is achieved. However, the errors \(\epsilon_F\) and \(\epsilon_k\) are relatively large on \(C_3\) because the logarithmic function of \(t_{mk}\) for \(F_c(z)\) and \(F_r(z)\) in (23) has a singularity at \(z = 0\) and the boundary curve \(C_3\) comes nearest the singular point.

Fig. 4 shows contour lines of (a) \(\text{Im}(e^{-i\theta}F_\theta(z))\), (b) \(\text{Im}(-\log F_c(z))\) and (c) \(\text{Im}(|\arg F_r(z)|)\), which illustrate streamlines of (a) a uniform, (b) a vortex and (c) a point source (or sink) flows past three cylindrical objects of outlines \(C_1\), \(C_2\) and \(C_3\). Here \(e^{-i\theta}F_\theta(z)\), \(-\log F_c(z)\) and \(|\arg F_r(z)|\) are the complex potentials of the flows. The vortex and the point source are at the origin \(z = 0\). In addition, various flows with vortices and point sources and/or sinks in a uniform flow are easily obtained by the superposition of complex potentials\(^{19}\).

3. ANOTHER NORMALIZATION CONDITION

Let \(v\) be a finite point arbitrarily given in \(D\), i.e., \(v \in D\), \(|v| < \infty\). Consider the conformal mappings of \(D\) onto the parallel, circular and radial slit domains under the different normalization condition \(f_\theta(v) = f_c(v) = f_r(v) = \infty\).

3.1 The parallel slit domain

Theorem 4 For an angle \(\theta\) arbitrarily assigned to the real axis, there exists a unique analytic function \(f_\theta(z)\) such that (i) conformally maps \(D\) onto the parallel slit domain and (ii) satisfies \(f_\theta(v) = \infty\) with the Laurent expansion near \(z = v\) of the form

\[
f_\theta(z) = \frac{1}{z-v} + a_1(z-v) + a_2(z-v)^2 + \cdots.
\]  

(28)
Table 1 Numerical results of the conformal mapping (Example 1, $N = 64$, $q = 0.8$, $\kappa = 4.7 \times 10^4$, $\theta = \pi/3$)

<table>
<thead>
<tr>
<th></th>
<th>$\epsilon_{F_{\ell}}$</th>
<th>$\epsilon_{P_{\ell}}$</th>
<th>$P_{\ell}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $F_\theta(z)$</td>
<td>$C_1$</td>
<td>5.2E-08</td>
<td>1.2E-08</td>
</tr>
<tr>
<td></td>
<td>$C_2$</td>
<td>2.6E-08</td>
<td>1.3E-08</td>
</tr>
<tr>
<td></td>
<td>$C_3$</td>
<td>7.5E-08</td>
<td>4.5E-09</td>
</tr>
<tr>
<td>(b) $F_\ell(z)$</td>
<td>$C_1$</td>
<td>2.1E-07</td>
<td>1.8E-08</td>
</tr>
<tr>
<td></td>
<td>$C_2$</td>
<td>3.9E-08</td>
<td>2.3E-08</td>
</tr>
<tr>
<td></td>
<td>$C_3$</td>
<td>8.6E-05</td>
<td>6.2E-09</td>
</tr>
<tr>
<td>(c) $F_z(z)$</td>
<td>$C_1$</td>
<td>6.7E-08</td>
<td>1.3E-08</td>
</tr>
<tr>
<td></td>
<td>$C_2$</td>
<td>2.1E-08</td>
<td>8.5E-08</td>
</tr>
<tr>
<td></td>
<td>$C_3$</td>
<td>2.5E-05</td>
<td>4.9E-09</td>
</tr>
</tbody>
</table>

We express the mapping function as

$$f_\theta(z) = \frac{1}{z - v} + i e^{i\theta} a(z),$$

where the analytic function $a(z)$ should satisfy the following conditions.

(i) Normalization condition. From the Laurent expansion (28),

$$\lim_{z \to v} \left( f_\theta(z) - \frac{1}{z - v} \right) = 0, \quad \text{i.e., } \quad a(v) = 0.$$  

(ii) Boundary condition. Since $f_\theta(z)$ maps $C_1, \ldots, C_n$ onto the parallel slits $S_1, \ldots, S_n$ of the angle $\theta$,

$$\Im(e^{-i\theta} f_\theta(z)) = p_m, \quad \text{i.e., } \quad \Re a(z) - p_m = -\Im \frac{e^{-i\theta}}{z - v},$$

$$z \in C_m, \quad m = 1, \ldots, n.$$  

We approximate the analytic function $a(z)$ by

$$a(z) \simeq A(z) = Q_0 + \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log(z - \zeta_{lj}),$$

and impose the following requirements.

(i) Single-valuedness condition. The approximate function (32) is single-valued in $D$ if and only if

$$\sum_{j=1}^{N_l} Q_{lj} = 0, \quad l = 1, \ldots, n.$$  

(ii) Normalization condition. From the normalization condition (30), we require

$$A(v) = Q_0 + \lim_{z \to v} \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log(z - \zeta_{lj}) = 0,$$
so that

\[ Q_0 = - \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log(v - \zeta_{lj}). \]  

(iii) Collocation condition. We require \( A(z) \) to satisfy the boundary condition (31) at a finite number of boundary points, and have the collocation condition

\[ \sum_{l=1}^{n} \sum_{j=1}^{N_l} Q_{lj} \log \left| \frac{z_{mk} - \zeta_{lj}}{v - \zeta_{lj}} \right| - P_m = - \text{Im} \frac{e^{-i\theta}}{z_{mk} - v} \]

\[ z_{mk} \in C_m, \; k = 1, \ldots, N_m, \; m = 1, \ldots, n. \]  

Equations (33), (35) constitute a set of simultaneous linear equations in \( Q_{lj}, P_m \).

3.2 The circular slit domain

**Theorem 5** There exists a unique analytic function \( f_c(z) \) such that it (i) conformally maps \( D \) onto the circular slit domain and (ii) satisfies \( f_c(0) = 0, f_c(v) = \infty \) with the Laurent expansion near \( z = v \) of the form

\[ f_c(z) = \frac{1}{z - v} + b_0 + b_1(z - v) + b_2(z - v)^2 + \cdots. \]  

We express the mapping function as

\[ f_c(z) = \frac{z}{v(z - v)} \exp a(z), \]
where $a(z)$ should satisfy the following conditions.

(i) Normalization condition. From the Laurent expansion (36),
\[
\lim_{z \to v} (z - v)f_c(z) = 1, \quad \text{i.e.,} \quad a(v) = 0.
\] (38)

(ii) Boundary condition. Since $f_c(z)$ maps $C_1, \ldots, C_n$ onto the circular slits $S_1, \ldots, S_n$ of the radii $r_1, \ldots, r_n$,
\[
|f_c(z)| = r_m, \quad \text{i.e.,} \quad \text{Re} \ a(z) - \log r_m = -\log \left| \frac{z}{v(z - v)} \right|,
\]
\[
z \in C_m, \quad m = 1, \ldots, n.
\] (39)

The problem is now to find $a(z)$ satisfying (38), (39), together with $r_m$.

We approximate $a(z)$ by (32), and obtain the single-valuedness condition (33), the normalization condition (34), and the collocation condition
\[
\sum_{l=1}^{N_1} \sum_{j=1}^{N_l} Q_{lj} \log \left| \frac{z_{mk} - \zeta_{lj}}{v - \zeta_{lj}} \right| - \log R_m = -\log \left| \frac{z_{mk}}{v(z_{mk} - v)} \right|,
\]
\[
z_{mk} \in C_m, \quad k = 1, \ldots, N_m, \quad m = 1, \ldots, n.
\] (40)

Equations (33), (40) constitute a set of simultaneous linear equations in $Q_{ij}$, $\log R_m$.

3.3 The radial slit domain

**Theorem 6** There exists a unique analytic function $f_r(z)$ such that it (i) conformally maps $D$ onto the radial slit domain and (ii) satisfies $f_r(0) = 0$, $f_r(v) = \infty$ with the Laurent expansion near $z = v$ of the form
\[
f_r(z) = \frac{1}{z - v} + c_0 + c_1(z - v) + c_2(z - v)^2 + \cdots.
\] (41)

We express the mapping function as
\[
f_r(z) = \frac{z}{v(z - v)} \exp(ia(z)),
\] (42)

where $a(z)$ should satisfy the following conditions.

(i) Normalization condition. From the Laurent expansion (41),
\[
\lim_{z \to v} (z - v)f_r(z) = 1, \quad \text{i.e.,} \quad a(v) = 0.
\] (43)

(ii) Boundary condition. Since $f_r(z)$ maps $C_1, \ldots, C_n$ onto the radial slits $S_1, \ldots, S_n$ of the arguments $\theta_1, \ldots, \theta_n$,
\[
\arg f_r(z) = \theta_m, \quad \text{i.e.,} \quad \text{Re} \ a(z) - \theta_m = -\arg \frac{z}{v(z - v)},
\]
\[
z \in C_m, \quad m = 1, \ldots, n.
\] (44)

The problem is now to find $a(z)$ satisfying (43), (44), together with $\theta_m$.

We approximate $a(z)$ by (32), and obtain the single-valuedness condition (33), the normalization condition (34), and the collocation condition
\[
\sum_{l=1}^{N_1} \sum_{j=1}^{N_l} Q_{ij} \log \left| \frac{z_{mk} - \zeta_{lj}}{v - \zeta_{lj}} \right| - \theta_m = -\arg \frac{z_{mk}}{v(z_{mk} - v)},
\]
\[
z_{mk} \in C_m, \quad k = 1, \ldots, N_m, \quad m = 1, \ldots, n.
\] (45)
Equations (33), (45) constitute a set of simultaneous linear equations in $Q_{ij}, \Theta_m$.

### 3.4 A unified scheme

We assume that each $C_l$ is starlike with respect to $\zeta_0$, and using (33) rewrite (32) and (34) into

$$A(z) = Q_0 + \sum_{i=1}^{n} \sum_{j=1}^{N_i} Q_{ij} \frac{z - \zeta_{ij}}{z - \zeta_0}, \quad Q_0 = -\sum_{i=1}^{n} \sum_{j=1}^{N_i} Q_{ij} \frac{v - \zeta_{ij}}{v - \zeta_0},$$

which is continuous in $D$ when the principal value is used. We now have a unified scheme for the numerical conformal mappings.

**Scheme 2**  The approximate mapping functions are given by

$$F_\theta(z) = \frac{1}{z - v} + i e^{i\theta} A(z),$$

$$F_c(z) = \frac{z}{v(z - v)} \exp A(z), \quad (46)$$

$$F_i(z) = \frac{z}{v(z - v)} \exp (iA(z)),$$

$$A(z) = Q_0 + \sum_{i=1}^{n} \sum_{j=1}^{N_i} Q_{ij} \frac{z - \zeta_{ij}}{z - \zeta_0}, \quad Q_0 = -\sum_{i=1}^{n} \sum_{j=1}^{N_i} Q_{ij} \frac{v - \zeta_{ij}}{v - \zeta_0}, \quad (47)$$

where the unknown coefficients $Q_{ij}$, together with the constants $P_m, R_m$ or $\Theta_m$ are determined by solving the linear equations

$$\sum_{j=1}^{N_i} Q_{ij} = 0, \quad l = 1, \ldots, n, \quad (48)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{N_i} Q_{ij} \log \left| \frac{z_{mk} - \zeta_{ij}}{v - \zeta_{ij}} \right| - S_m = -t_{mk}, \quad (49)$$

$$z_{mk} \in C_m, \quad k = 1, \ldots, N_m, \quad m = 1, \ldots, n, \quad (50)$$

where

$$S_m = \begin{cases} P_m & \log R_m, \\
\Theta_m & t_{mk} = \begin{cases} \text{Im} \frac{e^{-i\theta}}{z_{mk} - v} & \text{for } F_\theta(z) \\
\log \left| \frac{z_{mk}}{v(z_{mk} - v)} \right| & \text{for } F_c(z). \\
\arg \frac{z_{mk}}{v(z_{mk} - v)} & \text{for } F_i(z) \end{cases} \end{cases} \quad (51)$$

The coefficient matrix is the same for all the mapping functions. In addition,

$$F_\theta(\infty) = ie^{i\theta} Q_0, \quad Q_0 = -ie^{-i\theta} F_\theta(\infty),$$

$$F_c(\infty) = (1/v) \exp Q_0, \quad Q_0 = \log(v F_c(\infty)),$$

$$F_i(\infty) = (1/v) \exp(iQ_0), \quad Q_0 = -i \log(v F_i(\infty)).$$

### 3.5 An example

**Example 2**  The problem domain $D$ is the exterior of three disks,

$$C_l : |z - \zeta_0| = \rho_l, \quad \rho_1 = 1.5, \quad \rho_2 = 1.0, \quad \rho_3 = 0.5,$$

$$\zeta_0 = 2 \exp \frac{(2l - 1)\pi i}{3}, \quad l = 1, 2, 3,$$
Fig. 5 Numerical conformal mappings onto the (a) parallel \((\theta = \pi/4)\), (b) circular and (c) radial slit domains.

and the normalization point is \(v = 4\).

Collocation points and charge points are placed by (24), and errors are estimated by (25), (26) and (27).

Fig. 5 illustrates the numerical conformal mapping of \(D\) onto the (a) parallel, (b) circular and (c) radial slit domains. Table 2 shows numerical results of the conformal mapping. We can see that high accuracy is achieved. The relatively large errors \(\epsilon_{F_{d}}\) and \(\epsilon_{F_{c}}\) on \(C_{1}\) are due to the singularity of the logarithmic function in (51), where \(C_{1}\) comes nearest the singular point \(z = 0\) in this case. Fig. 6 shows contour lines of (a) \(\operatorname{Im}(e^{-\imath H_{0}}(z))\), (b) \(\operatorname{Im}(-\log F_{c}(z))\) and (c) \(\operatorname{Im}(\arg F_{c}(z))\), which illustrate streamlines of (a) a dipole source, (b) a vortex pair and (c) a point source and sink pair flows past three cylindrical objects. The dipole is at \(z = v\), and the vortex and the source and sink pairs are at \(z = 0, z = v\).

4. CONCLUDING REMARKS

We have presented a unified scheme for numerical conformal mappings of unbounded multiply connected domains onto the parallel, circular and radial slit domains under two different conditions. The basic idea is to approximate analytic functions by a linear combination of complex logarithmic functions based on the charge simulation method or the fundamental solution method. It gives approximate mapping functions of simple form with high accuracy. Numerically, we have only to solve a system of linear equations with the same coefficient matrix under each condition. The method is suited for domains of curved boundaries in general though the case of circular domains
Table 2 Numerical results of the conformal mapping (Example 2, $N = 64$, $q = 0.8$, $\kappa = 1.2 \times 10^8$, $\theta = \pi/4$)

<table>
<thead>
<tr>
<th></th>
<th>$F_\eta(z)$</th>
<th>$F_c(z)$</th>
<th>$F_r(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_1$</td>
<td>$C_1$</td>
<td>$C_1$</td>
</tr>
<tr>
<td></td>
<td>$3.4E-08$</td>
<td>$3.8E-05$</td>
<td>$2.5E-05$</td>
</tr>
<tr>
<td></td>
<td>$7.4E-10$</td>
<td>$9.8E-09$</td>
<td>$5.1E-09$</td>
</tr>
<tr>
<td>$C_2$</td>
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<td>$6.8E-08$</td>
<td>$6.1E-08$</td>
</tr>
<tr>
<td></td>
<td>$3.2E-10$</td>
<td>$1.9E-08$</td>
<td>$1.7E-08$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>$3.3E-09$</td>
<td>$1.4E-08$</td>
<td>$5.8E-09$</td>
</tr>
<tr>
<td></td>
<td>$9.9E-10$</td>
<td>$1.5E-08$</td>
<td>$1.5E-09$</td>
</tr>
</tbody>
</table>

was illustrated as a typical example.

We proposed similar methods for conformal mappings (i) onto the linear slit domain\(^{24}\) where the angles of each slit can arbitrarily be assigned, which was treated by Shiba\(^{25}\) as a generalization of the parallel slit domain, and (ii) onto the circular and radial slit domain\(^{26}\) where circular and radial slits can simultaneously exist in the same domain, which was treated by Koebe\(^{8}\) as a generalization of the circular and the radial slit domains. Ogata proposed the charge simulation method for a compressible fluid flow problem\(^{27}\). Ogata et al. proposed also the fundamental solution method for the conformal mapping of periodic structure domains\(^{28}\), and for Stokes flow problems with obstacles in a periodic array\(^{29-31}\), where various types of periodic fundamental solutions were introduced.

Conformal mappings are important in science and engineering, especially in potential flow problems. Sakajo\(^{32}\) pointed out that numerical conformal mappings of multiply connected domains may play an important role in the study of dynamics of point vortices\(^{33}\), which is applicable to many problems such as environmental flows, bio-fluids, etc. Conformal mappings onto the spiral slit domain\(^{9}\) and in particular the circular domain\(^{4}\) are interesting problems left for future studies.

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REFERENCES

Fig. 6 (a) A dipole source, (b) a vortex pair and (c) a point source and sink pair flows past three obstacles.


