Invariant Manifolds and Lagrangian Coherent Structures in the Planar Circular Restricted Three-Body Problem

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For the sake of spacecraft mission design, it is indispensable to develop a low energy transfer of spacecrafts using very little fuel for interplanetary transport network. The Planar Circular Restricted Three-Body Problem (PCR3BP) has been a fundamental tool for the analysis of such a space mission design. In this paper, we explore stable and unstable invariant manifolds associated with the collinear Lagrange points \( L_1, L_2 \) of the PCR3BP, in which geometrical structures of the invariant manifolds are clarified on a Poincaré section. Further, we compute the Finite Time Lyapunov Exponent fields (FTLE fields) to obtain Lagrangian Coherent Structures (LCS) as the ridges of the FTLE fields. In particular, we compare the LCS with the invariant manifolds on the Poincaré section from the viewpoint of the numerical integration times.

1. INTRODUCTION

The motion of a spacecraft under gravitational fields of solar planets may be properly analyzed by the Planar Circular Restricted Three-Body Problem (PCR3BP), assuming that the mass of the spacecraft may be negligible in comparison with the masses of large (primary) planets such as the Sun and Jupiter and also that the orbits of the planets and the spacecraft are restrained on the same plane. For the space mission design, the idea of the Interplanetary Transport Network has been focused on, in which a low-energy trajectory may be generated by patching invariant orbits of the PCR3BP\(^{3,4}\). As is well known, there exist five equilibrium points called the Lagrange points, where three of the points are collinear along the \( x \) axis, while the other points form an equilateral triangle with two primaries. The low-energy pathways by connecting cylindrical stable and unstable invariant manifolds are called tubes\(^1\), which are associated with unstable periodic orbits called Lyapunov orbits near the collinear Lagrange points.

In this paper, we will analyze the stable and unstable invariant manifolds of the PCR3BP for the case in which a spacecraft explores under the gravitational fields of the Sun and Jupiter. First we will compute the invariant manifolds associated with \( L_1 \) and \( L_2 \). Then, we will show the geometrical structures of the stable and unstable invariant
manifolds on a Poincaré section in the Sun region. We will also focus on the Lagrangian Coherent Structures (LCS) as a tool to demonstrate the geometrical structures of the invariant manifolds. To do this, we will compute the Finite Time Lyapunov Exponents fields (FTLE fields) by an adaptive-time stepping Runge-Kutta-Fehlberg method, and then, it is shown that the LCS can be given as a ridge of the FTLE fields. It was shown by Gawlik et al 5) that the invariant manifolds of the Planar (Circular and Elliptic) Restricted Three-Body Problems may be numerically detected as the LCS; however, they just computed the FTLE fields for the short time interval. Then, in this paper, we will explore the LCS of the PCR3BP in details by computing the FTLE fields for the long time interval on the Poincaré section.

2. THE PLANAR CIRCULAR RESTRICTED THREE-BODY PROBLEM

2.1 MATHEMATICAL MODEL

The motion of a spacecraft submitted to the attraction of two planets can be analyzed in the context of the Planar Circular Restricted Three-Body Problem (PCR3BP) as in Fig. 1, in which we assume that two bodies of large primary planets move along a circular orbit with constant angular velocity around the common mass center and also that a spacecraft with an infinitesimal mass moves in the plane of the circle subject to the gravitational forces due to the two primary planets. In this paper, we assume that the collision points between the spacecraft and the planets are removed. Let $m_1$ and $m_2$ be the masses of the primary planets.

![Fig. 1 PCR3BP](image1)

![Fig. 2 Hill's region](image2)

The nondimensional system can be made by choosing the unit of mass as $m_1 + m_2$, the unit of length as the distance between $m_1$ and $m_2$ and the unit of time so that the planets period becomes $2\pi$. By the nondimensionalization, the constant of gravitation $G$ can be set to 1. We can define the mass parameter by $\mu = m_2/(m_1 + m_2)$, and it follows that equations of motion in the rotating frame with the planets may be given, in local
coordinates \((x, y, v_x, v_y)\) of the velocity phase space \(M\), by

\[
\begin{aligned}
\dot{x} &= v_x, \\
\dot{y} &= v_y, \\
\dot{v}_x - 2v_y &= -\bar{U}_x, \\
\dot{v}_y + 2v_x &= -\bar{U}_y,
\end{aligned}
\tag{1}
\]

where \(\bar{U}(x, y)\) denotes the effective potential as

\[\bar{U}(x, y) = -\frac{1}{2}(x^2 + y^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} - \frac{1}{2}\mu(1 - \mu).\]

In the above, \(\bar{U}_x\) and \(\bar{U}_y\) indicate the partial derivatives by \(x\) and \(y\).

It follows from the equations of motion that there exist five equilibrium (Lagrange) points on the \(x\) axis \((L_1, L_2, L_3)\) as well as on the lines of the regular triangle \((L_4, L_5)\). The collinear Lagrange points \((L_1, L_2, L_3)\) are at the bottom of the effective potential.

2.2 HILL’S REGION

The total energy is given by the sum of the kinetic energy and the effective potential as

\[E(x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + v_y^2) + \bar{U}(x, y),\]

which is preserved in the PCR3BP. Fixing the energy \(E\) to some value \(E_0\), one can define a subset \(D \subset M\), called energy surface, by

\[D = \{(x, y, v_x, v_y) \in M \mid E(x, y, v_x, v_y) = E_0\}.\]

Then, one can also define Hill’s region, where the spacecraft can move, by projecting the energy surface \(D\) onto the \(x-y\) plane.

In this paper, we consider the mass of the Sun (S) by \(m_1\) and the one of Jupiter (J) by \(m_2\). Then, we set the parameter \(\mu = 9.537 \times 10^{-4}\), where the energy \(E = E_0(= -1.51686)\) is so chosen that the spacecraft can be transferred between the planets. The lines \(x = x_1(= 0.93239043)\) and \(x = x_2(= 1.06880367)\) split the Hill’s region into three regions; the vicinity of Sun (Sun region), the vicinity of Jupiter (Jupiter region) as well as the exterior region, where \(x_1, x_2\) are the \(x\)-coordinates of \(L_1, L_2\). We illustrate the Hill’s region and the forbidden region in Fig.2.

3. IN Variant Manifolds and the Poincaré Map

3.1 Lyapunov Orbits and Invariant Manifolds

In order to analyze the vicinity of the invariant manifolds associated with the collinear Lagrange points, one can linearize the equations of motion (1) around the equilibrium point \((x_e, 0, 0, 0)\). Associated with the eigenvalues \(\pm \lambda\) and \(\pm i\nu\) of the linearized equations, one may introduce the new coordinates \((\xi, \eta, \zeta_1, \zeta_2)\) regarding the eigenvectors of
the linearized equations and we finally obtain the linearized equations\(^1\):\(^4\):

\[
\begin{align*}
\dot{\xi} &= \lambda \xi, \\
\dot{\eta} &= -\lambda \eta, \\
\dot{\zeta}_1 &= \nu \zeta_2, \\
\dot{\zeta}_2 &= -\nu \zeta_1,
\end{align*}
\]

(3)

together with the energy

\[
E_l = \lambda \xi \eta + \frac{\nu}{2} (\zeta_1^2 + \zeta_2^2).
\]

Fig. 3 Projections onto the $\xi - \eta$ space  
Fig. 4 Projections onto the $\zeta_1 - \zeta_2$ space

As shown in Fig.3 and Fig.4, a collinear Lagrange point has the saddle x center structure and there exists the unstable orbit near the Lagrange point called the Lyapunov orbit. Further, there are stable and unstable invariant manifolds associated with the Lyapunov orbit; namely, the stable invariant manifold may tend asymptotically to the Lyapunov orbit, and the unstable invariant manifold may leave from the Lyapunov orbit, as illustrated in Fig.5. It follows from the stable and unstable invariant manifolds that the orbits can be divided into transit and non-transit orbits. These invariant manifolds are called "tubes" since the manifolds are homeomorphic to the cylinder $S \times \mathbb{R}$. In this paper, we primarily focus on the Sun region and we illustrate that the stable manifold $W^s_{L_1}$ tends to the Lyapunov orbit of the $L_1$ from Sun region and also that the unstable manifold $W^u_{L_1}$ leaves from the Lyapunov orbit of the $L_1$ in Fig.6.

3.2 TRANSIT AND NON-TRANSIT ORBITS

Setting the Poincaré section $U$ as

\[
U = \{(x, y, v_x, v_y) \in D \mid y = 0, \ x < 0, \ v_y(x, v_x, E_0) < 0\},
\]

the first intersection of the stable and unstable invariant manifolds on the Poincaré section, namely, $\Gamma^{1,s}, \Gamma^{1,u}$ may be illustrated in Fig.7. Notice that there exist four homoclinic points, which are the intersections of the stable and unstable invariant manifolds. These invariant manifolds play an essential role in the spacecraft mission design, because if the spacecraft starts from a point inside the invariant manifolds on the Poincaré sec-
tion, the orbit may become a transit orbit, in which the spacecraft is to transfer from the Sun region to Jupiter region. Otherwise the orbit is to be non-transit and hence the spacecraft may remain in the Sun region. We illustrate the subsets of the stable and unstable manifolds on $U$ in Fig.7. Set the initial values $p_t$ and $p_n$ on $U$, where $p_t$ is an inside point $(x, v_x) = (-0.6, -0.1)$ of the stable manifold and $p_n$ an outside point $(x, v_x) = (-0.7, -0.1)$ of the stable manifold. Then, the orbit with the initial point $p_t$ is to be transit, while the orbit with the initial point $p_n$ non-transit as in Fig.8.

3.3 Poincaré Maps

Let $\phi : \mathbb{R} \times D \rightarrow D$ be the flow of the dynamical system given in equation (1) and let $\tau_U(p)$ for $p \in U \subset D$ be the time taken for a trajectory $\phi(t, p)$ to the first return to $U$. Then, we define the Poincaré map by

$$\Pi : U \rightarrow U; \quad p \mapsto \Pi(p) = \phi_{\tau_U(p)}(p).$$

We show the Poincaré plots on the Poincaré section $U$ with 250 initial points, which
are randomly chosen in $U$ in Fig.9. Apparently, one can see that there exist chaotic seas as well as the KAM tori on the Poincaré section $U$. In particular, we denote by the subset $\Gamma^i,\sigma$ the $i$-th intersections of the stable manifold with $U$ through the Poincaré map.

Fig. 9 Invariant manifolds on the Poincaré section

4. LAGRANGIAN COHERENT STRUCTURES IN THE PRC3BP

4.1 FINITE TIME LYAPUNOV EXPONENT FIELDS

Following the paper by Shadden, Lekien and Marsden$^6$, let us review the notions of the Finite Time Lyapunov Exponent Fields and the Lagrangian Coherent Structures.

Let $D$ be a subset of a phase space $M$. Given a dynamical system as

$$
\begin{cases}
\dot{x}(t; t_0, x_0) = v(x(t; t_0, x_0), t), \\
x(t_0; t_0, x_0) = x_0,
\end{cases}
$$

where $x(t; t_0, x_0)$ is a smooth solution curve starting at an initial point $x_0 \in D$ at time $t_0$ and $v(x, t)$ is a given vector field. Then, the point $x_0$ moves to another point after a time interval $T$ by the flow map as

$$\phi^{t_0+T}_{t_0} : D \to D; \quad x_0 \mapsto \phi^{t_0+T}_{t_0}(x_0) = x(T; t_0, x_0).$$

We recall that the Finite Time Lyapunov Exponent (FTLE) is a finite time average of the maximum expansion rate for a pair of particles, with neighboring in the initial time, advected in the flow. Now, we let a perturbed point $y = x + \delta x(0)$, where $\delta x(0)$ is infinitesimal. After the time interval $T$, the perturbation is to be

$$\delta x(T) = \phi^{t_0+T}_{t_0}(y) - \phi^{t_0+T}_{t_0}(x) = \frac{d\phi^{t_0+T}_{t_0}(x)}{dx} \cdot \delta x(0) + O(\|\delta x(0)\|^2).$$

Neglecting the higher order terms $O(\|\delta x(0)\|^2)$, the magnitude of the perturbation becomes

$$\|\delta x(T)\| = \sqrt{\langle \delta x(0), \Delta \delta x(0) \rangle},$$

where $\Delta$ is a symmetric matrix given by

$$\Delta = \left( \frac{d\phi^{t_0+T}_{t_0}(x)}{dx} \right)^* \frac{d\phi^{t_0+T}_{t_0}(x)}{dx},$$

and $\langle \cdot, \cdot \rangle$ is the inner product.
which is a finite time version of the (right) Cauchy-Green tensor.

The maximum stretching is to be given when $\delta x(0)$ is so chosen that it is aligned with the eigenvector $\lambda_{\text{max}} (\Delta)$ associated with the maximum eigenvalue of $\Delta$; namely, letting $\delta x(0)$ be an initial perturbation aligned with the eigenvector associated with $\lambda_{\text{max}} (\Delta)$, it follows that

$$\max_{\delta x(0)} \| \delta x(T) \| = \sqrt{\lambda_{\text{max}} (\Delta)} \| \delta x(0) \|,$$

which may be restated by

$$\max_{\delta x(0)} \| \delta x(T) \| = e^{\sigma^T_{t_0}(x)|T|} \| \delta x(0) \|.$$

The Finite Time Lyapunov Exponent (FTLE) field $\sigma^T_{t_0} : D \subset M \to \mathbb{R}$ associated with a finite integration time $T$ is defined by

$$\sigma^T_{t_0}(x) = \frac{1}{|T|} \ln \sqrt{\lambda_{\text{max}} (\Delta)}.$$

### 4.2 LAGRANGIAN COHERENT STRUCTURES

Recall also that the Lagrangian Coherent Structure may be defined as a second-derivative ridge of an FTLE field $\sigma^T_{t_0}$, which is given by the Hessian of the FTLE field as

$$\Sigma = \frac{d^2 \sigma^T_{t_0}(x)}{dx^2}.$$

More formally, the second-derivative ridge of $\sigma^T_{t_0}$ is defined by an injective curve $c : (a, b) \to D$ that satisfies the following conditions:

- **(i)** The vectors $c'(s)$ and $\nabla \sigma^T_{t_0}(c(s))$ are parallel,
- **(ii)** $\Sigma(n, n) = \min_{\|u\|=1} \Sigma(u, u) < 0$, where $n$ is a unit normal vector to the curve $c(s)$ and $\Sigma$ is thought of as a bilinear form evaluated at the point $c(s)$.

In the vicinity of the stable and unstable manifolds, the FTLE fields take high values, from which the LCS can be detected. In particular, a ridge of the backward-time FTLE field, which one can obtain by the negative integration time $T$, is called the attracting Lagrangian Coherent Structure corresponding to the time-dependent analogue of the unstable manifold, while a ridge of the forward-time FTLE field with the positive integration time $T$ the repelling Lagrangian Coherent Structure corresponding to the stable manifold.

### 4.3 NUMERICAL ANALYSIS

In order to numerically compute the FTLE fields, we advect a regularly spaced rectilinear grid of tracers forward in time by $T$ by using the Runge-Kutta-Fehlberg integrator, in which we discretize the matrix $\frac{d}{dx} \phi^{t_0+T}_{t_0}(x)$ as

$$\frac{\partial [\phi^{t_0+T}_{t_0}(x)]_i}{\partial x_j} \approx \frac{[\phi^{t_0+T}_{t_0}(x + \Delta x_j)]_i - [\phi^{t_0+T}_{t_0}(x - \Delta x_j)]_i}{2\Delta x_j},$$
where \( x_i \) and \( x_j \) are components of \( x \) by the approximative central difference.

In this paper, recall that the subspace \( D \subset M = \mathbb{R}^4 \) is given as
\[
D = \{ x \in M \mid E(x) = E_0 \}
\]
and set \( 250 \times 250 \) grids on \( U \subset D \) in the \((x, v_x)\) coordinates. Then, we compute the FTLE fields for the case of \( T = 10, \ 30, \ 70 \) by choosing \( t_0 = 0 \). The resultant numerical FTLE fields are illustrated in Fig.10, 11 and Fig.12.

![Fig. 10 FTLE (T = 10)](image)

![Fig. 11 FTLE (T = 30)](image) ![Fig. 12 FTLE (T = 70)](image)

As is easily seen from the results, the attracting LCS can be detected as the ridges of the FTLE fields. In particular, it is shown in Fig.10 that the first cut of stable manifold \( \Gamma^{1,s} \) is revealed in the FTLE field for \( T = 10 \). This may be obvious in comparison with the numerical results in Fig.7. As in Fig.11, the stable manifolds \( \Gamma^{1,s}, \Gamma^{2,s}, \) and \( \Gamma^{3,s} \) obtained in Fig.9 are revealed as the LCS associated with the FTLE field for \( T = 30 \). Similarly, one can detect more complicated LCS as the many times cuts, which are illustrated in the FTLE field for \( T = 70 \).

Interestingly, comparing the FTLE field for \( T = 70 \) in Fig.12 with the Poincaré plots
obtained in Fig.9, one can see that the subsets of the FTLE field with low values for the long time interval may correspond to the regions of the KAM tori on the Poincare section, while the one with high values the chaotic regions.

5. SUMMARY

In this paper, we explored the Finite Time Lyapunov Exponent fields and Lagrangian Coherent Structures of the invariant manifolds called tubes associated with the Lyapunov orbits around the collinear Lagrange points. We first showed the stable and unstable invariant manifolds associated with the collinear Lagrange points in Jupiter and the Sun regions. In particular, we illustrated the invariant manifolds on the Poincaré section in the Sun region, where there exist the homoclinic points as the intersections of the stable and unstable manifolds. Second, computing the FTLE field of the PCR3BP, we introduced the LCS as a second-derivative ridge of the FTLE field. Then, we demonstrated that the invariant manifolds can be detected in the Lagrangian Coherent Structures, which can be detected from the numerical FTLE fields; we computed the FTLE fields for the cases of $T = 10, 30, 70$, together with the repelling LCS corresponding to the stable manifolds. Finally, in the numerical analysis of comparing the FTLE fields for the case $T = 70$ with the Poincaré plots on the Poincaré section, we clarified that the regions with low-valued FTLE correspond to the regions of the KAM tori, while the regions with high-valued FTLE field correspond to the stable invariant manifolds. It is expected that this theory may be quite useful in computing time-dependent dynamical systems such as the Planar Elliptic Restricted Three-Body Problem, especially for the case of long time integrations.

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