Confidence Intervals for the Euler-Maruyama Approximate Solutions of Stochastic Delay Differential Equations

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Stochastic delay differential equations (SDDEs) are used for models of phenomena, the future states of the systems depend on both the present states and their past states. For SDDEs, several approximate solutions have been considered. In this paper, we investigate the Euler-Maruyama approximate solutions for SDDEs and estimate the mean square error of approximate solutions.

1 Introduction

Stochastic models have been used in many areas of science (medicine, biology, ecology, physics, economics, etc.) because systems in those are often influenced by various types of noise. Under suitable conditions, stochastic differential equations are useful formulations for describing models above. In 1964, Ito and Nisio considered stochastic differential equations which depend on "infinite past process" in [5], which is an originator of stochastic delay differential equations (SDDEs). Following the research, many properties have been studied (see [6] and [13]). Using SDDEs, we can consider models of phenomena whose future states depend on not only the present states but the past states. Such models are more realistic and SDDEs give mathematical formulations for them. For example, Mao et al. studied population dynamics with SDDEs, which are continuous models of the delay Lotka-Volterra models with environmental noises in [12]. Using SDDEs, they describe models that population growths depend on present and previous populations and environmental effects. Concerning mathematical finance models, Arriojasa et al. considered a stochastic delay model for stock prices, which is a generalization of Black-Scholes model and showed the no-arbitrage property of the model and the completeness of the market in [1]. For more models in other areas of science described by SDDEs, see [2], [10] and references therein.

Solutions of SDDEs do not always have Markov property, and their representations are more complicated than those of stochastic differential equations. Hence, approximate solutions of SDDEs have been also studied (e.g. [11], [3] and [9]). In those researches, rates of convergences of approximate solutions of SDDEs are shown. In [11], Mao and Sabanis use the Euler-Maruyama method to obtain the approximate solutions of SDDEs and their convergence rates. As we see (4) in Theorem 1 later, they showed the approximate solutions converge strongly to the true solutions of SDDEs with order 1/2 for the time step size of the approximations. However, we cannot obtain the exact solutions of SDDEs by simulation studies. To predict behaviors of sample paths of the solutions, we investigate confidence intervals of the exact solutions with the approximate solutions by simulation studies. We remark that such
confidence intervals are discussed for stochastic differential equations in [7] and [8]. In this paper, we study the case of SDDEs with one time delay. Recently, approximate solutions of SDDEs with several time delays have been considered ([4] and [9]). For the case, we can consider the same problems, however proofs are complicated. Details of the proof for this case will appear elsewhere. For SDDEs, this approach is the first one as far as the author knows, and it will be worth while discussing the one time delay case.

2 The Euler-Maruyama approximation for SDDE

We first present settings of SDDEs and conditions on the existence of the unique strong solution. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a probability space with the standard \(m\)-dimensional Brownian motion \(B(t)\) consisting of \(m\) dependent one-dimensional Brownian motions. We denote by \(C([\tau, 0]; \mathbb{R}^n)\) the space of continuous functions \(\phi : [-\tau, 0] \to \mathbb{R}^n\) with the norm \(\|\phi\| = \sup_{-\tau \leq \theta \leq 0} \|\phi(\theta)\|\). We also denote by \(C_{\mathcal{F}_t}([a, b]; \mathbb{R}^n)\) the family of \(\mathcal{F}_t\)-measurable \(C([a, b]; \mathbb{R}^n)\)-valued random variables. Let \(f(x_0, x_1)\) and \(g(x_0, x_1)\) be \(\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n)\)-measurable functions with value in \(\mathbb{R}^n\) and \(\mathbb{R}^{n \times m}\), respectively. Let \(\tau\) and \(T\) be positive constants and consider the following \(n\)-dimensional SDDE:

\[
dX(T) = f(X(t), X(\delta(t)))dt + g(X(t), X(\delta(t)))dB(t), \quad 0 \leq t \leq T
\]

with an initial data \(\{X(t), -\tau \leq t \leq 0\} = \{\xi(t), -\tau \leq t \leq 0\} \in C_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)\), where \(\delta(t)\) is a Borel measurable function such that \(-\tau \leq \delta(t) \leq t\), which stands for a time lag.

In this paper, we assume the followings:

(A1) There exists \(K_0\) such that \(E\|\xi(u)\|^2 = K_0 < \infty\).

(A2) There exists \(\rho > 0\) such that \(|\delta(t) - \delta(s)| \leq \rho|t - s|\) for \(-\tau \leq \delta(t) \leq t, t \geq 0\).

(A3) There exist \(K_1 > 0\) and \(\gamma \in (0, 1]\) such that for any \(-\tau \leq s < t \leq 0\)

\[E|\xi(t) - \xi(s)|^2 \leq K_1(t - s)^\gamma.\]

(A4) There exists \(K_2 > 0\) such that for \(x, \tilde{x}, y \in \mathbb{R}^n\)

\[|f(x, y) - f(\tilde{x}, \tilde{y})|^2 \vee |g(x, y) - g(\tilde{x}, \tilde{y})|^2 \leq K_2(|x - \tilde{x}|^2 + |y - \tilde{y}|^2).\]

Under the conditions above, there exists the unique strong solution of the SDDE (1) ([11], [13]).

We next present the Euler-Maruyama scheme for the SDDE (1). For a fixed sufficiently large integer \(N\), we set a time step size \(\Delta \in (0, 1)\) such that \(\Delta = \tau/N\). We define a discrete Euler-Maruyama approximate solution of the SDDE (1) by

\[
\begin{align*}
\tilde{y}(k + 1) &= \tilde{y}(k\Delta) + f(\tilde{y}(k\Delta), \tilde{y}(I_\Delta[\delta(k\Delta)]\Delta))\Delta + g(\tilde{y}(k\Delta), \tilde{y}(I_\Delta[\delta(k\Delta)]\Delta))\Delta B_k \\
\tilde{y}(0) &= \xi(t), \quad -\tau \leq t \leq 0,
\end{align*}
\]

(2)
where \( k = 0, 1, 2, \ldots \), \( \Delta B_k = B((k + 1)\Delta) - B(k\Delta) \) and \( I_{\Delta}[u] \) denotes the integer part of \( \delta(k\Delta)/\Delta \). We remark that \(-\tau \leq I_{\Delta}[\delta(k\Delta)] \leq k\Delta \) for each \( k \). To construct the continuous approximate solution, we let

\[
    z_0 = \sum_{k=0}^{\infty} \mathbb{1}_{[k\Delta, (k+1)\Delta)}(t) \bar{y}(k\Delta); \quad z_1 = \sum_{k=0}^{\infty} \mathbb{1}_{[k\Delta, (k+1)\Delta)}(t) \bar{y}(I_{\Delta}[\delta(k\Delta)]\Delta).
\]

Using them, we define a continuous Euler-Maruyama approximate solution by

\[
    y(t) = \begin{cases} \xi(t), & -\tau \leq t \leq 0, \\
    \xi(0) + \int_0^t f(z_0(s), z_1(s))ds + \int_0^t g(z_0(s), z_1(s))dB(s), & 0 \leq t \leq T. \end{cases} \tag{3}
\]

We remark that \( y(k\Delta) = \bar{y}(k\Delta) \) for each \( k \).

In [11], Mao and Sabanis showed the following convergence theorem:

**Theorem 1 (Mao-Sabanis)** Under the conditions (A1)-(A4), the approximate solution (3) converges to the solution of SDDE \((1)\) in the sense

\[
    E\left[ \sup_{0 \leq t \leq T} |X(t) - y(t)|^2 \right] \leq C\Delta + o(\Delta), \tag{4}
\]

where \( C \) is a positive constant independent of \( \Delta \).

### 3 Mean square error rate for the Euler-Maruyama approximate solution of SDDE

We investigate a confidence interval of the exact solution \( X(t) \) with using the approximate solution \( y(t) \). We follow arguments in [7] and [8]. Theorem 1 and Chebyshev's inequality imply that as \( n \to \infty \),

\[
    P\left\{ \sup_{0 \leq t \leq T} |X(t) - y(t)| \leq \epsilon \right\} \geq 1 - E\left[ \sup_{0 \leq t \leq T} |X(t) - y(t)|^2 \right]/\epsilon^2
    \geq 1 - O(N^{-1})/\epsilon^2. \tag{5}
\]

The right hand side of (5) gives less information to obtain the confidence interval. We hence need to obtain more precise estimations. Using notations in the assumptions, we let \( C_1 = 2\max\{K_2, |f(0, 0)|^2, |g(0, 0)|^2\} \) and \( C_2 = 3\{K_0 + C_1T(T + 4)\} \exp\{6C_1T(T + 4)\} \). For the SDDE (1), we obtain the following mean square errors for the convergence.

**Theorem 2** Under the conditions (A1)-(A4), we obtain that

\[
    P\left[ \sup_{0 \leq t \leq T} |X(t) - y(t)| > \epsilon \right] \leq \frac{1}{\epsilon^2} 4K_2T(T + 4)(M_1 + M_2) \exp\{8K_2T(T + 4)\} \Delta^7, \tag{6}
\]

where \( M_1 = 2C_2(1 + \Delta)(1 + C_2) \) and \( M_2 = 4C_1(1 + 2C_2)(\rho + 1)\{(1 + \rho)\Delta + 1\} + 2K_1(1 + \rho)^7 \).
Outline of proof.

To show the theorem, we use results in [11] and present them without proofs. The condition (A4) implies that $|f(x, y)|^2 + |g(x, y)|^2 \leq C_1(1 + |x|^2 + |y|^2)$. Using this and (3), we obtain the following:

**Proposition 1**

$$\sup_{-\tau \leq t \leq T} E|y(t)|^2 \leq 3 \left\{ \sup_{-\tau \leq r \leq 0} E|\xi(r)|^2 + C_1 T(T + 1) \right\} \exp\{6C_1T(T + 1)\}. \tag{7}$$

Using this proposition, we can estimate the $L^2$-mean of the difference between $y(t)$ and $z_0(t)$. For any $t \in [0, T]$, we can choose $k$ such that $t \in [k\Delta, (k + 1)\Delta)$ and obtain that

$$y(t) - z_1(t) = y(t) - y(k\Delta) = \int_{k\Delta}^{t} f(z_0(s), z_1(s))ds + \int_{k\Delta}^{t} g(z_0(s), z_1(s))dW(s).$$

This and the proposition above imply the following:

**Lemma 1** For any $t \in [0, T]$,

(i) $E|y(t) - z_0(t)|^2 \leq 2C_1(1 + 2C_2)(1 + \Delta)\Delta$, \tag{8}

(ii) $E|y(\delta(t)) - z_1(t)|^2 \leq [4C_1(1 + 2C_2)(1 + \rho)\{1 + (1 + \rho)\Delta\} + 2K_1(1 + \rho)^\gamma]\Delta^\gamma. \tag{9}$

**Proof of Theorem 2**

Denote the right hand sides of (8) amd (9) by $M_1\Delta$ and $M_2\Delta^\gamma$, respectively. According to (3), we have that

$$|X(t) - y(t)|^2$$

$$= \left| \int_0^t \{f(X(s), X(\delta(s))) - f(z_0(s), z_1(s))\}ds \right|^2 + \int_0^t \{g(X(s), X(\delta(s))) - g(z_0(s), z_1(s))\}dW(s)^2$$

$$\leq 2 \left| \int_0^t \{f(X(s), X(\delta(s))) - f(z_0(s), z_1(s))\}ds \right|^2 + 2 \left| \int_0^t \{g((s), X(\delta(s))) - g(z_0(s), z_1(s))\}dW(s) \right|^2$$

$$\leq 2 \int_0^t |f(X(s), X(\delta(s))) - f(z_0(s), z_1(s))|^2 ds + 2 \int_0^t |g((s), X(\delta(s))) - g(z_0(s), z_1(s))|dW(s) \right|^2.$$
Using Doob’s maximal inequality for martingales, we obtain that for each $t_i \leq T$

$$E \left[ \sup_{0 \leq t \leq t_i} |X(t) - y(t)|^2 \right] \leq 8K_2(T + 4) \int_0^{t_i} E \left[ \sup_{0 \leq r \leq s} |X(r) - y(s)|^2 \right] ds + 4K_2T(T + 4) \left\{ \sup_{0 \leq t \leq T} E[|y(s) - z_0(s)|^2] + \sup_{0 \leq t \leq T} E[|y_0(s) - z_1(s)|^2] \right\} \leq 8K_2(T + 4) \int_0^{t_i} E \left[ \sup_{0 \leq r \leq s} |X(r) - y(s)|^2 \right] ds + 4K_2T(T + 4)(M_1 \Delta + M_2 \Delta^\gamma).$$

Applying Gronwall’s inequality, we obtain that

$$E \left[ \sup_{0 \leq t \leq T} |X(t) - y(t)|^2 \right] \leq 4K_2T(T + 4)(M_1 \Delta + M_2 \Delta^\gamma) \exp\{8K_2T(T + 4)\} \leq 4K_2T(T + 4) \exp\{8K_2T(T + 4)\}(M_1 + M_2) \Delta^\gamma).$$

This and Chebyshev’s inequality imply that

$$P \left\{ \sup_{0 \leq t \leq T} |X(t) - y(t)| < \epsilon \right\} \geq 1 - \frac{1}{\epsilon^2} E \left[ \sup_{0 \leq t \leq T} |X(t) - y(t)|^2 \right] \geq 1 - \frac{1}{\epsilon^2} 4K_2T(T + 4)(M_1 + M_2) \exp\{8K_2T(T + 4)\} \Delta^\gamma,$$

which shows Theorem 2. □

The theorem implies a confidence interval where the exact solution of the SDDE (1) lies with a probability determined by parameters of the simulation study.

**Example 1.** We consider the case where the end point is $T = 1$ and the time delay is constant as that $\delta(t) = t - 0.1$. For the SDDE (1), we assume that $f \equiv 0$ and $g \equiv 0.1$. We also assume that $K_0 = 1, K_1 = 1, K_2 = 0.1^2, \rho = 1$ and $\gamma = 1$ for parameters in the assumptions. Setting the time step size as $\Delta = 0.1/10^4 = 10^{-5}$, we obtain that

$$P \left[ \sup_{0 \leq t \leq 1} |X(t) - y(t)| < 0.074 \right] \geq 0.95.$$  

To obtain hazard rates 0.9 and 0.99, we take $\epsilon = 0.052$ and 0.165, respectively.

This example shows that $X(t)$ lies in the interval $(y(t) - 0.074, y(t) + 0.074)$ with at least probability 0.95 when time step size is $10^{-5}$. By the confidence interval, we can predict the behavior of sample paths of $X(t)$ in a quantitative way. We note that the confidence interval is random.

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