Breakup of a Liquid Column Jet in a Static Electric Field

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We analytically investigate breakup phenomena of a viscous liquid column jet in a static electric field. Under a long wave approximation, nonlinear equations of the jet radius, velocity and electric surface charge density are derived. Assuming a constant axial electric field strength, these equations are numerically solved for the initial-boundary condition that a semi-spherical jet emanates from a nozzle exit. It is shown that there exist two types of breakup modes - dripping and cone-jet modes - depending upon the parameters \( \Lambda (=\text{electrical force/liquid force}) \) and \( \text{Pe}(=\text{convective current/conductive current}) \). Then a critical curve is found in the \( \Lambda - \text{Pe} \) parameter space, across which the breakup is transferred from the dripping mode to the cone-jet mode as the increase of \( \Lambda \) and/or the decrease of \( \text{Pe} \). In the dripping mode, the produced drop size decreases as the increase of \( \Lambda \) for larger \( \text{Pe} \), while there is a region of \( \Lambda \) where the drop size increases as increasing \( \Lambda \) for smaller \( \text{Pe} \).

1. INTRODUCTION

Techniques of electrospray and electrospinning are available to produce fine liquid drops and thin fibers of submicron order\(^1\). These are based on fluid motions in static electric fields known in the electrohydrodynamics (EHD)\(^2\). Analytical studies in EHD have already started from the pioneering work of Rayleigh\(^3\) in 19th century. He examined capillary wave instabilities on a charged droplet to show a limitation of the stability to a total surface charge. Such a droplet is more unstable for deformations and forms a conical surface when the charge becomes sufficiently large. Later, this conical shape was analytically examined by Taylor\(^4\) and called Taylor cone after his name.

It is experimentally shown that such phenomena also appear on a jet emanating from a nozzle\(^5,6\). When the electric field strength is sufficiently weak, liquid drops are successively produced due to the surface tension instability. However, for sufficiently large electric field strength, the jet forms the Taylor cone near the nozzle and a series of fine liquid drops or a thin thread is produced from a tip of the cone. The former is sometimes called the dripping mode, though the term ‘dripping’ should be strictly distinguished from the term ‘jetting’ depending upon the Weber number\(^6\). On the other hand, the latter is called the cone-jet mode.

In the previous theoretical investigations the following two have been mainly considered: behavior of a steady semi-infinite jet which is spatially thinning and accelerated due to the axial electric field\(^7,9\) and temporal instabilities of disturbances on a uniform infinite jet when the axial electric field is neglected\(^10,11\). However, in practice, the jet is not only unsteady but also finite in length. Therefore, in this paper, we investigate evolutions of the jet emanating from a nozzle under the axial electric field and reveal a parameter region governing the breakup modes and produced drop sizes.

The organization of this paper is as follows: in Section 2 we show the formulation of the present problem and derive jet equations under a long wave approximation. In Section 3, linear instabilities
of disturbances are examined on the jet which is weakly nonuniform due to the axial electric field. In Section 4, an initial-boundary value problem for the finite jet is numerically solved and, finally, we summarize our results in Section 5.

2. FORMULATION

We consider the jet in the static electric field as shown in Fig.1, where the liquid is viscous and slightly conducting. The jet is placed inside the concentric metal sheath with radius $R$. According to the leaky dielectric model\(^7\), we assume that the charge stays only on the surface of the jet. Then the surface charge is subject to the axial (tangential) force due to the constant electric field strength $E_w$ along the sheath which is generated by the applied voltage $V$. In the analysis, we do not consider the motion of surrounding gas, the gravitational force and the self-induction due to the surface charge.

Assuming the $z - r$ axisymmetric coordinate system, the jet radius is prescribed by $r = h(z, t)$ ($h < R$). The liquid density is denoted by $\rho$, the velocity by $\mathbf{u} = (u(z,r,t), v(z,r,t))$ and the surface charge density by $\sigma_e(z,t)$, while the electric fields outside ($h < r < R$) and inside ($0 < r < h$) the jet, respectively, by $\mathbf{E}^{(o)}(z,r,t)$ and $\mathbf{E}^{(i)}(z,r,t)$.

The basic equations consist of the continuity and momentum equations of the liquid jet for $0 < r < h$:

$$\nabla \cdot \mathbf{u} = 0, \quad \rho(\partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u}) = \nabla \cdot \mathbf{D},$$

where the stress tensor $\mathbf{D} = -p I + \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ with kinematic viscosity $\mu$ and unit matrix $I$ is introduced, while the electric field is continuous and irrotational both inside ($0 < r < h$) and outside ($h < r < R$) the jet because of no electric charge and no magnetic field (the magnetic field acts passively in a slightly conducting fluid even if it exists):

$$\nabla \cdot \mathbf{E}^{(o)} = \nabla \cdot \mathbf{E}^{(i)} = 0, \quad \nabla \times \mathbf{E}^{(o)} = \nabla \times \mathbf{E}^{(i)} = 0.$$

On the other hand, the kinematical boundary conditions at $r = h$ consist of

$$\partial F/\partial t + \mathbf{u} \cdot \nabla F = 0, \quad \mathbf{E}^{(o)} \cdot \mathbf{n} = \mathbf{E}^{(i)} \cdot \mathbf{n}, \quad \epsilon^{(o)} \mathbf{E}^{(o)} \cdot \mathbf{n} - \epsilon^{(i)} \mathbf{E}^{(i)} \cdot \mathbf{n} = \sigma_e,$$

where $F \equiv r - h$ and $\mathbf{n}$ and $\mathbf{t}$ are the unit normal and tangential vectors on the surface, while $\epsilon^{(i)}$ and $\epsilon^{(o)}$ are the dielectric constants inside and outside the jet. The dynamical conditions are given in the normal and tangential components, respectively,

$$[\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n}]^{(o)}_{(i)} \equiv \gamma \kappa, \quad [\mathbf{t} \cdot \mathbf{D} \cdot \mathbf{n} + \mathbf{t} \cdot \mathbf{T} \cdot \mathbf{n}]^{(o)}_{(i)} = 0,$$

where $[\cdot]^{(o)}_{(i)} = (\cdot)^{(o)} - (\cdot)^{(i)}$ denotes the jump in the quantity $(\cdot)$ across the interface. In the above, the surface tension $\gamma$, the curvature $\kappa = h^{-1}[1 + (\partial h/\partial z)^2]^{-1/2} - (\partial^2 h/\partial z^2)[1 + (\partial h/\partial z)^2]^{-3/2}$ and the Maxwell stress tensor $\mathbf{T} = \epsilon[\mathbf{E} \mathbf{E} - (1/2) \mathbf{I} \mathbf{E} \cdot \mathbf{E}]$ are introduced. Besides, on the surface, the surface charge density is governed by\(^2\)

$$\partial \sigma_e/\partial t + \mathbf{u}_n \cdot \nabla_n \sigma_e + \nabla_s \cdot (\sigma_s \mathbf{u}_s) + \sigma_s (\mathbf{n} \cdot \mathbf{u}) \kappa = -[\mathbf{K} \mathbf{E} \cdot \mathbf{n}]^{(o)}_{(i)}.$$
where $K$ denotes the electric conductivity, $\mathbf{u}_n = \mathbf{n} (\mathbf{n} \cdot \mathbf{u})$, $\mathbf{u}_s = t (\mathbf{t} \cdot \mathbf{u})$, $\nabla_n = \mathbf{n} (\mathbf{n} \cdot \nabla)$ and $\nabla_s = \nabla - \nabla_n$.

Assuming the jet to be sufficiently thin compared with wave length of deformation, we can introduce the following long wave approximation (slender jet approximation):

$$ u = u_0(z,t) + r^2 u_2(z,t) + \cdots, \quad E_{z}^{i} = E_{z0}^{i}(z,t) + r^2 E_{z2}^{i}(z,t) + \cdots, $$

which leads to the leading terms of $v$ and $E_{e}^{i}$ from the continuity equations in Eqs.(1) and (2)

$$ v = -(r/2)(\partial u_0/\partial z) - \cdots, \quad E_{e}^{i} = -(r/2)(\partial E_{z0}^{i}/\partial z) - \cdots. $$

Making use of the above expansions into Eqs.(1) to (5), we finally obtain the following non-dimensional equations in the lowest order of the approximation (the suffix 0 has been omitted):

$$ \partial h/\partial t + u \partial h/\partial z = -(h/2)(\partial u/\partial z), $$

$$ \partial u/\partial t + u \partial u/\partial z = -(1/Wb)(\partial \sigma_e/\partial z) + (1/Re)(3/h^2)[(\partial (h^2 \partial u/\partial z)/\partial z] $$

$$ + \Lambda[(2\sigma_e E_z/h) + \beta E_z (\partial E_z/\partial z) + \sigma_e (\partial \sigma_e/\partial z)], $$

$$ \partial \sigma_e/\partial t + u \partial \sigma_e/\partial z = -(\sigma_e/2)(\partial u/\partial z) - (1/Pe)(1/2h)[(\partial (h^2 E_z)/\partial z], $$

where we note that $\sigma_e \simeq \epsilon^{(0)} E_0^{(0)}$ and $E_z^{(0)} \simeq E_z^{(0)} \equiv E_z$ from the kinematical condition in (3). In the above representations, the non-dimensional parameters $Wb = \rho U^2 h_0/\gamma$, $Re = \rho U h_0/\mu$, $\Lambda = \epsilon^{(0)} E_0^2/(\rho U^2)$, $\beta = \epsilon^{(i)}/\epsilon^{(0)} - 1$ and $Pe = U \epsilon^{(i)}/(Kh_0)$ are introduced. In particular, $\Lambda$ means the ratio of the electric force due to a static electric field to the fluid force like a dynamic pressure, while Pe called the electrical Peclet number means the ratio of the convective current to the conductive current since the ratio is given by $U \epsilon^{(i)} E_0 2\pi h_0/(KE_0 \pi h_0^2)$ with the characteristic electric field strength $E_0$, velocity $U$ and length $h_0$.

On the other hand, we note that $E_z$ on the surface is related to $E_w$ as follows7,8:

$$ E_z = E_w - \ln(R/h)[(\partial (h \sigma_e)/\partial z] + \sigma_e h \partial h/\partial z. $$

However, in the present analysis, we set $E_z \simeq E_w$ (const.) for simplicity12, which might be valid when $h \lesssim R$ and $\partial h/\partial z$ varies moderately. Thus the problem is reduced to solving Eqs.(8a) to (8c) for an appropriate initial and boundary condition, provided that $E_z$ is constant and, therefore, the derivative of $E_z$ always vanishes. In the following analysis, we always consider when $Wb = 10$, $Re = 100$ and $E_w = E_z = 0.5$. Before going on to the numerical analysis of these equations when the jet length is finite, we show some results of the linear instability for an infinite jet.

3. LINEAR INSTABILITY

Because of the tangential force between the surface charge density and the axial electric field, the jet becomes gradually thinning and accelerated to the downward. Assuming this nonuniformity to be weak, we consider the following disturbances on the steady flow:

$$ h = \bar{h}(Z) + \delta h(z,Z,t), \quad u = \bar{U}(Z) + \delta u(z,Z,t), \quad \sigma_e = \bar{\sigma}_e(Z) + \delta \sigma_e(z,Z,t), $$

where $\delta$ is a small parameter. In the above, according to the multiple-scale perturbation method13, the long scaled variable $Z (= \delta z)$ has been introduced and, so that, $\partial/\partial Z$ is replaced by $\partial/\partial Z + \delta \partial/\partial Z$. Since the wave number of disturbances is spatially influenced by the nonuniformity of the steady flow, the disturbances can be set as

$$ \bar{h}, \bar{u}, \bar{\sigma}_e \propto \exp[i(k(Z)z - \omega t)]. $$
Fig. 2: Profiles of the steady jet for $Pe = 50$, where the solid line denotes the case for $\Lambda = 0.1$ and the broken line for $\Lambda = 0.2$.

Using (10) into Eqs.(8a) to (8c), in the lowest order of the approximation, we can obtain the equations of variables with $\vec{\cdot}$ from the steady parts and those with $\tilde{\cdot}$ from the unsteady parts. As a result, the steady state is governed by the equations

\[
\begin{align*}
\bar{u}\bar{h}^2 &= 1, \quad \bar{\sigma}_e = \bar{h}[E_z(1 - \bar{h}^2)/(2 Pe) + 1], \\
\bar{u}^2 &= -2/(Wb \bar{h}) + \Lambda[4E_z \int_0^z (\bar{\sigma}_e/\bar{h}) dZ + \bar{\sigma}_e^2] + (1 - \Lambda + 2/Wb),
\end{align*}
\]  

(12)

where we have assumed $\bar{h} = \bar{u} = \bar{\sigma}_e = 1$ at $z = 0$. On the other hand, making use of (11) into the unsteady equations (not shown) of the variables with $\tilde{\cdot}$, we have the following dispersion relation from a compatibility condition:

\[
\Omega^3 + (3i/Re)k^2\Omega^2 + \Omega[\Lambda \bar{\sigma}_e^2/2 + \bar{h}(1/\bar{h}^2 - k^2)/(2 Wb)]k^2 + \Lambda E_z k^2(\bar{h} \bar{\sigma}_e - 2iE_z)/(2 Pe) = 0,
\]

(13)

where $\Omega = \omega - ku$. Solving the above equation for $\Omega$ as a function of real $k$ based on the steady solutions in Eqs.(12), we can find that the disturbance is temporally unstable when the imaginary part of $\Omega$ ($=\Omega_I$) is positive, since both $k$ and $u$ are always assumed to be real.

Figure 2 shows the profiles of the steady state jet for $Pe = 50$, where the solid line denotes the case for $\Lambda = 0.1$ and the broken line for $\Lambda = 0.2$. The jet radius gradually reduces with $z$ due to the tangential electric force, whose effect is slightly enhanced for larger $\Lambda$. For the same parameters, the instability diagrams at $z = 0$ are shown in Fig.3. In this figure, it is found that the jet is unstable for long disturbances, where the critical wave number $k = k_c$ giving the maximum growth rate of $\Omega_I$ becomes larger for larger $\Lambda$. Resulting from this, it is expected that the liquid drop size becomes smaller as $\Lambda$ increases in the dripping mode, since the drop size would be determined by the disturbance with the most unstable wavenumber.

Figure 4 shows the variations of $k_c$ for different $z$ when $Pe = 5$ in (a) and 50 in (b), where the lines denote the cases for $\Lambda = 0.05$, 0.1 and 0.2. It is found from the figure that $k_c$ increases as the increase of $z$ as well as $\Lambda$, whose tendency is more enhanced in larger $Pe$. Thus, the produced drop size would be also smaller as the increase of $z$ as long as the dripping mode is preserved. Even in this case the long wave approximation would be still valid, since the jet radius also decreases as the increase of $z$. Because of the long wave instability as in Fig.3, any short wave components of
Fig. 4: Variations of $k_e$ at different $z$ for $\Lambda = 0.05, 0.1$ and 0.2, where $Pe = 5$ in (a) and $Pe = 50$ in (b).

initial disturbances will not grow in the wave propagations and, so that, only long wave components dominantly appear after a long time, where the long wave approximation is valid.

4. NUMERICAL ANALYSIS

In the numerical analysis, we adopt the following initial-boundary condition for the semi-spherical jet emanating from the nozzle exit at $z = 0$:

$$k(z, 0) = \sqrt{1 - z^2}, \quad u(z, 0) = 1, \quad \sigma_e(z, 0) = 1.$$  \hspace{1cm} (14)

For the above condition, Eqs.(8a), (8b) and (8c) are solved by the finite difference method, where, noting that the advection velocity $u$ exists in common on the left hand side in each equation, the calculations are carried out by using the CIP method for the advection phase and the time splitting method for the non-advection phase\textsuperscript{14}. In the calculations, the numerical grid sizes are taken to be $\Delta t = 0.0001$ and $\Delta z = 0.1$ in order to retain the numerical accuracy within the relative error of 1.5% in the volume ratio. The parameters $Wb = 10$, $Re = 100$, $E_t = E_w = 0.5$ are still fixed, while $Pe$ and $\Lambda$ are chosen as the control parameters. We first show in Fig.5 the profiles of the jet and surface charge density and axial velocity distributions at the breakup time $t_b = 18.6$ when $\Lambda = 0.06$, $Pe = 5$. As is seen in this figure, the jet produces a large liquid drop at the top and breaks up by pinching off. This breakup is called the dripping mode where the surface tension is dominant. The surface charge abruptly decreases at the pinching and becomes rather large on the drop surface, while the axial velocity has a sharp peak near the pinching. However, in Fig.6 where $\Lambda$ increases large to unity without changing the other parameters, we can see that the jet makes a cone shape profile (Taylor cone) near the nozzle exit and subsequently becomes thinner to the downward without any liquid drop. This breakup is called the cone-jet mode where the electric force becomes superior to the surface tension. We note in this mode the breakup time $t_b = 2.4$ is much shorter than that in the dripping mode. Both surface charge and axial velocity increase as approaching the tip of the jet and, as a result, the tangential force subject to the axial electric field is increased and, in turn, the flow velocity is increased and the jet becomes thinner and thinner.

Such two different modes appear not only for different $\Lambda$ but also for different $Pe$. Figure 7 shows how these two modes appear in the $\Lambda - Pe$ parameter space. It is found from the figure that the transition from the dripping mode to the cone-jet mode appears across the critical curve when
Fig. 5: The surface profile, surface charge density and axial velocity distributions for the dripping mode at the breakup time $t = 18.6$ for $\Lambda = 0.06$, $\text{Pe} = 5$.

Fig. 6: The surface profile, surface charge density and axial velocity distributions for the cone-jet mode at the breakup time $t = 2.4$ for $\Lambda = 1$, $\text{Pe} = 5$.

Fig. 7: The critical curve between the dripping and cone-jet modes in ($\Lambda$, $\text{Pe}$) space.

Fig. 8: The breakup profiles in the marginal region for $\text{Pe} = 50$, where $\Lambda = 0.14$ in (a) and $\Lambda = 0.19$ in (b).
\( \Lambda \) increases and/or Pe decreases. Figure 8 shows the breakup profiles in the marginal parameter region when \( \Lambda = 0.14 \) and 0.19 for Pe = 50. We can find that the large drop on the tcp of the jet is drastically reduced to the appearance like a microdrop at the tip of the cone when \( \Lambda \) slightly increases across the curve. We note that this appearance of microdrop disappears for larger \( \Lambda \) as in Fig.6, which may be closely related to the difference between the electrospay and electrosprinning.

Finally, we show the produced drop size in the dripping mode. Figure 9 shows the averaged diameter \( D_{\text{ave}} \) of the drop for different \( \Lambda \) and Pe, where \( D_{\text{ave}} \) is defined by \((D_1D_2^2)^{1/3}\) with the axial diameter \( D_1 \) and radial diameter \( D_2 \) of the drop. In the figure we can see that \( D_{\text{ave}} \) decreases as the increase of \( \Lambda \) for Pe \( \geq 6 \), while there is a region of \( \Lambda \) where \( D_{\text{ave}} \) increases as the increase of \( \Lambda \) for Pe \( \leq 5 \). We note that the tendency of \( D_{\text{ave}} \) which decreases as the increase of \( \Lambda \) agrees with the linear prediction where the critical wave number increases as the increase of \( \Lambda \) at a fixed \( z \) as in Fig.4. However, the existence of the region increasing \( D_{\text{ave}} \) with \( \Lambda \) is not predicted by the linear instability, though the linear theory cannot determine in advance which modes of dripping and cone-jet appear. Since the infinite jet is assumed in the linear instability, the discrepancy might be resulting from the top of the jet, where inflow or outflow of the surface charge from the bulk of the jet strongly arises because of the derivative term of \( h \) (see Eq.(8c)) whose effect is more enhanced for smaller Pe.

Fig. 9: Variations of the averaged drop sizes \( D_{\text{ave}} \) for different values of \( \Lambda \) and Pe in the dripping mode.

5. CONCLUSIONS

The evolution of the jet emanating from the nozzle is numerically examined when the axial electric field is constant. Typical breakup profiles are shown in the dripping and cone-jet modes and the existing region of these two modes is found in the parameter space of \( \Lambda (=\text{static electric force/liquid dynamic force}) \) and \( \text{Pe} (=\text{convective current/conductive current}) \). It is found that the critical curve exists in the parameter space, across which the transition of the dripping mode to the cone-jet mode appears when \( \Lambda \) increases and/or Pe decreases.

On the other hand, it is found that the produced drop size in the dripping mode decreases as the increase of \( \Lambda \) for larger Pe, which agrees with the linear prediction. However, there is a certain region of \( \Lambda \) where the drop size increases with the increase of \( \Lambda \) for smaller Pe, which is not predicted by the linear instability.

ACKNOWLEDGMENT

This work has been partially supported by the Grant-in-Aid for Science Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan (No.21560177).

REFERENCES


