A Mixed Poisson Model for Random Damage Accumulation of Tunnel Concrete Linings

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A new probabilistic model describing temporally random damage accumulation of tunnel concrete linings is proposed, in which independent two compound Poisson processes are used for (i) accumulation of daily small damages and (ii) accumulation of large damages due to severe frost damages, earthquakes or other unusual phenomena. The model is based upon a random differential equation of Itô type driven by the two compound Poisson processes, whose solution and statistical moments are analytically derived. It is clarified that the proposed model can reproduce damage accumulation processes whose statistical tendency shows good agreement with actual damage data.

1. INTRODUCTION

In Japan, many road tunnels by a conventional construction method have been constructed mainly in 1960’s or 1970’s. Recently, most of such constructed tunnels have deteriorated owing to aging and many environmental factors, which seriously increases the risk of bearing the fatal damage. Especially, road tunnels in northern area of Japan, under severe damages due to cold weather, have been deteriorated more rapidly than other areas in Japan. To execute better management of tunnels so that we can avoid the risk of accidents caused by collapse of tunnels, we may need to revise the traditional system of maintenance of the tunnels and to introduce a theoretical approach for optimizing the maintenance. Damage accumulation data obtained from tunnels in Hokkaido area shows quite large scatters, as reported in 4), 5), 6) and 7). Thus, we need to construct a probabilistic model for describing temporally random damage accumulation process so that we can predict damage growth in future based upon a probabilistic standpoint.

Some of the authors have discussed a probabilistic model for random damage accumulation based upon a stochastic differential equation of Itô type, in which a driving noise is supposed to be a Wiener process. Such a model is frequently called diffusive model, since its solution process is classified as a Markov diffusion process. In the diffusive model, an empirically obtained damage growth equation is randomized by taking into account some random factors with respect to environmental loading; earthquake and a cold winter damage. The most important feature of the diffusive model is that it can well-reproduce the shape of probability distribution of damage degree, which can be well approximated by a log-normal distribution. However, the temporal growth of damages obtained by the diffusive model shows very large fluctuations, which is clearly not suited for actual damage growth property.

In order to remove such fluctuations, some of authors have also proposed a new probabilistic model using a compound Poisson process as a driving noise, which we call Poisson model hereinafter. It has been clarified that (i) the damage accumulation process produced by the Poisson
model monotonically increases almost certainly, i.e., we can perfectly remove the fluctuation of the solution inherent in the diffusive model and (ii) statistical properties observed in actual damage data can be almost well-reproduced by the Poisson model. The Poisson model has been further applied to derive an optimal repair strategy based upon minimization of total maintenance cost\(^{(19)}\), which has been achieved by applying the stochastic control theory.

Damage accumulation in the Poisson model is classified into two parts. One is an accumulation of small scale growth due to daily environmental factors, which is approximated as deterministic under the assumption that fluctuation is small with respect to such environmental factors. The other is an accumulation of large scale growth due to unusual environmental factors such as severe frost effects or earthquakes, which is assumed to show significant scatters. However, we do not have a clear evidence, at the present stage, for ensuring that the fluctuation due to daily environmental factors is small. Further, since the currently obtained damage data shows quite large scatters whereas its mean growth rate is not so large, we may face difficulty for estimating some model parameters appearing in the Poisson model. Therefore, in this paper, we newly revise the Poisson model by introducing another compound Poisson process describing small scale damage growth caused by usual environmental factors. Our newly proposed model is based upon a stochastic differential equation of Itô type driven by two compound Poisson processes.

2. PROBABILITY MODELS FOR TEMPORALLY RANDOM DAMAGE ACCUMULATION

Let \( X(t) \) be a quantified damage degree of tunnel concrete linings\(^{4}\) at time \( t \). Figure 1 shows damage degree data currently obtained in Hokkaido area including 162 tunnels\(^{4\,5}\), where each damage degree is quantified as an averaged value of damages obtained for each span. Although the temporal variation is significantly scattered, its mean behavior shows almost exponential growth\(^{6}\). Thus, we suppose that the temporal growth of \( X(t) \) is, in the sense of mean, described by the following differential equation;

\[
\frac{dX(t)}{dt} = \mu_0 X(t),
\]

where \( \mu_0 \) is a positive parameter corresponding to the so-called propagating resistance in the fracture mechanics.

![Fig.1 Detected damage degree for 162 tunnels in Hokkaido area.](image)

Some of authors have been proposed a probabilistic model\(^{10\,8}\), in which the propagating resistance \( \mu_0 \) is randomized by a stochastic process \( W_Z(t) \) with mean zero. Then, the randomized growth equation is transformed into the following stochastic differential equation of Itô type;

\[
dX(t) = \mu_0 X(t)dt + X(t-)dZ(t),
\]

where \( Z(t) \) is a stochastic process formally defined as \( Z(t) = \int_0^t W_Z(s)ds \) and \( X(t-) \) represents a left-continuous version of \( X(t) \) defined as \( X(t-) = \lim_{s \to t^-} X(s) \).
We suppose that \( Z(t) \) has increments independent of past so that the solution process \( X(t) \) has a Markov property. If we further assume that \( Z(t) \) is continuous in probability, \( Z(t) \) is a Lévy process (generally not temporally homogeneous).

2.1 DIFFUSIVE MODEL

If \( Z(t) \) almost certainly has a continuous path, then \( Z(t) \) must be intrinsically a Wiener process according to the well-known Lévy-Itô decomposition theorem. Some of authors have discussed a probabilistic model based upon such a situation\(^6\), i.e., the damage growth equation is supposed as

\[
dX(t) = \mu_0 X(t) dt + \sigma_B X(t) dB(t),
\]

where \( \sigma_B \) is a constant. The model described by Eq.(3) is called **diffusive model** as mentioned in Section 1.

The diffusive model has been widely used especially in the field of mathematical finance. Since the probability distribution of its solution process is a log-normal distribution, it can well approximate a probability distribution of actual data\(^7\). However, the sample behavior of a diffusive model shows quite large fluctuations since the obtained damage accumulation data shows very large scatterers (as shown in Section 4.3).

2.2 POISSON MODEL

In order to avoid such a problem inherent in the diffusive model, some of authors have also developed another probabilistic model\(^8\). The Lévy-Itô decomposition theorem shows that \( Z(t) \) is intrinsically a compound Poisson process if \( Z(t) \) is a pure jump process with finite number of jumps in each bounded interval. A compound Poisson process, here denoted by \( C(0)(t) \), is expressed as

\[
C(0)(t) = \sum_{k=1}^{N(0)(t)} Y_k^{(0)},
\]

where \( N(0)(t) \) is a homogeneous Poisson process with an intensity \( \lambda(0) \) and \( \{Y_k^{(0)}\}_{k=1,2,\ldots} \) is a family of i.i.d. positive random variables. Its first and second moments are given as

\[
E\{C(0)(t)\} = \lambda(0) q_1^{(0)} t, \quad E\left\{\left(C(0)(t)\right)^2\right\} = \lambda(0) q_2^{(0)} t,
\]

where \( q_1^{(0)} \) and \( q_2^{(0)} \) are first and second moments of jumps \( Y_k^{(0)} \), i.e.,

\[
q_1^{(0)} = E\left\{Y_k^{(0)}\right\}, \quad q_2^{(0)} = E\left\{\left(Y_k^{(0)}\right)^2\right\}.
\]

It should be noted that these moments do not depend on \( k \) because of i.i.d. property. Thus, the driving noise \( Z(t) \) is constructed as

\[
Z(t) = C(0)(t) - \lambda(0) q_1^{(0)} t,
\]

which leads to the following stochastic differential equation of Itô type;

\[
dX(t) = \mu X(t) dt + X(t-)dC(0)(t) \quad \left( \mu = \mu_0 - \lambda(0) q_1^{(0)} \right).
\]

We can give a physical meaning for the damage growth equation (8) as (i) the first term of the right-hand side represents accumulation of daily small-scale damages with small scatterers and (ii) the second term represents significantly random large-scale damage growth caused by serious frost damage, salt damage and other unusual factors such as earthquakes.

2.3 A NEWLY REVISED MODEL: MIXED POISSON MODEL

Within the actual data obtained up to the present stage, we can not clearly state that the randomness associated with the damage accumulation due to usual environmental factors, which is expressed in the first term of Eq.(5), is small. Thus, in this paper, we newly revise the Poisson model so that such random feature is also incorporated by introducing another compound Poisson
process. That is, we newly propose the following random differential equation of Itô type describing random damage accumulations:

$$dX(t) = X(t-)dC^{(1)}(t) + X(t-)dC^{(2)}(t),$$

where $C^{(1)}(t)$ and $C^{(2)}(t)$ are compound Poisson processes expressed as

$$C^{(j)}(t) = \sum_{k=1}^{N^{(j)}(t)} Y^{(j)}_k \quad (j = 1, 2).$$

The compound Poisson process $C^{(1)}(t)$ represents small-scale random fluctuations caused by usual environmental factors, in which $N^{(1)}(t)$ is a homogeneous Poisson process with an intensity $\lambda^{(1)}$ and $\{Y^{(1)}_k\}_{k=1,2,...}$ is a family of i.i.d. random variables with a probability distribution function $F^{(1)}(y)$ representing a growth rate corresponding to such small-scale damage accumulations. On the other hand, $C^{(2)}(t)$ represents large-scale and significantly random damage accumulations caused by unusual environmental factors, in which $N^{(2)}(t)$ is a homogeneous Poisson process with an intensity $\lambda^{(2)}$ and $\{Y^{(2)}_k\}_{k=1,2,...}$ is a family of i.i.d. random variables with a probability distribution function $F^{(2)}(y)$ representing a growth rate corresponding to such large-scale damage growth. We basically suppose

$$\lambda^{(1)} \gg \lambda^{(2)}, \quad q_1^{(1)} \ll q_1^{(2)} \quad \left( q_1^{(j)} = \mathbb{E}\{Y^{(j)}_k\} \right).$$

![Figure 2: Sample behavior of damage processes generated by Eq. (9).](image)

In what follows, we assume that $C^{(1)}(t)$ and $C^{(2)}(t)$ are statistically independent, provided that rare events associated with large scale growth $C^{(2)}(t)$ are independent of daily environmental factors. It should be emphasized that the solution $X(t)$ is a Markov process under such assumption.

Hereinafter, we call a model described by Eq. (9) mixed Poisson model. Figure 2 shows some sample functions generated by a mixed Poisson model with $\lambda^{(1)} = 5.0[\text{year}^{-1}]$, $\lambda^{(2)} = 0.5[\text{year}^{-1}]$, $q_1^{(1)} = 2.0 \times 10^{-3}$, $q_1^{(2)} = 4.0 \times 10^{-2}$, in which jump size distribution is supposed as a Pareto distribution and the initial damage is fixed as $X_0 = 1.0$.

### 3. Probabilistic Properties Associated with Mixed Poisson Model

Giving attention to that (i) the solution $X(t)$ of Eq. (9) varies only by discrete jumps caused by two compound Poisson processes and (ii) $C^{(1)}(t)$ and $C^{(2)}(t)$ are statistically independent, we can express an increment of a transformed process $\log X(t)$ by applying the generalized Itô formula as follows:

$$\log X(t) - \log X(0) = \sum_{0 < s \leq t} \{ \log(X(s-) + \Delta X(s)) \} = \sum_{k_1=1}^{N^{(1)}(t)} \log(1 + Y^{(1)}_{k_1}) + \sum_{k_2=1}^{N^{(2)}(t)} \log(1 + Y^{(2)}_{k_2}).$$
Therefore, the solution process $X(t)$ can be expressed as follows:

$$X(t) = X_0 \prod_{k_1=1}^{N^{(1)}(t)} \left( 1 + Y_{k_1}^{(1)} \right) \prod_{k_2=1}^{N^{(2)}(t)} \left( 1 + Y_{k_2}^{(2)} \right),$$  \hspace{1cm} (12)$$

in which $X_0 = X(0)$ represents an initial damage degree. Using Eq.(12), we can easily calculate the first and second moments of the solution $X(t)$ as

$$\text{E}\{X(t)\} = X_0 \exp \left\{ \left( \lambda^{(1)} d_1^{(1)} + \lambda^{(2)} d_1^{(2)} \right) t \right\},$$  \hspace{1cm} (13)$$

$$\text{E}\{X(t)^2\} = X_0^2 \exp \left\{ \left( \lambda^{(1)} q_2^{(1)} + \lambda^{(2)} q_2^{(2)} \right) t \right\},$$  \hspace{1cm} (14)$$

where

$$q_1^{(j)} = \text{E}\{Y_{k}^{(j)}\}, \quad q_2^{(j)} = \text{E}\left\{ (Y_{k}^{(j)})^2 \right\} \quad (j = 1, 2).$$  \hspace{1cm} (15)$$

Let $F_X(x,t)$ be a transition probability distribution function of $X(t)$ under the condition of $X(0) = X_0$. Using Eq.(12), we can express it as

$$F_X(x,t) = P(X(t) \leq x | X(0) = X_0) = P \left( \sum_{k_1=1}^{N^{(1)}(t)} \log \left( 1 + Y_{k_1}^{(1)} \right) + \sum_{k_2=1}^{N^{(2)}(t)} \log \left( 1 + Y_{k_2}^{(2)} \right) \leq \log \left( x / X_0 \right) \right).$$

Let us assume that $F_Y^{(1)}(y)$ and $F_Y^{(2)}(y)$ are both differentiable. Differentiating the equation above and using the well-known property associated with a Poisson process, we can obtain the following transition probability density function of $X(t)$;

$$f_X(x,t) = \frac{\partial}{\partial x} F_X(x,t) = \frac{1}{x} e^{-\lambda} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{n!}{j!} C_j \left( \frac{\lambda^{(1)}}{\lambda} \right)^{n-j} \left( \frac{\lambda^{(2)}}{\lambda} \right)^j f_W^{(1)} \ast f_W^{(2)} \ast \left( \log \frac{x}{X_0} \right),$$  \hspace{1cm} (16)$$

where $\lambda = \lambda^{(1)} + \lambda^{(2)}$, $C_j$ represents a binomial coefficient, $f_W^{(j)}$ is a probability density function of $W^{(j)} \equiv \log \left( 1 + Y_{k}^{(j)} \right)$ $(j = 1, 2)$, "*" is an operator to take a convolution and $g^n$ represents an $n$-fold convolution of a function $g$. It should be noted that we can rewrite Eq.(16) as follows;

$$f_X(x,t) = \frac{1}{x} e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} f_W^{n\ast} \left( \log \frac{x}{X_0} \right),$$  \hspace{1cm} (17)$$

$$f_W(w) = e^{\alpha w} \left\{ \lambda^{(1)} f_Y^{(1)} (e^w - 1) + \lambda^{(2)} f_Y^{(2)} (e^w - 1) \right\},$$  \hspace{1cm} (18)$$

where $f_Y^{(j)}(y) = dF_Y^{(j)}(y)/dy$ $(j = 1, 2)$ is a probability density function of $\{Y_{k}^{(j)}\}_{k=1,2,\ldots}$.

4. NUMERICAL ESTIMATION BASED UPON ACTUAL DATA

4.1 PARAMETER ESTIMATION BASED UPON ACTUAL DATA

Here we estimate model parameters appearing in the mixed Poisson model based upon actual damage data obtained in Hokkaido area in Japan.

At the present stage, we do not have enough data for estimating statistical quantities associated with $X(t)$ for each $t$. Some of the authors have developed a method for estimating mode parameters under such situation\(^7\), in which locally stationary property is assumed in estimating statistical moments of $X(t)$. Let $\hat{E}\{X(t)\}$ and $\hat{E}\{X(t)^2\}$ be estimated first and second moments through such a method. Based upon Eq.(13) and Eq.(14), we suppose that both moments have logarithmic linearity, i.e.,

$$\log \hat{E}\{X(t)\} = a + mt, \quad \log \hat{E}\{X(t)^2\} = 2a + st,$$  \hspace{1cm} (19)$$
where $a = \log X_0$.

Figures 3 and 4 show estimated moments (dashed circles) and obtained linear regressions. Although the estimated moments show large fluctuations, first and second moments are almost well approximated by linear regressions with estimated coefficients

$$\hat{a} = 0.4528, \quad \hat{n} = 1.82 \times 10^{-2} \text{[year}^{-1}], \quad \hat{s} = 5.17 \times 10^{-2} \text{[year}^{-1}].$$

It should be noted that the estimated initial damage degree takes on a positive value. Such initial damage can be caused in a construction procedure owing to many inevitable factors.

![Fig.3 Linear regression for $E\{X(t)\}$.](image1)
![Fig.4 Linear regression for $E\{X(t)^2\}$.](image2)

We basically determine model parameters so that first and second moments of $X(t)$ coincide with the actual data with linear regressions, i.e., setting $X_0 = \exp(\hat{a})$ in Eqs. (14) and (15), we have

$$\lambda^{(1)}q_1^{(1)} + \lambda^{(2)}q_1^{(2)} = \hat{n}, \quad \lambda^{(1)}q_2^{(1)} + \lambda^{(2)}q_2^{(2)} = \hat{s}. \quad (20)$$

To determine parameters, we suppose the following situations;

- Supposing that large-scale damage growth due to rare events occurs, in the sense of mean, every two years, i.e., we set as $\lambda^{(2)} = 0.5 \text{[year}^{-1}]$. On the other hand, we here suppose that the intensity of small-scale damage growth is about five times of $\lambda^{(2)}$, i.e., $\lambda^{(1)} = 2.5 \text{[year}^{-1}].$
- The jump size distribution functions $F^{(1)}_Y(y)$ and $F^{(2)}_Y(y)$ are both of log-normal type.
- The ratio of mean jump size of $C^{(1)}(t)$ per unit time to $C^{(2)}(t)$ is given as $\lambda^{(1)}q_1^{(1)} : \lambda^{(2)}q_1^{(2)} = 3 : 7$.
- The coefficient of variation of log $Y_k^{(2)}$ is set as about -0.35, which is selected so that the scatter of damage degree almost cover the data range.

4.2 SAMPLE BEHAVIOR OF THE MIXED POISSON MODEL

Figure 5 shows 15 samples of $X(t)$ generated by the mixed Poisson model (solid curves) and actual damage data (solid circles) as shown in Fig., in which model parameters are determined as discussed in Section 4.1. Further, since we can not deterministically identify the initial damage degree $X_0$ owing to many uncertain factors in construction procedures, we assume that it obeys an exponential distribution with the estimated mean $\exp(\hat{a})$. We apply a stochastic Euler scheme with mesh size $\Delta t = 10^{-3}$ for integrating Eq.(9).

The result shows that the proposed mixed Poisson model can well-reproduce sample behavior of the damage accumulation process $X(t)$ and it never produces ‘fluctuations’ in the damage accumulation process, i.e., samples of $X(t)$ never decrease.
4.3 COMPARISON WITH OTHER MODELS

In order to clarify that the proposed mixed Poisson model is predominant in comparison with other models for random damage accumulation discussed so far.

Figure 6 shows 15 samples generated by the diffusive model (solid curves) and actual damage data (solid circles), in which model parameters are determined so that the first and second moments coincide with the estimated moments as discussed in Section 4.1, i.e., also coincide with the moments of the mixed Poisson model. We can find that generated samples show quite large fluctuations since the temporal growth of scatter in actual data is significantly large compared with mean growth. Such sample behavior is clearly not suited for actual damage accumulation phenomena.

On the other hand, Fig. 7 shows 15 samples generated by the Poisson model (solid curves) and actual damage data (solid circles). The parameters in the Poisson model is determined so that (i) the first and second moments coincide with moments of actual data and (ii) the jump size distribution of $C^{(0)}(t)$ is of log-normal type, whose variance takes on the same value as that of $C^{(2)}(t)$ in the mixed Poisson model. Under such parameter estimation, the deterministic growth rate $\mu$ in Eq. (8) takes on a negative value ($\mu = -7.0 \times 10^{-3}[\text{year}^{-1}]$). Consequently, the sample behavior show ‘fluctuation’ similarly as the diffusive model.

In the Poisson model, the scatter of $X(t)$ is produced only by $C^{(0)}(t)$. Then, in order to realize large scatter in the temporal growth, we have to give large scatter to $C^{(0)}(t)$, which simultaneously raise the mean growth rate of $X(t)$. Consequently, we have to select negative $\mu$ so that reduce the mean growth rate of $X(t)$ itself. The result given by Fig. 7 indicates that the proposed mixed Poisson model is quite useful for describing random damage accumulations.
4.4 PROBABILITY DISTRIBUTION OF THE SOLUTION

The probability distribution of the solution process $X(t)$ is described by the transition probability density function Eq. (17). Since it includes many convolution terms, we here examine the probability distribution of $X(t)$ through Monte Carlo simulation.

Figure 8 shows a relative frequency distribution of $10^4$ samples of $X(t)$ at $t = 30$ [year] generated based upon the mixed Poisson model, in which model parameters are set as same as in Fig. 8. We can see that the obtained frequency distribution shows antisymmetric shape and very long right tail. Such features are closely similar to the probability distribution of actual damage data\textsuperscript{415}.

![Graph showing relative frequency distribution](image)

Fig. 8 Frequency distribution of generated samples of $X(30)$ based upon the mixed Poisson model.

5. CONCLUSIONS

In this paper, we have newly constructed a probabilistic model for damage accumulation in tunnel concrete linings, in which independent two compound Poisson processes are introduced for describing randomness associated with accumulation of small scale damage under daily environmental loadings and that associated with accumulation of large scale damage under unusual environmental loadings. Model parameters have been estimated based upon actual damage data obtained for tunnels in Hokkaido area and statistical properties have been examined through Monte Carlo simulations.

The proposed model is considered to be predominant compared with preceding models – diffusive model and Poisson model – in the point that the proposed model has a flexibility in determining model parameters even though the actual data shows quite significant scatters, which has been demonstrated through numerical examples.

In near future, performance preservation for tunnel structures will be, undoubtedly, more and more important. Thus, much more discussion will be needed with respect to applications to maintenance strategies for tunnel concrete linings.

REFERENCES


