A Procedure to Find a Discrete Counterpart of a Quasi-Invariant for Two-Dimensional Beta-Plane Turbulence

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Zonostrophy, which was discovered by Balk in 1991, is a quasi-invariant for the two-dimensional quasi-geostrophic equation on a beta-plane. Zonostrophy is useful for explaining the anisotropic energy cascade that favors zonally elongated structures, that is, the Rhines effect. In the present study, we propose a general procedure to numerically obtain a quasi-invariant for discrete wavenumber systems, which correspond to finite physical domains, and derive a discrete counterpart of zonostrophy, which we refer to as the third quasi-invariant. We also conduct numerical experiments to compare the conservation properties of the third quasi-invariant and the zonostrophy. In addition, we show that the third quasi-invariant is closely related to the concept of near-resonant triad interactions, which was introduced by Lee and Smith in 2007.

1. INTRODUCTION

Large-scale flows in the atmosphere and the ocean are considered to be two-dimensional, to the first approximation due to the rotation of the Earth and the density stratification. Two-dimensional flows have a remarkable feature whereby coherent structures tend to be organized from turbulent states as reported by McWilliams (1984)\(^1\). This self-organization process is a manifestation of the inverse cascade of energy, which is related to the conservation of enstrophy.

Including the effect of the latitudinal gradient of the Coriolis parameter, the $\beta$-effect, changes the energy cascade process. Rhines (1975)\(^2\) revealed that the energy cascade becomes anisotropic due to the $\beta$-effect and the flow pattern tends to have a zonally elongated structure. The anisotropic energy cascade due to the $\beta$-effect is now known as the Rhines effect.

In order to understand the Rhines effect, Vallis & Maltrud (1993)\(^3\) introduced the following scaling argument. A dumbbell-shaped wave-turbulence boundary in the wave number space is derived by equating the turbulent eddy turnover time and the inverse Rossby wave frequency. Inside the dumbbell-shaped region, modes are not efficiently excited by the turbulence because of the high frequency of the Rossby waves. As a result, inverse energy cascade causes a detour around the dumbbell-shaped region, which leads to energy accumulation toward components corresponding to zonal flow.
Although the above argument explains the Rhines effect well and the existence of the dumbbell-shaped region in the two-dimensional energy spectrum was confirmed by numerical experiments\(^3\), it remains difficult to evaluate quantitatively the degree to which the dumbbell-shaped region behaves as an obstacle to inverse energy cascade. Using a new quasi-invariant discovered by Balk (1991)\(^4\) for two-dimensional weak-nonlinear flow on a \(\beta\)-plane, Balk (2005)\(^5\) shed new light on this long-standing problem by showing that energy must accumulate toward components corresponding to zonal flow if inverse energy cascade occurs.

Whether the explanation of the Rhines effect presented by Balk (2005)\(^5\) is reasonable depends on how well the new quasi-invariant, which we hereinafter refer to as zonostrophy, following Nazarenko & Quinn (2009)\(^6\), is conserved. Balk (1991)\(^4\) proved that zonostrophy is conserved approximately in the limit of small wave amplitudes. A more detailed consideration of the conservation property of zonostrophy was presented by Balk & van Heerden (2006)\(^7\).

The proof for conservation, however, is not applicable to finite spaces, in which Fourier spectra become discrete, as shown later herein. In spite of this, Nazarenko & Quinn (2009)\(^6\) conducted numerical experiments, which are of course based on a discrete model, and showed that zonostrophy is well conserved in the course of time evolution of turbulence on a \(\beta\)-plane. Hence, some explanation is required to fill the gap between discrete systems and continuous systems. Furthermore, such an explanation is also desired for zonostrophy conservation to be applied to real geophysical fluids, because the horizontal extent of the real atmosphere and the ocean are, strictly speaking, both finite and their Fourier spaces are discrete.

In the present paper, we introduce another new quasi-invariant, which is considered to be a discrete counterpart of zonostrophy. We also conduct numerical experiments to compare the conservation properties of the new quasi-invariant and zonostrophy. Moreover, we show that the new quasi-invariant is closely related to the concept of the near-resonant triad interactions, which is introduced by Lee and Smith (2007)\(^8\) to explain the anisotropic energy cascade and the generation of zonal pattern on a \(\beta\)-plane.

The remainder of the present paper is organized as follows. In section 2, we describe how to deduce the new quasi-invariant for the two-dimensional flow on a \(\beta\)-plane after briefly summarizing zonostrophy. The conservation properties of the new quasi-invariant and zonostrophy are compared by conducting numerical experiments in section 3. A discussion and conclusions are presented in section 4.

2. DERIVATION OF QUASI-ININVARIANTS

2.1 GOVERNING EQUATION

The system under consideration is a two-dimensional flow on a \(\beta\)-plane, which is governed by the following quasi-geostrophic vorticity equation:

\[
\frac{\partial}{\partial t_*} (\nabla^2_* \psi_* - \alpha_*^2 \psi_*) + \beta_* \frac{\partial \psi_*}{\partial x_*} + \frac{\partial \psi_*}{\partial x_*} \frac{\partial \nabla^2_* \psi_*}{\partial y_*} - \frac{\partial \psi_*}{\partial y_*} \frac{\partial \nabla^2_* \psi_*}{\partial x_*} = 0. \tag{1}
\]

Here, \(t_*\) is the time, \(x_*\) is the eastward coordinate, \(y_*\) is the northward coordinate, \(\nabla^2_*\) is the horizontal Laplacian, \(\psi_*\) is the stream-function, \(\beta_*\) is the \(y_*\)-derivative of the Coriolis parameter, \(\alpha_*\) is the reciprocal of the Rossby radius of deformation, and the subscript * indicates dimensional
variables. We non-dimensionalize (1) in the following manner:

\[
(x_*, y_*) = L_* (x, y), \quad t_* = T_* t, \quad \psi_* = (L_*^2/T_*) \psi,
\]

\[
\nabla_*^2 = \nabla^2 / L_*^2, \quad \alpha_* = \alpha / L_*, \quad \beta_* = \beta / (T_* L_*),
\]

where \(L_*\) and \(T_*\) are arbitrary length and time scales, respectively. The quasi-geostrophic vorticity equation, (1), then becomes

\[
\frac{\partial}{\partial t} (\nabla^2 \psi - \alpha^2 \psi) + \beta \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x^2} = 0.
\]

(3)

If we assume that the domain is infinite, the Fourier transform of (3) yields the following evolution equation:

\[
\frac{\partial \hat{\psi}_k}{\partial t} = \omega_k \hat{\psi}_k + \frac{1}{2\pi} \iint \iint \left( |k|^2 + \alpha^2 \right)^{-1} W_{k_1, k_2} \delta(-k + k_1 + k_2) \hat{\psi}_{k_1} \hat{\psi}_{k_2} dk_1 dk_2.
\]

(4)

Here,

\[
\hat{\psi}_k(t) = \frac{1}{2\pi} \iint \psi(x, y, t) e^{-ikx} dx,
\]

(5)

where \(x\) is the horizontal position vector \((x = (x, y))\), \(k, k_1,\) and \(k_2\) are the wave number vectors \((k = (k_1, l_1), k_1 = (k_1, l_1), \text{and} k_2 = (k_2, l_2))\), \(\delta\) is the Dirac delta function, \(\omega_k\) is the angular frequency of the Rossby wave with wavenumber \(k\), which is given as,

\[
\omega_k = \frac{b_1}{|k|^2 + \alpha^2},
\]

(6)

and \(W_{k_1, k_2}\) is defined as

\[
W_{k_1, k_2} = \frac{1}{4\pi} (k_1 l_2 - k_2 l_1)(|k|^2 - |k_1|^2).
\]

(7)

### 2.2 INVARIANTS AND ZONOSTROPHY

Equation (4) has two invariants, the total energy \((E)\) and the total enstrophy \((\Omega)\), which are second-order quantities of the coefficients \(\hat{\psi}_k\) and are written as follows:

\[
E = \iint \varepsilon_k dk \quad \text{and} \quad \Omega = \iint |k|^2 \varepsilon_k dk,
\]

(8)

respectively. Here,

\[
\varepsilon_k = \frac{1}{2} (|k|^2 + \alpha^2)|\hat{\psi}_k|^2.
\]

(9)

The conservation of these two invariants plays a dominant role in explaining the process of the inverse cascade of energy. However, the invariants do not explain the anisotropic energy cascade corresponding to the Rhines effect.

Balk (1991) discovered a quantity that is approximately conserved and may potentially explain the anisotropic energy cascade. The quantity, \(M\), is defined as follows:

\[
M = \iint \phi_k \varepsilon_k dk
\]

\[
-\frac{i}{3} \iint \iint \iint \frac{\phi_{k_1} W_{k_1, k_2} + \phi_{k_2} W_{k_2, k_1} + \phi_{k_1} W_{k_1, k_2} \delta(k + k_1 + k_2) \hat{\psi}_{k_1} \hat{\psi}_{k_2} dk_1 dk_2}{\omega_k + \omega_{k_1} + \omega_{k_2}}.
\]

(10)
where

$$\phi_k = \frac{\eta_k}{\omega_k}, \quad \text{and} \quad \eta_k = \beta \left\{ \arctan \left[ \frac{\alpha (l - k \sqrt{3})}{|k|^2} \right] - \arctan \left[ \frac{\alpha (l + k \sqrt{3})}{|k|^2} \right] \right\}. \quad (11)$$

The conservation property of $M$ is shown as follows. Taking the time-derivative of (10) and applying (4) yield

$$\frac{dM}{dt} = O(\dot{\psi}^4), \quad (12)$$

where $\dot{\psi}$ are the amplitudes of $\hat{\psi}_k$s. In other words, $M$ is conserved approximately to the third order of $\dot{\psi}$ when $\dot{\psi}$ are small. The smallness of $\dot{\psi}$, however, should be defined more clearly. Let us assume that the distribution of $\hat{\psi}_k$ is approximated as

$$|\hat{\psi}_k| = \begin{cases} \Psi & (|k| - K_r \leq \Delta K_r/2) \\ 0 & (|k| - K_r > \Delta K_r/2), \end{cases} \quad (13)$$

where $\Psi$ is a constant, $K_r$ is the representative wavenumber for $\hat{\psi}_k$ and $\Delta K_r$ is the representative width of the distribution of $\hat{\psi}_k$ in the wavenumber space. We further assume that $K_r \gg \alpha$. If we write the non-dimensionalized representative velocity scale as $U$, a scaling estimate for $U$ is given as follows:

$$U \sim |\nabla \psi| \sim \left| \int K_r \hat{\psi}_k \, dk \right| \sim \Psi K_r^2 \Delta K_r. \quad (14)$$

Similarly, we obtain the following estimates:

$$M \sim \phi_{K_r} \Psi^2 K_r^3 \Delta K_r + \phi_{K_r} \Psi^3 K_r^7 (\Delta K_r)^2 / \beta, \quad \frac{dM}{dt} \sim \phi_{K_r} \Psi^4 K_r^9 (\Delta K_r)^3 / \beta. \quad (15)$$

Here, $\phi_{K_r}$ is the representative value of $|\phi_k|$ for $|k| \sim K_r$. Using (14) and (15), the ratio of $dM/dt$ to $M$ is estimated as follows:

$$\frac{1}{M} \frac{dM}{dt} \sim \frac{\Psi^2 K_r^7 (\Delta K_r)^2 / \beta}{1 + \Psi K_r^4 \Delta K_r / \beta} \sim \frac{U K_r \cdot (U K_r^2 / \beta)}{1 + U K_r^2 / \beta} = \frac{U K_r \cdot Rh}{1 + Rh}, \quad (16)$$

where $Rh = U K_r^2 / \beta$ is the Rhines number. Then, if $Rh \ll 1$, we obtain

$$\left| \frac{1}{M} \frac{dM}{dt} \right| \ll U K_r. \quad (17)$$

In other words, the time scale for $M$ to change is much longer than the eddy-turnover time, which is defined as $1/(U K_r)$. Furthermore, if $Rh \ll 1$, the second term of the right-hand side of (10) is negligible compared with the first term of the right-hand side. Consequently, when $Rh \ll 1$, the quadratic part of $M$, that is,

$$Z = \int \phi_k \varepsilon_k \, dk, \quad (18)$$

can be regarded as a well-conserved quantity in the time scale of the eddy-turnover time. Nazarenko & Quinn$^6$ referred to $Z$ as zonostrophy, based on its importance in the zonation process. A similar scaling argument for the conservation of $Z$ was presented by Balk & van Heerden (2006)$^7$.

Assuming zonostrophy conservation, the anisotropic energy cascade is explained as follows$^5$. Noting that

$$\phi_k \to 2 \sqrt{3} \alpha \quad (|k| \to \infty), \quad (19)$$
we consider a linear combination of energy and zonostrophy,

\[ Z - 2\sqrt{3}\alpha E = \iint (\phi_k - 2\sqrt{3}\alpha) \varepsilon_k dk. \]  

(20)

If both energy and zonostrophy are conserved, the linear combination is also conserved. Figure 1 shows the distribution of \( \phi_k - 2\sqrt{3}\alpha \) in the wavenumber space. Here, we set \( 1/\alpha = \pi \) and the values are normalized to unity at \( (k, l) = (1, 0) \). In the long wave limit \( (|k| \to 0) \), \( \phi_k - 2\sqrt{3}\alpha \) diverges to infinity along the \( k \)-axis \( (l = 0) \), whereas \( \phi_k - 2\sqrt{3}\alpha \) remains finite along the \( l \)-axis \( (k = 0) \). Accordingly, each contour curve is dumbbell-shaped. It is this asymmetric distribution of \( \phi_k - 2\sqrt{3}\alpha \) that leads to the anisotropic energy cascade. Provided that energy inversely cascades toward the region of smaller wave numbers, energy must accumulate toward the \( l \)-axis in order to conserve \( Z - 2\sqrt{3}\alpha E \) because of the sharp increase of the weighting function, \( \phi_k - 2\sqrt{3}\alpha \), around the \( k \)-axis. This means that energy must cascade anisotropically and accumulate toward components corresponding to zonal flow.

2.3 EXTENSION TO FINITE DOMAINS

The relevance of zonostrophy to the anisotropic energy cascade depends on the special form of \( \phi_k \), which is defined by (11). The special form originates from the requirement that the integrand in the second term of the right-hand side of (10) does not diverge to infinity when the resonance condition

\[ k + k_1 + k_2 = 0 \quad \text{and} \quad \omega_k + \omega_{k_1} + \omega_{k_2} = 0, \]  

(21)

is satisfied. Based on the definition given in (11), since

\[ \eta_k + \eta_{k_1} + \eta_{k_2} = 0 \]  

(22)

whenever the resonance condition (21) is satisfied, it can be shown that the integrand in the second term of the right-hand side of (10) does not diverge to infinity. The mathematical details are described in Balk (1991)\(^4\) and Balk & van Heerden (2006)\(^7\).
Next, let us try to extend the concept of zonotrophy to finite domains in which Fourier spectra become discrete. For simplicity, we consider a domain, \((x, y) \in [0, 2\pi] \times [0, 2\pi]\), with the periodic boundary condition in both the \(x\) and \(y\) directions. Since the wavenumber space is discrete, the governing equation for the Fourier coefficients is written as

\[
\frac{1}{i} \frac{\partial \hat{\psi}_k}{\partial t} = \omega_k \hat{\psi}_k + i \sum_{k_1} \sum_{k_2} \sum_{l_1} \sum_{l_2} (|\mathbf{k}|^2 + \alpha^2)^{-1} W_{k_1, k_2} \delta_{k, k_1 + k_2} \hat{\psi}_{k_1} \hat{\psi}_{k_2}, \tag{23}
\]

instead of (4). Here, \(\delta_{k, k_1 + k_2}\) is the Kronecker delta.

For this system, it is possible to consider the discrete analog of the quantity, \(M\), as follows:

\[
M_{\text{discrete}} = \sum_k \sum_l \phi_k \epsilon_k
- \frac{i}{3} \sum_k \sum_{k_1} \sum_{k_2} \sum_{l_1} \sum_{l_2} \frac{\phi_k W_{k_1, k_2} + \phi_{k_1} W_{k, k_2} + \phi_{k_2} W_{k_1, k} \delta_{k_1, k_1 + k_2} \hat{\psi}_{k_1} \hat{\psi}_{k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}. \tag{24}
\]

Even for \(M_{\text{discrete}}\), the approximate conservation property,

\[
\frac{d}{dt} M_{\text{discrete}} = O(\psi^4), \tag{25}
\]

holds. Whenever \(\alpha^2\) is an irrational number, however, it is not necessary to define \(\phi_k\) using (11), because the resonance condition given by (21), is satisfied only when

\[
(k = 0 \quad \text{and} \quad |\mathbf{k}| = |\mathbf{k}_2|)
\]

or \((k_1 = 0 \quad \text{and} \quad |\mathbf{k}_2| = |\mathbf{k}|)\)

or \((k_2 = 0 \quad \text{and} \quad |\mathbf{k}| = |\mathbf{k}_1|)\),

as shown in appendix. In this case, for the second term of the right-hand side of (24) to be finite, \(\phi_{k_8}\) need only satisfy the following constraint:

\[
\phi_{(k, l)} = \phi_{(-k, l)} = \phi_{(k, -l)} = \phi_{(-k, -l)}. \tag{27}
\]

In other words, arbitrary selection of \(\phi_{(k, l)}\) \((k, l \geq 0)\) is possible in this case, and the justification for choosing the special form of zonotrophy is lost. Thus, deeper consideration is necessary in order to explain the anisotropic energy cascade using the quantities that are conserved approximately in the discrete wavenumber space.

In order to consider a counterpart of zonotrophy in the discrete wavenumber space, let us introduce a second-order quantity of \(\hat{\psi}_{k_8}\):

\[
\Phi = \sum_k \sum_l f_k \epsilon_k, \tag{28}
\]

where \(f_k\) is an arbitrary real function of \(k\). As discussed above, if \(\alpha^2\) is an irrational number and \(f_k\) satisfies

\[
f_{(k, l)} = f_{(-k, l)} = f_{(k, -l)} = f_{(-k, -l)}, \tag{29}
\]

then the following equation holds:

\[
\frac{d}{dt} \left[ \Phi - \frac{i}{3} \sum_k \sum_{k_1} \sum_{k_2} \sum_{l_1} \sum_{l_2} \frac{f_k W_{k_1, k_2} + f_{k_1} W_{k, k_2} + f_{k_2} W_{k_1, k} \delta_{k_1, k_1 + k_2} \hat{\psi}_{k_1} \hat{\psi}_{k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}} \right] = O(\psi^4). \tag{30}
\]
Integration of (30) with respect to time yields
\[
\Phi = \Phi_0 + \frac{i}{3} \sum_{k} \sum_{l} \sum_{k_1} \sum_{l_1} \sum_{k_2} \sum_{l_2} \frac{f_k W_{k_1,k_2} + f_k W_{k_2,k_3} + f_k W_{k_1,k} \delta_{-k_1,k_2} \psi_k \psi_{k_1} \psi_{k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}
+ O(\psi^4),
\]
(31)
where \( \Phi_0 \) is a constant of integration. When the Rhines number is much smaller than unity, \( \psi_k \)'s vary approximately as follows:
\[
\dot{\psi}_k(t) \approx |\dot{\psi}_k| e^{i\omega_k t},
\]
(32)
according to (23). Then, the second term of the right-hand side of (31) is regarded as a superposition of oscillations. Based on the above consideration, we introduce an index of conservation of \( \dot{\psi}_k \), as follows:
\[
I = \sum_{k} \sum_{l} \sum_{k_1} \sum_{l_1} \sum_{k_2} \sum_{l_2} \left( \frac{f_k W_{k_1,k_2} + f_k W_{k_2,k_3} + f_k W_{k_1,k} \delta_{-k_1,k_2} |\dot{\psi}_k| |\dot{\psi}_{k_1}| |\dot{\psi}_{k_2}|}{\omega_k + \omega_{k_1} + \omega_{k_2}} \right)^2.
\]
(33)
If we can choose the function \( f_k \) for \( I \) to be much smaller than \( \Phi^2 \), \( \Phi \) is well conserved. Although this condition may appear to be trivially satisfied when \( R_h \ll 1 \), arbitrary choices of \( f_k \) can make \( I \) very large because of the near-resonance.

2.4 EIGENVALUE ANALYSIS

In this subsection, we propose a procedure to determine \( f_k \) so that \( I \) as defined by (33) is as small as possible. Let us consider a truncated wavenumber space, \( k, l = 0, \pm 1, \pm 2, \ldots, \pm K_T \). Here, \( K_T \) is the truncation wavenumber. Next, we define \( m \) as follows:
\[
m = M^2 \{ M(K_T + l) + (K_T + k) \} + M(K_T + l_1) + (K_T + k_1) + 1,
(\text{for } (k, l, k_1, l_1 = 0, \pm 1, \pm 2, \ldots, \pm K_T),
\]
where \( M = 2K_T + 1 \) and we introduce the vector \( u = (u_m), \ (m = 1, 2, \ldots, M^4) \) as follows:
\[
u_m = \frac{f_k W_{k_1,k_2} + f_k W_{k_2,k_3} + f_k W_{k_1,k} \delta_{-k_1,k_2} |b_k| |b_{k_1}| |b_{k_2}|}{\omega_k + \omega_{k_1} + \omega_{k_2}}, \ (k_2 = -k - k_1).
\]
(34)
Using \( u \), (33) can be rewritten as
\[
I = u \cdot u = u^T u.
\]
(35)
Next, we define \( n \) as follows:
\[
n = \begin{cases} 
1 & \text{for } (k = 0 \text{ and } l = 1, 2, \ldots, K_T), \\
(K_T + 1)k + l & \text{for } (k = 1, 2, \ldots, K_T \text{ and } l = 0, 1, 2, \ldots, K_T),
\end{cases}
\]
and introduce the vector \( \mathbf{F} = (F_n) \ (n = 1, 2, \ldots, N) \), where \( N = K_T^2 + 2K_T \). Using \( \mathbf{F} \), we write \( f_k \) that satisfies (29) as
\[
f_{(\pm k, \pm l)} = F_n \ (k, l = 0, 1, \ldots, K_T; \text{excluding the case of } k = l = 0).
\]
Here, we set \( f_{(0,0)} = 0 \) because the \( \hat{\psi}_{(0,0)} \) component does not affect the dynamics. Then, we can write \( u \) as
\[
\mathbf{u} = L \mathbf{F},
\]
where \( L \) is a \( M^2 \times N \) matrix. Substituting (36) into (35) yields
\[
I = \mathbf{F}^T S \mathbf{F},
\]
where \( S = L^T L \) is a symmetric \( N \times N \) matrix. In other words, \( I \) is expressed as a positive semidefinite symmetric bilinear form of the vector \( \mathbf{F} \), which enables us to use eigenvalue analysis. Solving the following eigenvalue problem (numerically, in practice),
\[
S \mathbf{v} = \lambda \mathbf{v},
\]
we can obtain \( N \) eigenvalues, \( \lambda_i \, (i = 1, 2, \ldots, N) \), and corresponding eigenvectors, \( \mathbf{v}_i \, (i = 1, 2, \ldots, N) \). Here, the eigenvalues and the eigenvectors are chosen to satisfy
\[
0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N, \quad \mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} \, (i, j = 1, 2, \ldots, N).
\]
Defining an \( N \times N \) matrix, \( P \), as
\[
P = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N],
\]
\( S \) is written as
\[
S = P \mathbf{D} P^T,
\]
where \( D \) is a diagonal matrix:
\[
D = \begin{pmatrix}
\lambda_1 & & \\
& \lambda_2 & \\
& & \cdots \\
& & & \lambda_N
\end{pmatrix}.
\]

Next, we present an example of the eigenvalues and eigenvectors of the matrix \( S \). In order to calculate these eigenvalues and eigenvectors, we must specify the truncation wavenumber, \( K_T \), and the non-dimensionalized deformation radius, \( 1/\alpha \). Here, we set \( K_T = 85 \) and \( 1/\alpha = \pi \). We chose this irrational value of \( \alpha \) for the resonance condition not to be satisfied except for the trivial case (26). Even for this choice, however, there is still a rare possibility that the resonance might occur in the computation because of the finite digit. We checked that the present computational settings do not fall into this rare case. Since the matrix \( S \) depends on \( |\hat{\psi}_k| \), we must also specify the distribution of \( |\hat{\psi}_k| \). In order to simplify the problem as far as possible, we set the distribution of \( \hat{\psi}_k \) so that \( |\hat{\psi}_k| \) has no azimuthal dependence in the wavenumber space and has a uniform one-dimensional energy spectrum. Then, \( \hat{\psi}_k \) should be written as follows:
\[
|\hat{\psi}_k| = \gamma(k^2 + \ell^2)^{-1/4}(k^2 + \ell^2 + \alpha^2)^{-1/2}.
\]
Here, the constant, \( \gamma \), is set such that the total energy is equal to 0.5.

Based on the above settings, we can now calculate the eigenvalues and eigenvectors of the matrix \( S \) solving an eigenvalue problem of \( N \times N = 7395 \times 7395 \) numerically with LAPACK subroutines. Figure 2 shows the eigenvalues, and figure 3 shows the distributions of the eigenvectors in the
Fig. 2 Fifty smallest eigenvalues \((\lambda_1, \lambda_2, \ldots, \lambda_{50})\) of matrix \(S\).

Fig. 3 Contour plot of eigenvectors in the wavenumber space (a) \(v_1\), (b) \(v_2\), and (c) \(v_3\) associated with eigenvalues \(\lambda_1, \lambda_2, \) and \(\lambda_3\), respectively. (d) is a magnified view of (c) for \(|k| \leq 10\) and \(|l| \leq 10\).
wavenumber space. The first two eigenvalues, $\lambda_1$ and $\lambda_2$, shown in figure 2 are much smaller than other eigenvalues and can be regarded as being equal to zero, considering round-off errors in computation. The eigenvectors corresponding to these two zero eigenvalues, $\mathbf{v}_1$ and $\mathbf{v}_2$, are shown in figures 3(a) and 3(b). Each of these eigenvectors has a parabolic shape in the wavenumber space. This reflects the fact that each of these two eigenvectors is a linear combination of the vector that corresponds to energy, $f_k = 1$, and the vector that corresponds to enstrophy, $f_k = k^2 + l^2$. An arbitrary linear combination of these two vectors becomes an eigenvector corresponding to an eigenvalue of zero because each of these vectors satisfies the following equation:

$$f_k W_{k_1,k_2} + f_{k_1} W_{k,k_2} + f_{k_2} W_{k_1,k} = 0 \quad (k_2 = -k - k_1),$$

and yields $I = 0$ based on the definition given in (33). The other eigenvalues shown in figure 2 are definitely non-zero and the corresponding eigenvectors have anisotropy in the wavenumber space. Figure 3(c) shows the eigenvector corresponding to the third eigenvalue, $\mathbf{v}_3$, for example. An anisotropic distribution appears near the origin of the wavenumber space and is clearly seen in the magnified view shown in figure 3(d).

2.5 THE THIRD QUASI-ININVARIANT

A quantity is considered to be well conserved when its variation is much smaller than its value. In order to make the quantity, $\Phi$, which is defined by (28), have this property, we consider the constrained minimization problem described below. Letting $\mathbf{F}$ be the independent variable, we minimize $I = \mathbf{F}^T S \mathbf{F}$ with the constraint

$$\Phi = \mathbf{e} \cdot \mathbf{F} = C. \quad (41)$$

Here, $C$ is a constant, and the vector $\mathbf{e} = (\epsilon_n) \ (n = 1, 2, \ldots, N)$ is defined as follows:

$$\epsilon_n = \begin{cases} 
\epsilon_{(k,l)} + \epsilon_{(-k,l)} + \epsilon_{(k,-l)} + \epsilon_{(-k,-l)} & (k, l > 0) \\
\epsilon_{(k,0)} + \epsilon_{(-k,0)} & (k > 0, \ l = 0) \\
\epsilon_{(0,l)} + \epsilon_{(0,-l)} & (k = 0, \ l > 0).
\end{cases} \quad (42)$$

The solution of this minimization problem is, however, a trivial one. Since the matrix $S$ has two zero eigenvalues, $\lambda_1$ and $\lambda_2$, as described in the previous subsection, the minimum of $I$ is zero and the corresponding $\mathbf{F}$ is an arbitrary linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ that satisfies (41).

In order to exclude this trivial solution, we impose another constraint:

$$\mathbf{v}_1 \cdot \mathbf{F} = 0 \quad \text{and} \quad \mathbf{v}_2 \cdot \mathbf{F} = 0. \quad (43)$$

The minimization of $I = \mathbf{F}^T S \mathbf{F}$ under the constraints of (41) and (43) can be solved easily using the Lagrange multiplier method. The vector $\mathbf{F}$ that gives the minimum of $I$ is written as

$$\mathbf{F} = \frac{C(\tilde{D} \tilde{D}^T) \mathbf{e}}{\mathbf{e}^T (\tilde{D} \tilde{D}^T) \mathbf{e}}, \quad (44)$$

where

$$\tilde{D} = \begin{pmatrix}
0 & 0 \\
0 & {\lambda_3}^{-1} \\
& \ddots \\
& & {\lambda_N}^{-1}
\end{pmatrix}. \quad (45)$$
Fig. 4 (a) Contour plot of $\log_{10}(|f_k|)$ corresponding to $F$ in the wavenumber space. The thick solid contour labeled "0" indicates the location at which $f_k = 0$. Thin solid and dashed contours indicate the locations at which $f_k$ is positive and negative, respectively. (b) Contour plot of $\log_{10}(|f_k|)$ corresponding to $F'$ in the wavenumber space (dashed curve). Contours in solid curves are the same as those in figure 1.

Using the same settings for $K_T$, $\alpha$, and $|\hat{v}_k|$ as used in the computation of the eigenvalues and the eigenvectors in the previous subsection, we compute the vector $F$, using (44). Despite the constraint of (43), the distribution of the resulting vector, like $v_3$ in figure 3(c), becomes very close to that of a linear combination of $v_1$ and $v_2$ in wavenumber space where $|k|$ is large, as shown in figure 4(a). In order to remove this component, we introduce the following vector: $F' = (F'_n) \ (n = 1, 2, \ldots, N)$ as

$$F' = F + c_1v_1 + c_2v_2,$$

and choose $c_1$ and $c_2$ so as to satisfy

$$F'_{(85,85)} = F'_{(85,42)} = 0.$$

Here, although the choice of the point of $(k, l) = (85, 85)$ and $(85, 42)$ may seem arbitrary, the distribution of $F'$ is almost independent of the choice, as long as $k$ and $l$ are sufficiently large. Figure 4(b) shows the distribution of $F'$ in the wavenumber space with the distribution of $\phi_k - 2\sqrt{2}\alpha$ for comparison, which is already shown in figure 1. As shown in figure 4(b), these two distributions are very similar, where $|k| \approx 30$. We can then regard the second-order quantity corresponding to $F'$,

$$\Phi' = e \cdot F',$$

as a counterpart of zonostrophy in finite domains in which Fourier spectra become discrete. Hereinafter, we refer to $\Phi'$ as the third quasi-invariant in order to distinguish $\Phi'$ from zonostrophy, because $\Phi'$ originates from the minimization problem, excluding the energy and enstrophy component.
3. NUMERICAL EXPERIMENTS

3.1 GOVERNING EQUATION AND EXPERIMENTAL SETUP

In this section, we investigate the conservation property of the third quasi-invariant introduced in the previous section, computing the time evolution of the quasi-geostrophic equation numerically. The equation to be integrated is (3) with the $2\pi$-periodic boundary condition in both the $x$ and $y$ directions. In practice, however, in order to avoid spurious accumulation of enstrophy near the truncation wavenumber for the computation, we introduce a hyper-viscosity term to the right-hand side of (3) as

$$
\frac{\partial}{\partial t} (\nabla^2 \psi - \alpha^2 \psi) + \beta \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \nabla^2 \psi}{\partial x} = (-1)^{p+1} \nu_p \nabla^2 (p+1) \psi,
$$

(49)

and (49) is numerically integrated instead. Then, the exact invariants, the total energy and the total enstrophy, are no longer conserved exactly over the course of time evolution. Here, $p$ is the order of hyper-viscosity, and $\nu_p$ is the hyper-viscosity coefficient. We set $p = 5$ and $\nu_p = 1 \times 10^{-18}$ in the following computations.

In order to integrate (49) numerically, we adopted the Fourier spectral method with a truncation wavenumber of $K_T = 85$ for the spatial discretization. The nonlinear term is computed using the transform method with alias-free grids of $256 \times 256$. The time integration scheme is the classical fourth-order Runge–Kutta scheme with time step $\Delta t = 8 \times 10^{-4}$.

The initial condition is a random vorticity field, the one-dimensional energy spectrum $E_K$ of which is given as follows:

$$
E_K \propto \left( \frac{2}{\sqrt{K/K_0 + \sqrt{K_0/K}}} \right)^m,
$$

(50)

where $K = |k| = \sqrt{k^2 + \beta^2}$ is the total wave number. We set $K_0 = 15$ and $m = 1000$. The phase of each Fourier component is set randomly. The total energy of the initial state is determined by setting the root-mean-square velocity, $u_0$, to be unity.

As for the two parameters, $\alpha$ and $\beta$, we set $\alpha = 1/\pi$ as in the previous section, and two values of $\beta$ are examined in order to determine the dependence of the conservation property on $\beta$. In the following subsections, we show the results for $\beta = 200$ and 450. The corresponding Rhines wavenumbers, $K_\beta = \sqrt{\beta/(2u_0)}$, are 10 and 15, respectively. We hereinafter refer to the experiment with $\beta = 200$ as the "$K_\beta = 10$ experiment" and the experiment with $\beta = 450$ as the "$K_\beta = 15$ experiment".

3.2 RESULTS

For both $K_\beta = 10$ and $K_\beta = 15$ experiments, numerical time integration of (49) is conducted until $t = 30$. In other words, the integration period is 450 times the initial eddy-turnover time because the initial eddy-turnover time is $1/(u_0 K_0) = 1/15$. Figure 5 shows the instantaneous vorticity $(\nabla^2 \psi)$ fields at the beginning of the time integration (figure 5(a)) and the end of the time integration (figure 5(b) is for the $K_\beta = 10$ experiment and figure 5(c) is for the $K_\beta = 15$ experiment). The initial random vorticity field (figure 5(a)) evolves to have the zonally elongated structure due to the Rhines effect (figures 5(b) and 5(c)). The $y$-directional wavenumber of the
Fig. 5 Snapshots of the vorticity field in nonlinear time-evolutions. (a) is the initial field \( t = 0 \), (b) is a snapshot at \( t = 30 \) for the \( K_\beta = 10 \) experiment, and (c) is a snapshot at \( t = 30 \) for the \( K_\beta = 15 \) experiment. The contour intervals are 15 for (a) and 8 for (b) and (c). Positive regions are shaded.

Fig. 6 Time-evolution of the one-dimensional energy spectrum. (a) is for the \( K_\beta = 10 \) experiment, and (b) is for the \( K_\beta = 15 \) experiment. The solid, long-dashed, and short-dashed curves denote the spectra at \( t = 0, 2, \) and 30, respectively, in both figures.

Fig. 7 Snapshots of two-dimensional energy spectra in nonlinear time-evolutions. (a) initial condition \( t = 0 \), (b) at \( t = 30 \) for the \( K_\beta = 10 \) experiment, and (c) at \( t = 30 \) for the \( K_\beta = 15 \) experiment.
zonally elongated structure in the final state appears to be roughly equal to the corresponding Rhines wavenumber, $K_\beta$, for each experiment.

Corresponding to the structural change in the vorticity field in the course of time evolution, the energy spectrum evolves from the initial distribution that is defined by (50). Figure 6 shows the time evolutions of the one-dimensional energy spectra for both experiments. As time passes, the energy spectrum spreads and its peak shifts to lower wavenumbers for each experiment. This inverse energy cascade is, however, suppressed to some extent by the Rhines effect. Since this suppression effect becomes stronger as $\beta$ increases, the $K_\beta = 15$ experiment (figure 6(b)) has less energy in the low wavenumber region in the final state than the $K_\beta = 10$ experiment (figure 6(a)). Two-dimensional energy spectrum for the initial state and the final states for both the experiments are shown in figure 7. The initial isotropic distribution (figure 7(a)) evolves into an anisotropic distribution in which energy accumulates around the $l$-axis corresponding to the zonal structure of the vorticity field for each experiment (figures 7(b) and 7(c)).

Thus far, we have seen the basic characteristics of the time evolutions. We next examine the conservation property of the (quasi-)invariants introduced in the previous section. Figure 8 shows the time evolutions of the energy, the enstrophy, the zonostrophy, and the third quasi-invariant for each $K_\beta$ experiment. Each quantity is normalized by its initial value for its variation to be seen clearly. For comparison, figure 8 also shows the time evolution of a non-conserved quantity, which is defined by (28) with the following setting of the coefficient: $f_k = (k^2 + l^2 + \alpha^2)^{-1}$. This non-conserved quantity is the same as that introduced by Nazarenko & Quinn (2009)\textsuperscript{6}. In both figures 8(a) and 8(b), the energy decreases slightly, and the enstrophy decreases significantly due to the hyper-viscosity. On the other hand, both the zonostrophy and the third quasi-invariant exhibit different behaviors. In figure 8(a), the zonostrophy and the third quasi-invariant grow significantly and have amplitudes that are approximately 10 times greater than their initial amplitudes. This amplitude growth is greater than that of the non-conserved quantity, which means that the zonostrophy and the third quasi-invariant are far from being conserved. These quasi-invariants are not conserved in the case of figure 8(a), where the initial value of the Rhines number is $(K_0/K_\beta)^2 = (15/10)^2 = 9/4 > 1$ because the conservation of the quasi-invariants is based on the assumption that the Rhines number is much smaller than unity, as described in the previous section. When the Rhines number is smaller, however, the quasi-invariants are conserved better. In the case of figure 8(b), where the initial value of the Rhines number is $(K_0/K_\beta)^2 = 1$, the amplitudes of both of the quasi-invariants increase in the early stage ($t \lesssim 5$); after that, however, the growth saturates and the amplitude varies only slightly. Compared with the large growth of the non-conserved quantity, the quasi-invariants are considered to be well conserved in this case. As shown above, although the behavior of both of the quasi-invariants depends largely on the initial value of the Rhines number, the third quasi-invariant shows very similar time evolutions to those of the zonostrophy in the cases shown in figures 8(a) and 8(b). Furthermore, the third quasi-invariant exhibits slightly less variation than the zonostrophy in both cases.

Before concluding this section, let us briefly discuss two issues concerning the above experiments. First, let us consider the difference between the present experiments and those of Nazarenko & Quinn (2009)\textsuperscript{6}. The main difference, of course, is that, in the present study, we examine the time evolution of both the third quasi-invariant, which is introduced in the previous section, and the zonostrophy. There is another difference. In the present study, the hyper-viscosity term is
Fig. 8 Time-evolutions of energy (solid curve), enstrophy (long-dashed curve), zonostrophy (short-dashed curve), third quasi-invariant (solid curve with open circles), and a non-conserved quantity (see the text; solid curve with open triangles) (a) is for the $K_\beta = 10$ experiment, and (b) is for the $K_\beta = 15$ experiment. Each quantity is normalized by its initial value.

introduced in (49), which prevents the exact invariants, the total energy and enstrophy, from being conserved. The hyper-viscosity, however, enables us to run the time evolution for a longer period without spurious accumulation of enstrophy near the truncation wavenumber. This longer time-integration period made it possible to confirm that the quasi-invariants are nearly conserved in the case of figure 8(b), in which the initial Rhines number is small. Second, we consider the relevance of the explanation described in the previous section for the anisotropic energy cascade. The explanation depends on the conservation of the quasi-invariants. As shown above, in the $K_\beta = 10$ experiment, the zonation of the vorticity field can be observed even though the quasi-invariants are far from being conserved. In this case, it may be possible to interpret the result such that the energy cascade in the wavenumber region of $K < K_{\beta}$ is constrained by the conservation of the quasi-invariants to some extent, which leads to the zonation of the vorticity field.

4. DISCUSSION AND SUMMARY

The main subject of the present paper is to introduce a discrete counterpart of zonostrophy that was discovered by Balk (1991)\(^4\). For this purpose, we introduced the third quasi-invariant and showed that, in the course of nonlinear time evolutions, the third quasi-invariant has a similar (or slightly better) conservation property than the zonostrophy. This reflects the fact that the distributions of the coefficients corresponding to the two quasi-invariants are similar, as shown in figure 4. The distribution of the coefficient for the third quasi-invariant, however, does depend on the truncation wavenumber, $K_T$. Figures 9(a) through 9(c) show the distributions of the coefficients corresponding to the third quasi-invariant with truncation wavenumbers of $K_T = 20$, 42, and 85, respectively. For comparison, the coefficients corresponding to the zonostrophy are shown together and the shown wavenumber range is $|k| \leq 20$ and $|l| \leq 20$. The distribution of the coefficient corresponding to the third quasi-invariant clearly approaches that corresponding
to the zonostrophy as $K_T$ becomes large. Based on this observation, the third quasi-invariant is thought to converge to the zonostrophy in the limit of $K_T \to \infty$. If the speculation is true, the third quasi-invariant may appear to be merely a rough approximation of the zonostrophy and so have little meaning. However, the third quasi-invariant provides a general procedure to obtain a form of quasi-invariant. This procedure will be applicable for other systems in which zonostrophy-like quasi-invariants have not been found analytically. We will explore this possibility in a future study. Moreover, it may be possible to extend the above procedure to obtain still other quasi-invariants, by imposing additional constraints in the minimization problem in section 2. For the quasi-geostrophic vorticity equation (1), however, it has been proved that there is no quasi-invariant other than zonostrophy in the continuous wavenumber system (see Balk (2005)). In fact, we can solve the minimization problem with an additional constraint $\mathbf{F}_3 \cdot \mathbf{F} = 0$, where $\mathbf{F}_3$ is the vector corresponding to the coefficient of the third quasi-invariant. The solution was found to have almost the same distribution in the wavenumber space as that of the third quasi-invariant, and provides no additional information on anisotropy of energy cascade.

Before concluding this paper, let us discuss the relation of the third quasi-invariant to near-resonant triad interactions. This concept was introduced by Lee & Smith (2007), and is defined as a subset of triad interactions that satisfy $|\Delta \omega_k| < \kappa$, where $\kappa(>0)$ is a certain small number and $\Delta \omega_k = \beta^{-1}(\omega_k + \omega_{k_1} + \omega_{k_2})$. Including only a certain subset of triad interactions in numerical time-integrations of two-dimensional $\beta$-plane turbulence, Lee & Smith (2007) showed that near-resonant interactions play an important role in the generation of zonal structure. Importing their idea into the minimization procedure shown in section 2, we can show that near-resonant interactions are also important in the third quasi-invariant. The third quasi-invariant was defined so as to minimize $I$, which is defined by (33), under the constraints of (41) and (43). Let us define $I'$ to be identical to $I$, except that the summation for $k$, $l$, $k_1$, $l_1$, $k_2$, and $l_2$ is restricted to the combinations that satisfy the following expression: $|\Delta \omega_k| > 0.1$. This means that near-resonant triad interactions are excluded. We compute the coefficient for the third quasi-invariant to minimize $I'$ instead of $I$ while setting the truncation wavenumber to $K_T = 42$. Figure 10 shows the resulting distribution of the coefficient. Comparing figure 10 and figure 9(b) reveals that the dumbbell-shaped distribution
Fig. 10 Contour plot of $\log_{10}(|f_k|)$ in the wavenumber space calculated using a subset of triads (see the text for details). The thick solid contour labeled "0" indicates the locations at which $f_k = 0$. Thin solid and dashed contours indicate the locations at which $f_k$ is positive and negative, respectively.

in figure 9(b) disappears in figure 10. This implies that near-resonant interactions are essential to form the dumbbell-shaped distribution, which leads to the anisotropic energy cascade.

In conclusion, we believe that the third quasi-invariant introduced in the present manuscript can be useful tools for explaining the anisotropic energy cascade and the importance of near-resonant interactions in two-dimensional $\beta$-plane turbulence.

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In the present study, the ISPACK numerical library was used for the computations, and the GFD-DENNoui Library was used to draw the figures.

APPENDIX. PROOF OF THE ABSENCE OF COMPLETELY RESONANT TRIAD INTERACTIONS WHEN $\alpha^2$ IS AN IRRATIONAL NUMBER

Let us consider the complete resonance condition:

$$k_0 + k_1 + k_2 = 0,$$

$$l_0 + l_1 + l_2 = 0,$$

and

$$\frac{k_0}{k_0^2 + l_0^2 + \alpha^2} + \frac{k_1}{k_1^2 + l_1^2 + \alpha^2} + \frac{k_2}{k_2^2 + l_2^2 + \alpha^2} = 0.$$  \hspace{1cm} (53)

Let $k_i, l_i$ ($i = 0, 1, 2$) be integers, and let $\alpha^2$ be an irrational number. Resonance interaction does not occur if any of $|k_i|$ ($i = 1, 2, 3$) is zero, because $W_{k_1,k_2}$, as defined by (8), becomes zero in this case. Therefore, we only consider the case of $|k_i| > 0$ ($i = 0, 1, 2$).

Clearing the fractions in (53) and then using (51), we obtain

$$\{k_0(k_0^2 + l_0^2)(k_2^2 + l_2^2) + k_1(k_2^2 + l_2^2)(k_0^2 + l_0^2) + k_2(k_0^2 + l_0^2)(k_1^2 + l_1^2)\}$$

$$-\alpha^2 \{k_0(k_0^2 + l_0^2) + k_1(k_1^2 + l_1^2) + k_2(k_2^2 + l_2^2)\} = 0.$$  \hspace{1cm} (54)
Since $\alpha^2$ is an irrational number and $(k_i, l_i) \ (i = 0, 1, 2)$ are integers, both terms in the left-hand side of (54) must vanish in order for (54) to hold. This requirement yields

$$\frac{k_0}{k_0^2 + l_0^2} + \frac{k_1}{k_1^2 + l_1^2} + \frac{k_2}{k_2^2 + l_2^2} = 0, \quad (55)$$

and

$$k_0(k_0^2 + l_0^2) + k_1(k_1^2 + l_1^2) + k_2(k_2^2 + l_2^2) = 0. \quad (56)$$

We can combine (51), (55), and (56) as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{k_0^2 + l_0^2} & \frac{1}{k_1^2 + l_1^2} & \frac{1}{k_2^2 + l_2^2} \\ \frac{1}{k_0^2 + l_0^2} & \frac{1}{k_1^2 + l_1^2} & \frac{1}{k_2^2 + l_2^2} \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (57)$$

If the solution is trivial ($k_i = 0 \ (i = 0, 1, 2)$), then resonant triad interaction does not occur because $W_{k_i,k_2}$ becomes zero in this case. Therefore, we seek a nontrivial solution. In order for a nontrivial solution to exist, the matrix in the left-hand side of (57) must be singular. After a trivial calculation, we can show that $k_i^2 + l_i^2 = k_j^2 + l_j^2 \ (i \neq j)$ must hold for at least one combination of $(i, j) \ (i \neq j)$. Suppose, for example, that $i = 0$ and $j = 1$. In this case, $k_0^2 + l_0^2 = k_1^2 + l_1^2$ must hold, and, from (51) and (56), this requires that $k_0^2 + l_0^2 = k_2^2 + l_2^2$ or $k_2 = 0$. Resonant triad interaction does not occur in either case, in the former case, $W_{k_1,k_2}$ becomes zero, and the latter case is merely a phase shift of the mode of $k_0$ by a zonal ($k_2 = 0$) component. For other combinations of $(i, j)$, the proof follows similar lines. Therefore, complete resonant triad interactions do not occur when $(k_i, l_i) \ (i = 0, 1, 2)$ are integers and $\alpha^2$ is an irrational number.

REFERENCES


