Effects of Rotation on a Point Sink Flow of Stratified Viscous Fluid

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We numerically study a point sink flow of stratified viscous fluid in a reservoir of finite depth under a rigid rotation of the whole system. In the initial stage after starting a discharge, selective withdrawal is formed and only a layer of fluid adjacent to the sink level is withdrawn. As time elapses more, this withdrawal layer increases its thickness gradually. The speed of such withdrawal-layer thickening is found to be a function of the Reynolds number only and does not depend on the other parameters if the time is non-dimensionalized using the rotating speed of the system. Based on our numerical results, we also discuss the effect of the earth’s rotation on sink flow in the actual situation.

1. INTRODUCTION

We consider a point sink flow in a reservoir of finite depth. When a fluid is stratified in the vertical direction (in which the gravity acts), only a layer of fluid adjacent to the sink level is withdrawn. This phenomenon is called selective withdrawal, and there exists wide practical application to water quality management\(^1,2\) such as withdrawal of cooling water for power plants. Therefore many researchers have studied it using a point sink flow of linearly stratified fluid as its simple model\(^3-5\).

When lengthscale or timescale we are concerned is relatively large, effect of the earth’s rotation must be taken into account\(^6,7\). In the actual situation of dam reservoir where soil control is necessary, selective withdrawal is utilized for long time in order to prevent muddy water being discharged. In this case the intake and sink flow are continued for more than several hours or days so that the effect of the earth’s rotation is significant. However, there are not many prior studies which examined this rotational effect and no numerical study can be found which investigated this effect systematically. In the present study, therefore, we investigate this point sink flow of stratified viscous fluid in a rotating system numerically. In particular, we want to clarify features of withdrawal-layer thickening near the sink caused by a strong swirling flow because this thickening can spoil the effect of selective withdrawal. It is found that the speed of thickening is a function of the Reynolds number only and does not depend on the other parameters if the time is non-dimensionalized using the rotating speed of the system. We also discuss the effect of the earth’s rotation in the actual situation on the basis of our numerical results.

This paper is structured as follows. In Section 2, we formulate basic equations in order to specify four parameters characterizing the problem. Section 3 presents numerical results and Section 4 gives a discussion on actual situation.

2. BASIC EQUATIONS

Consider a flow induced by an impulsively started point sink of stratified fluid in a space between two
horizontal free-slip planes located at $z = \pm d$ ($0 < r < \infty$, $0 < \theta < 2\pi$, $-d < z < d$; $r - \theta - z$ is a cylindrical coordinate system) in which the point sink is located at $(r, z) = (0, 0)$, the gravitational force $g$ acts in the negative $z$ direction and the whole system is rotating at angular speed $f/2$ around the $z$ axis. It should be noted that the position of the axis of rotation, which is taken to be on $r = 0$ here, has no effect on the flow as long as it is in the $z$ direction (the governing equations (3)-(7) are independent of the position of the axis of rotation). The problem geometry is sketched in Fig. 1. Initially, the fluid is quiescent and linearly stratified in the vertical $z$ direction so that the initial density distribution is given by

$$\bar{\rho}(z) = \rho_0(1 - bz),$$

where $\rho_0$ is the density at $z = 0$ and $b$ is a positive constant. We are here interested in the flow after starting the discharge $Q$ from the point sink at time $t = 0$. This flow has no $\theta$ dependence, and hence, is axisymmetric about the $z$ axis. The Boussinesq approximation is applied considering the fluid satisfying $bd << 1$, which is often the case of most practical situations. The flow is then symmetric with respect to the $r$ and $z$ axes. Therefore, we analyze this problem only in the first quadrant of a $r-z$ plane by imposing the free-slip condition on the $r$ and $z$ axes in the fluid.

We express the time-dependent density distribution $\rho$ as

$$\rho(r, z, t) = \bar{\rho}(z) + \rho'(r, z, t),$$

where $\rho'$ is the density fluctuation from its initial value $\bar{\rho}$. We define $(u, v, w)$ as the velocity components in the $(r, \theta, z)$ directions and $p$ as the pressure. Then the governing equations incorporating the viscous and diffusive effects are given by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} + v^2 = -\frac{1}{\rho_0} \frac{\partial p}{\partial r} + \nu \left( \Delta u - \frac{u}{r^2} \right),$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} = \nu \left( \Delta v - \frac{v}{r^2} \right),$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = \frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{\rho'}{\rho_0} g + \nu \Delta w,$$

$$\frac{\partial \rho'}{\partial t} + u \frac{\partial \rho'}{\partial r} + w \frac{\partial \rho'}{\partial z} - \rho_0 b w = D \Delta \rho',$$

Fig. 1 Geometry of point sink flow. Calculation region is shaded.
where \( \Delta = \partial^3 / \partial r^3 + (1 / r) \partial / \partial r + \partial^3 / \partial z^3 \), \( \nu \) is the kinematic viscosity and \( D \) is the diffusivity of stratifying species. From the continuity (3), we can introduce the stream function \( \psi \) satisfying
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 \psi}{\partial z^2} = \frac{f}{\nu} \frac{\partial \psi}{\partial z},
\]
and by cross-differentiating (4) and (6), we can eliminate \( p \). Thus, the above set of equations (3)-(7) reduces to
\[
\frac{\partial \zeta}{\partial t} + \left( \frac{\partial \psi}{\partial r} \frac{\partial \zeta}{\partial r} - \frac{\partial \psi}{\partial z} \frac{\partial \zeta}{\partial z} - \zeta \frac{\partial \psi}{\partial r} \right) - 2 \frac{\partial \psi}{\partial z} = \frac{g}{\rho_0} \frac{\partial \rho'}{\partial z},
\]
(8)
\[
\frac{\partial \psi}{\partial t} + \left( \frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial r} - \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial z} \right) + \frac{f}{\nu} \frac{\partial \psi}{\partial z} = \nu \left( \frac{\Delta \psi}{r^2} \right),
\]
(9)
\[
\frac{\partial \rho'}{\partial t} + \frac{1}{r} \left( \frac{\partial \psi}{\partial z} \frac{\partial \rho'}{\partial r} - \frac{\partial \psi}{\partial r} \frac{\partial \rho'}{\partial z} \right) + \frac{\rho \beta}{\nu} \frac{\partial \psi}{\partial z} = D \Delta \rho',
\]
(10)
\[
\zeta = \frac{1}{r} \left( \frac{\partial^3 \psi}{\partial r^3} - \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^3 \psi}{\partial z^3} \right).
\]
(11)

These equations (8)-(11) are to be solved under the boundary conditions
\[
\psi = 0, \zeta = 0, \rho' = 0 \quad (r > 0, z = 0),
\]
\[
\psi = -\frac{Q}{4 \pi}, \zeta = 0, \rho' = 0 \quad (r > 0, z = d),
\]
\[
\psi = -\frac{Q}{4 \pi}, \zeta = 0, \frac{\partial \rho'}{\partial r} = 0 \quad (r = 0, 0 < z < d),
\]
(12)
\[
\psi = -\frac{Q z}{4 \pi d} \quad (r = \infty, 0 < z < d),
\]
and the initial conditions
\[
\zeta = 0, \rho' = 0 \quad (r > 0, 0 < z < d). \quad (13)
\]

The system (8)-(13) is characterized by the four parameters: the dimensionless rotating frequency, the Froude number \( Fr \), the Reynolds number \( Re \), and the Schmidt number \( Sc \), defined by
\[
\frac{f}{N}, \quad Fr = \frac{Q}{Nd}, \quad Re = \frac{Q}{\nu d}, \quad Sc = \frac{\nu}{D},
\]
(14)
where \( N = \sqrt{bg} \) is the buoyancy frequency.

3. NUMERICAL RESULTS

In order to investigate the sink flow problem presented in Section 2, the set of equations (8)-(13) is solved numerically. The numerical method is the same as that in Ref.5 so that the reader is referred to that paper.

Figure 2 shows streamlines at \( Nt = 100 \) and 1000 of the sink flow both in the non-rotating \( (f = 0) \) and rotating systems \( (f / N = 0.02) \). Comparing time development of streamlines in these two cases, we see that the flow near the sink becomes steady at \( Nt = 100 \) in the non-rotating system, whereas it remains unsteady in the rotating system. The withdrawal-layer thickness (the vertical thickness of fluid layer flowing to the sink) therefore approaches steady-state value in the non-rotating system (Fig.2a), whereas it increases with time in the rotating system (Fig.2b). In order to discuss the physical mechanism why this thickening occurs in the rotating system, let us consider the forces acting on the fluid flowing to the sink. When the fluid moves toward the sink in the negative \( r \) direction, it is subject to Coriolis force in the positive \( \theta \) direction so that a swirling flow \( (\nu > 0) \) occurs. This swirling flow in the positive \( \theta \) direction induces both Coriolis and centrifugal forces in the positive \( r \) direction which prevent the fluid from flowing toward the sink. So the fluid in a stagnant region located above the
flowing region is also withdrawn in order to keep a constant volumetric discharge \( Q \).

Let us examine time dependence of the withdrawal-layer thickness \( \delta \) in more detail. In the present study, we define \( \delta \) as the \( z \) coordinate at which 80 per cent of fluid is withdrawn to the sink, or \( \delta = z \) at which \( \psi = 0.8 \times (-Q/4\pi) \). In the non-rotating system (\( f = 0 \)) the withdrawal-layer thickness approaches steady-state value \( \delta_0 \), and when the Reynolds number is high (\( Re \geq 1000 \)), \( \delta_0 \) is found to be

\[
\delta_0 = 0.43 Fr^{1/3} d .
\]  

(15)

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Fig. 2 Streamlines \( (0 < r < 20d , \ 0 < z < d) \) at \( Nt = 100 \) and \( 1000 \): \( (a) \ f = 0 \); \( (b) \ f / N = 0.02 \). The other parameters are fixed at \( Fr = 0.01 \), \( Re = 1000 \) and \( Sc = \infty \).
In the rotating system \((f > 0)\), the withdrawal-layer thickness increases with time as shown in Fig.2(b). Such a time dependence of \(\delta\) at \(r = d\) is shown in Fig.3 for \(Re=1000\). In this figure the ordinate is withdrawal-layer thickness \(\delta\) divided by \(\delta_0\) where \(\delta_0\) is defined by (15), and the abscissa is a dimensionless time \((ft)^{1/3}\). We find that the profile of \(\delta/\delta_0\) as a function of \((ft)^{1/3}\) can be represented by a single equation for high Reynolds number flows. Specifically,

\[
\frac{\delta}{\delta_0} = 3.2(ft)^{1/3} - 2.7
\]

is a good approximation of \(\delta/\delta_0\) when \((ft)^{1/3} > 1.5\).

Let us examine the effect of viscosity. Figure 4 shows \(\delta/\delta_0\) versus \((ft)^{1/3}\) for \(Re=10, 20, 40, 100\) and \(1000\) (with fixed values of \(f/N = 0.05, Fr = 0.01\) and \(Sc = \infty\)). It is clear from this figure that the gradient of \(\delta/\delta_0\) with respect to time \((ft)^{1/3}\) approaches different constant values for different Reynolds numbers and they become smaller as \(Re\) decreases. This fact indicates that the speed of withdrawal-layer thickening becomes slower as \(Re\) decreases. Such a slow thickening is caused by a slow increase of a swirling flow due to viscosity.

In order to check whether the above features of the withdrawal-layer thickening are also true of the other parameter values of \(f/N\) and \(Fr\), we plotted \(\delta/\delta_0\) for several different values of \(f/N\) and \(Fr\) in Fig.5. We see that the gradients of any profile are independent of \(f/N\) and \(Fr\). Thus, we can say that the increasing speed of the withdrawal-layer thickness depends only on a single parameter \(Re\). Specifically,

\[
\frac{d(\delta/\delta_0)}{d(ft)^{1/3}} = \alpha(Re),
\]

where \(\alpha(Re)\) is an increasing function of \(Re\) given by

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Fig. 3 Dimensionless withdrawal-layer thickness \(\delta/\delta_0\) at \(r = d\) versus \((ft)^{1/3}\) for \(Re=1000\) and \(Sc = \infty\). \(\times:\(f/N, Fr\) = (0.02, 0.005), \(\ominus:\(0.02, 0.01), \Delta:\(0.05, 0.005), \odot:\(0.05, 0.01), \triangledown:\(0.05, 0.05), \odot:\(0.1, 0.05)\).

Fig. 4 Dimensionless withdrawal-layer thickness \(\delta/\delta_0\) at \(r = d\) versus \((ft)^{1/3}\) for various Reynolds numbers with \(Sc = \infty\). The other parameters are fixed at \(f/N = 0.05\) and \(Fr=0.01\). The crosses are experimental results for \(Re=22\) by Whitehead89.
\[
\alpha(\text{Re}) = \begin{cases} 
0.4 - 0.5 & (\text{Re} = 10) \\
1.2 - 1.4 & (\text{Re} = 20) \\
2.2 - 2.4 & (\text{Re} = 40) \\
2.8 - 3.0 & (\text{Re} = 100) 
\end{cases}
\]

(18)

In figure 4 the experimental result of Whitehead\(^6\) for \(\text{Re}=22\) is also shown by the diagonal crosses. This result gives \(\alpha(22) = 1.6\), and we find a good agreement with our numerical results.

Finally, we discuss the effect of diffusion. This can be examined by setting the Schmidt number \(\text{Sc}\) finite. In many practical situations the stratification is formed by temperature difference and the corresponding Schmidt number is about 7. The profiles of the withdrawal-layer thickness \(\delta/\delta_0\) for \(\text{Sc}=7\) as well as those for \(\text{Sc}=\infty\) are presented in Fig.6. From this figure we see that the effect of diffusion is very small. Thus, our results in the non-diffusive case can be applied to practical situations only with a slight modification. Incidentally, for the smaller \(\text{Sc}\), the withdrawal-layer thickness \(\delta/\delta_0\) becomes much smaller.

4. DISCUSSION

Let us discuss the effect of the earth’s rotation in the actual situation of sink flows in the ocean or dam reservoir. The speed of the earth’s rotation is \(f = 1.0 \times 10^{-4}\) rad/s in mid-latitude region. The typical

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Fig. 5 Dimensionless withdrawal-layer thickness \(\delta/\delta_0\) at \(r=d\) versus \((ft)^{1/3}\) for various parameters with \(\text{Sc}=\infty\). Open, closed and open-dot markers denote results for \(\text{Re}=10, 20\) and 40, respectively. The other parameters \((f/N, Fr)\) are arranged as follows.

\[\bullet, \Diamond, \bigcirc: (0.05, 0.001); \quad \blacksquare, \square, \bigcirc: (0.1, 0.001); \]
\[\blacktriangledown, \nabla, \triangledown: (0.05, 0.01); \quad \blacktriangle, \triangle, \triangle: (0.1, 0.01); \]
\[\bullet, \bigcirc, \bigcirc: (0.1, 0.1).\]

Fig. 6 Dimensionless withdrawal-layer thickness \(\delta/\delta_0\) at \(r=d\) versus \((ft)^{1/3}\) for \(\text{Sc}=7\) (closed markers) and \(\infty\) (open markers). The remaining parameters are fixed at \(f/N = 0.05\) and \(Fr=0.01\).
value of the buoyancy frequency when the water temperature changes 1 degree by depth of 10 m is \( N = 0.005 \sim 0.02 \text{ rad/s} \). The corresponding dimensionless rotating frequency is \( f / N = 0.005 \sim 0.02 \). As for the other parameters, Fr is small (Fr < 0.1), Re is large (Re > 1000) and Sc is around 7. Since the effect of diffusion for Sc=7 is small according to our numerical results in Section 3 (Fig.6), the results of non-diffusive case for Re=1000 presented in Fig.3 can be used to estimate effect of the earth’s rotation in practical situations.

As an example, let us consider the case of \( d = 40 \text{ m}, Q = 1 \text{ m}^3/\text{s} \) and \( N = 0.01 \text{ rad/s} \). The corresponding dimensionless parameters are \( f / N = 0.01\), Fr=1.6\times10^{-3} and Re=O(10^4). From Fig.3, we see that the withdrawal-layer thickness \( \delta \) becomes one-and-a-half times \( \delta_0 \) by \( (ft)^{1/3} = 1.2 \) which corresponds to five hours. As time elapses more, \( \delta \) becomes twice and three-and-a-half times \( \delta_0 \) by \( (ft)^{1/3} = 1.45 \) and 2.05 which correspond to eight and twenty-four hours, respectively.

Thus, when sink flow is conducted for long times in the ocean or dam reservoir, we must be careful of the fact that the earth’s rotation can spoil the effect of selective withdrawal

5. CONCLUSION

We studied the point sink flow of rotating stratified fluid numerically and obtained the following results.

(1) After initiation of the discharge, the selective withdrawal is established first. However, the corresponding withdrawal-layer thickness increases gradually with time in proportion to \( (ft)^{1/3} \) due to the effect of rotation.

(2) The speed of the above thickening is a function of the Reynolds number only and does not depend on the other parameters if the time is non-dimensionalized properly using the rotating speed of the system.

(3) When fluid is withdrawn for long time as in the practical situation like pond or ocean, the effect of earth’s rotation is not negligible and the withdrawal-layer thickness can be doubled or tripled.

(4) The effect of diffusion has little effect on the flow.

REFERENCES