Application of Measurement-Integrated Simulation to Compressible Fluid Analysis

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In Japan Aerospace Exploration Agency (JAXA), work is presently underway for the complementary use of Experimental Fluid Dynamics (EFD) and Computational Fluid Dynamics (CFD). The main purpose is to improve the utility of both EFD and CFD and the data assimilation technique is expected to be a promising approach. In this study, a coupled system of high-speed CFD solver (FaSTAR) and measurement-integrated simulation (MIS) algorithm is constructed, and an identical twin experiment of flow field around a 2-dimensional airfoil with difference boundary condition is performed.

1. Introduction

At the Japan Aerospace Exploration Agency (JAXA), the complementary use of Experimental Fluid Dynamics (EFD) and Computational Fluid Dynamics (CFD) is currently being studied.

In flow analysis, EFD and CFD are commonly used. EFD is a direct method to obtain the state of real flow phenomena, and its reliability is ensured by calibration techniques. However, there are differences from real flight condition, such as Reynolds number, the walls and support of the wind tunnel, and model deformation. Moreover, it is impossible to obtain complete information about the flow state both spatially and temporally. On the other hand, CFD provides the full state of flow phenomena on grid points of the computational domain. However, its reliability is always a concern, especially for complicated phenomena such as turbulence, transition, separation, and reacting flow; so validation by experiment is required. These advantages and disadvantages of the two approaches are summarized in Table 1. A primary aim of EFD/CFD integration is to improve the accuracy and reliability of data by the complementary use of EFD and CFD.

Hayase et al. proposed a measurement-integrated simulation (MIS) algorithm, which was achieved by applying a flow observer (Figure 1). The MIS algorithm is a flow simulation scheme modified by adding the feedback signal proportional to the difference between the measurement of the real flow and the simulation. The main feature of the MIS algorithm, which distinguishes it from other existing observers, is that it uses the CFD scheme as a mathematical model of the physical flow. The validity of the MIS algorithm has been proven in several applications such as the Karman vortex street behind a
square cylinder, blood flow in an aneurismal aorta, and the reproduction of turbulent flow including fluctuations in a square duct.

However, the MIS algorithm has one problem with regard to the design of the feedback law, observer theory cannot be directly applied to the design. General observer theory assumes that the target system is described by the general state space model, or a set of the first-order ordinary differential equations with respect to the state variables, but the basic equations of the MIS algorithm, the discretized form of the Navier-Stokes equations with respect to the flow variables, are nonlinear and do not have minimal dimensions. Therefore, the feedback laws in former studies were designed by trial and error based on physical considerations for the feedback terms. One of the trials to design the feedback signal involved the eigenvalue analysis of linearized error dynamics, and some possibilities using this method have been shown. However, the computational cost is too high to use in various applications.

In this study, a high-speed CFD solver (FaSTAR), was coupled with an MIS algorithm based on an extended Kalman filter (EKF) algorithm, and a twin experiment for investigating the flow field around a two-dimensional airfoil with the difference boundary condition was performed. The twin experiment is a uses pseudo-measurement data derived from simulations instead of actual measurements. The experiment in this study is conducted to reproduce the pressure field using velocity pseudo-measurement data.

Table 1: Advantages and disadvantages of EFD and CFD

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<thead>
<tr>
<th></th>
<th>Advantage</th>
<th>Disadvantage</th>
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<tr>
<td>EFD</td>
<td>A direct way to obtain real flow phenomena</td>
<td>Differences from real flight conditions</td>
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<tr>
<td></td>
<td>Reliability is ensured by calibration techniques</td>
<td>Difficulty in obtaining complete information</td>
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<tr>
<td>CFD</td>
<td>Provides the full state of flow phenomena</td>
<td>Reliability is always a concern</td>
</tr>
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<td>Validation by experiment is required</td>
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2. Measurement-Integrated Simulation algorithm

The governing equations of the CFD model are generally written as

$$\frac{d\mathbf{Q}_N}{dt} = f(\mathbf{Q}_N)$$

(1)

where $\mathbf{Q}_N$ is a $5N$-dimensional conservative vector

$$\mathbf{Q}_N = \begin{pmatrix} \mathbf{Q}_1^T & \mathbf{Q}_2^T & \cdots & \mathbf{Q}_i^T & \cdots & \mathbf{Q}_N^T \end{pmatrix}^T$$

(2)

and $N$ and $\mathbf{Q}_i$ are the number of grid points and the conservative vector at grid $i$, respectively.

$$\mathbf{Q}_i = \begin{pmatrix} \rho_i & (\rho u)_i & (\rho v)_i & (\rho w)_i & e_i \end{pmatrix}^T$$

(3)

where $\rho$ is density, $u, v, w$ are velocity components, and $e$ is total energy. When the vector constructed from measurement data is denoted by $\mathbf{y}$, the basic equations of MIS are

$$\frac{d\mathbf{Q}_M}{dt} = f(\mathbf{Q}_N) + f_{MIS}$$

(4)

and

$$f_{MIS} = K \left( \mathbf{y} - h(\mathbf{Q}) \right)$$

(5)

where $h$ is the observation function, which indicates the correspondence between the measurement and numerical simulation, is the feedback matrix and is designed by trial and error in the former applications, and $f_{MIS}$ is the feedback term in the MIS algorithm and is denoted by a linear function of the difference of flow states between the measurement and numerical simulation $^{3-5}$.

The fundamental equation of the MIS algorithm is very similar to the equations of the EKF algorithm in continuous time. When the numerical model is derived as Eq.(1), the governing equations of the EKF for a continuous time system $^8$ are

$$\frac{d\mathbf{Q}}{dt} = f(Q) + K \left( \mathbf{y} - h(Q) \right)$$

(6)

$$K = \frac{1}{\Delta t} PH^T \left( HPH^T + R \right)^{-1}$$

(7)

$$\frac{d\mathbf{P}}{dt} = FP + PF^T - \Delta tKP + \Delta tGQG$$

(8)

where $F$ and $H$ are Jacobi matrices of $f(Q)$ and $h(Q)$, respectively, and $P, Q$, and $R$ are covariance matrices of the state estimation error, system noise, and measurement error, respectively.

These algorithms differ in two aspects. First, the MIS algorithm reproduces the flow field completely from limited measurement data without consideration of the measurement error, whereas the EKF considers the errors and noise. Second, the feedback matrix of the EKF can be designed using the Riccati equation. Basically, the MIS is a type of EKF without measurement error consideration.

The design method of the feedback matrix for the EKF algorithm can therefore be applied to the MIS algorithm as an EKF without measurement error consideration. $H^*$, the pseudo-inverse matrix of $H$, is
used in this method. $H^*$ is assumed as
\[ H^*H = I \]  
where $I$ is the unit matrix. The governing equations are a revised Eqs. (7) without measurement error consideration,
\[ K = \frac{1}{\Delta t} PH^T \left( HPH^T \right)^{-1} = \frac{1}{\Delta t} \left( H^*H \right) PH^T \left( HPH^T \right)^{-1} \]
\[ \Rightarrow K = \frac{1}{\Delta t} H^* \left( HPH^T \right) \left( HPH^T \right)^{-1} = \frac{1}{\Delta t} H^*. \]

Eq. (6) is of the same form of the ordinary governing equations of the MIS algorithm, Eqs. (4) and (5).

3. Numerical Experiment

3.1. Twin experiment

Twin experiments are a method of numerical experimentation commonly used for benchmark tests in data assimilation techniques. They use pseudo-measurement data derived from simulations instead of actual measurements. Figure 2 shows a schematic diagram of the twin experiment. The standard solution is original CFD data. Generally, the benchmark test of the integrated analysis system is a comparison of the simulated data with the standard solution.

3.2. Computational conditions

The twin experiment deals with the flow field around a two-dimensional airfoil, as shown in Figure 3. The flow field is assumed to be steady; hence, the simulated data are obtained as an asymptotic solution of Eq. (4). The details are given in Section 3.3. To achieve this, the high-speed CFD solver, FaSTAR, is coupled to the MIS algorithm. A uniform flow (Mach number = 0.8) with angle of attack $\alpha$ to the airfoil is used for the outer boundary condition. In the twin experiment, the angle of attack in the standard solution is $3^\circ$, and the integrated computation is carried out made using intentionally different angles of

![Figure 2: Schematic diagram of twin experiment](image-url)
attack. It is noted that the uniform flow used for outer boundary condition depends on the angle of attack $\tilde{\alpha}$. Then the computation with different angles of attack refer to the computation with different boundary condition. A no-slip condition is applied to the airfoil. The sample cases used in the present experiment are listed in Table 2. The standard solution is the velocity field over the entire computational domain. All computations were carried out using the JAXA Supercomputer System (JSS).

3.3. Steady Computation with Unsteady Computation Scheme

The flow chart of this computation is shown in Figure 4. In the numerical simulation of the flow field, the steady flow field is computed as the convergence solution of the unsteady computation scheme. Then, the governing equation, Eqs. (4) and (5), for the unsteady system is used for conducting the twin experiment for the steady flow field.

At first, Eqs. (4) and (5) is revised to the discretized form of time,

$$ Q_{n+1} = Q_n + \Delta t F_{Q_n} + \Delta t K_n \left( y_n - h(Q_n) \right). \tag{11} $$

The underlined terms are computed using the CFD solver. The flow field is then checked for convergence. If it does not converge, the computation proceeds to the next time step. Thus, the converged solution is the steady flow field.

![Diagram of twin experiment setup](image)

**Figure 3: Setup of this twin experiment**

<table>
<thead>
<tr>
<th>Table 2 Computational Condition</th>
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<tbody>
<tr>
<td>Number of Cell</td>
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<tr>
<td>Mach number of uniform flow</td>
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<tr>
<td>Angle of Attack</td>
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<td></td>
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<tr>
<td></td>
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<tr>
<td>Reynolds number</td>
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<td>Residual at convergence</td>
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3.4. Feedback gain matrix design

The feedback gain matrix $K$ in Eq. (5) is derived as a pseudo-inverse matrix of the Jacobi matrix of the observation function.

If the velocity components are employed as measurement data, $y$ in Eq. (5) is

$$\mathbf{y} = \begin{pmatrix} y_1^\top & y_1^\top & \cdots & y_i^\top & \cdots & y_N^\top \end{pmatrix}^\top$$

$$\mathbf{y}_i = \begin{pmatrix} u_i & v_i \end{pmatrix}^\top$$  \hspace{1cm} (12)

and the Jacobi matrix $H$ of the observation function $h$ is

$$H = \frac{\partial \mathbf{y}}{\partial \mathbf{Q}_N} = diag \left( H_1, H_2, \ldots, H_i, \ldots, H_N \right)$$  \hspace{1cm} (14)

$$H_i = \frac{\partial \mathbf{y}_i}{\partial \mathbf{Q}_i} = \frac{\partial (u_i, v_i)^\top}{\partial (\rho_i, (\rho u)_i, (\rho v)_i, (\rho w)_i, e_i)^\top}$$

$$= \frac{\partial (u_i, v_i)^\top}{\partial (\rho_i, u_i, v_i, w_i, p_i)^\top} \frac{\partial (\rho_i, u_i, v_i, w_i, p_i)^\top}{\partial (\rho_i, (\rho u)_i, (\rho v)_i, (\rho w)_i, e_i)^\top} = M_i L_i,$$  \hspace{1cm} (15)

where
\[ L_i = \frac{\partial (\rho, u, v, w, p, e)^T}{\partial (\rho, \rho u, \rho v, \rho w, e)^T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{u_i}{\rho_i} & 1 & 0 & 0 & 0 \\ -\frac{v_i}{\rho_i} & 0 & 1 & 0 & 0 \\ -\frac{w_i}{\rho_i} & 0 & 0 & 1 & 0 \\ \frac{1}{2}(\gamma - 1)(u_i^2 + v_i^2 + w_i^2) & -(\gamma - 1)u_i & -(\gamma - 1)v_i & -(\gamma - 1)w_i & \gamma - 1 \end{pmatrix} \]

and

\[ M_i = \frac{\partial (u, v)^T}{\partial (\rho, u, v, w, p)^T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \] (17)

Further, \( L_i \) is the transformation matrix from the conservative vector to the primitive vector, and the inverse matrix of \( L_i \), \( L_i^{-1} \) is the transformation matrix from the primitive vector to the conservative state vector, which is given by

\[ L_i^{-1} = \frac{\partial (\rho, \rho u, \rho v, \rho w, e)^T}{\partial (\rho, u, v, w, p)^T} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ u_i & \rho_i & 0 & 0 & 0 \\ v_i & 0 & \rho_i & 0 & 0 \\ w_i & 0 & 0 & \rho_i & 0 \\ \frac{1}{2}(u_i^2 + v_i^2 + w_i^2) & \rho_i u_i & \rho_i v_i & \rho_i w_i & \frac{1}{\gamma - 1} \end{pmatrix} \] (18)

and \( M_i^{-1} \), the pseudo-inverse matrix of \( M_i \), is

\[ M_i^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \] (19)

Then, \( H^* \), the pseudo-inverse matrix of \( H \), is

\[ H^* = \text{diag} \left( H_1^*, H_2^*, \ldots, H_l^*, \ldots, H_n^* \right) \] (20)

\[ H_i^* = L_i^{-1} M_i^{-1} = \begin{pmatrix} 0 & \rho_i & 0 & 0 \\ 0 & 0 & \rho_i & 0 \end{pmatrix}. \] (21)

Finally, we obtain the feedback gain matrix \( K \) as
\[ K = \text{diag} \left( K_1, K_2, \ldots, K_i, \ldots, K_N \right) \] 

\[ K_i = \frac{1}{\Delta t} H_i^T = \frac{1}{\Delta t} \begin{pmatrix} 0 & \rho_i & 0 & 0 \\ 0 & 0 & 0 & \rho_i \mu_i \\ 0 & 0 & 0 & \rho_i \nu_i \end{pmatrix} \] 

4. Result of Experiment

Figure 5 shows a comparison of the lift coefficients \( C_L \) with and without the MIS. In the distributions, the difference between the ordinary simulation (square) and standard solution (solid line) increases as the attack angle shifts from the standard solution (broken line). On the other hand, the difference between the data assimilation (circle) and the standard solution is still small, and is unaffected by shifts in the attack angle. As shown in this figure, the computation without the MIS is equivalent to a simple CFD computation, and its difference from the standard solution increases as the angle of attack recedes from the target value. On the other hand, the computations with the MIS successfully reproduce the standard solution.

The reconstructed pressure field with data assimilation is shown in Figures 6 and 7, with the largest difference around \( \alpha = 0^\circ \). Figure 6 shows the pressure distribution. The distribution of the standard solution is asymmetrical, but the distribution for the ordinary simulation is symmetrical. This difference is brought about by the attack angle. On the other hand, the distribution of the data assimilation when using the MIS algorithm is very similar to that of the standard solution although a different angle of

![Figure 5 Lift coefficient with angle of attack of standard solution, ordinary simulation and data assimilation with MIS algorithm.](image-url)
attack from that used for the standard solution is used. The difference between the ordinary simulation and data assimilation is attributable to the velocity measurement data used by the MIS algorithm. Figure 7 shows the pressure coefficient distribution on the airfoil. The distribution when the standard solution (broken line) is used differs from that when the ordinary simulation (chain line) is used. However, the distributions of the standard solution and the data assimilation are in good agreement.

Finally, the comparison of the CPU time for the convergence of the steady flow field is listed in Table 3. The number of steps involved in the convergence of the computation with the MIS is less than that without the MIS, and the CPU time with the MIS is less than that without the MIS. This implies that the MIS can be applied for large computation such as those required for three-dimensional problems.

Figure 6 Pressure distribution of reconstructed from velocity pseudo-measurement data.
Figure 7 Pressure coefficient on the airfoil of standard solution, ordinary simulation and data assimilation.

Table 3 Computational Cost

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<tr>
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<th>Without MIS</th>
<th>With MIS</th>
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<tbody>
<tr>
<td>CPU time for 1step</td>
<td>0.843</td>
<td>1.223</td>
</tr>
<tr>
<td>(Averaged of 10000 steps)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of step for convergence</td>
<td>403</td>
<td>79</td>
</tr>
<tr>
<td>CPU time for convergence</td>
<td>364.67</td>
<td>114.38</td>
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</table>

5. Conclusion

In this study, a high-speed CFD solver (FaSTAR) was coupled with an MIS algorithm, based on an EKF method, and a twin experiment of the flow field around a two-dimensional airfoil with the difference boundary condition was performed.

In the reconstruction of the unmeasured pressure field using the measured velocity field in all computational areas, the pressure distribution of data assimilation using the MIS algorithm is very
similar to that of the standard solution despite the large difference in the attack angle between the data assimilation and standard solution.

Future work will examine the effect of the measurement error and the measurement area. In this two-dimensional consideration, the computational cost of data assimilation with the MIS algorithm is similar to the cost of the ordinary simulation. Thus, we can consider the three-dimensional analysis with a realistic computational cost.

REFERENCES


