A Study of the Development and Topology of a Vortex with Inflow in Isotropic Homogeneous Turbulence

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This paper investigates local flow geometry characteristics of inflow vortices in isotropic homogenous turbulence. Eigenvalues and eigenvectors of the velocity gradient tensor are applied to specify the Galilei invariant local flow geometry, and the real part of complex eigenvalues is used to classify the vortex as inflow or outflow. However, the real part can classify only average inflow or outflow, whereas another property, which we term "sourcity," specifies uniformity of the radial flow direction. We analyze the characteristics of inflow vortices using sourcity with inflow from all or partial directions. The analysis shows that the rate of vortices with inflow from all directions is approximately 16% of all (average) inflow vortices, and that most inflow vortices include partial outflow. A vortex with inflow from all directions has greater symmetry. Although sourcity mathematically decreases for high-intensity swirling, it increases if swirling develops with vortical flow symmetry, especially for weak vortices.

1. INTRODUCTION

Vortices are observed in various types of turbulence and in many engineering fields, such as fluid machinery, power plants, windmills, and aircraft. They strongly affect flow characteristics, and are identified by vortex definitions based on several aspects of their physical characteristics\(^1\)\(^{-19}\). One of the primary characteristics of vortices is the flow geometry (topology) of swirling motion. The $\Delta$-definition\(^1\) derived from invariant swirling motion is an important and popular vortex definition, and is associated with other important vortex features such as the pressure minimum\(^2\)\(^{-4}\). However, it is required for understanding vortical phenomena and vortex-control technology to specify more detailed vortical geometry and its change.

Inflow in vortical flow is important for vortex stability and development, because it gives the compressive strain in the swirl plane and derives vortex stretching\(^20\),\(^21\). Inflow also prevents vorticity diffusion. A vortex with inflow from all directions in the swirl plane has enhanced vortical flow symmetry\(^22\) and may be stable\(^23\). Such a vortex affects flow characteristics in the considered region. However, the occurrence of both inflow and outflow

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in the swirl plane of a vortex reduces flow symmetry and the orthogonality of the vortical axis to the swirl plane in vortex stretching. Thus, detailed identification to classify the radial flow is necessary to analyze vortical phenomena.

To better understand and identify vortical structure, the rate of the strain tensor and vorticity vector (or tensor) have been analyzed to yield useful findings. For example, previous studies found that the vorticity vector tends to align with the eigenvectors of the second largest eigenvalue of the rate of the strain tensor in isotropic homogeneous turbulence or shear flow \(^{21, 24, 25}\). However, the swirl plane is generally different from the eigenspace of the rate of the strain tensor. The vorticity vector is not parallel to the vortical axis nor is it orthogonal to the swirl plane, and vorticity components parallel to the swirl plane distort the vortical axis. Thus, it is difficult to specify topological features of a vortex using this vector.

Because eigenvalues and eigenvectors of the velocity gradient tensor \(\nabla \nu\) specify local flow geometry that is Galilei invariant, they have been used to classify flow geometry and derive the \(\Delta\)-definition and other corresponding definitions\(^{5–11}\). The sign of the real part of complex eigenvalues is used to classify the vortex as inflow (convergent, stable) or outflow (divergent, unstable), and this classification has been applied to the mixing layer \(^{25}\) or channel flow \(^{26}\). Nevertheless, physical interpretation of the real or imaginary part of the complex eigenvalues was unclear, and it was difficult to use eigenvalues to specify vortical flow symmetry. It has been recently shown that they represent an arithmetic average of radial flow intensities and the geometrical average of azimuthal flows in a (swirl) plane. Physical properties (invariants) have been derived with respect to symmetry of these flows \(^{22}\), one of which is associated with the pressure minimum feature in terms of flow kinematics \(^{41, 27}\). It was then explained that a vortex classified as an inflow (convergent) vortex by the complex eigenvalues\(^{1, 9}\) may have either complete inflow from all directions or partial inflow with outflow in some directions. Thus, questions may arise about whether inflow vortices in turbulence may be of two types, and if so, whether these two types have significantly different rates and features.

The present study analyzes inflow vortices in detail. We apply symmetry properties along with “sourcity”\(^{22}\), which represents the geometric average of radial flow intensity. We also use “swirlity”\(^{22}\), which represents the geometric average of azimuthal flow intensity. In this study, we investigate the principal characteristics of vortices in isotropic homogenous decaying turbulence.

2. VORTICAL FLOW GEOMETRY AND CORRESPONDING PROPERTIES

Here, we summarize flow geometry (kinematics) derived from \(\nabla \nu\) \(^1\) and invariants associated with the symmetry of vortical flow, along with the definitions of sourcity and swirlity \(^{25}\).

In a reference (Cartesian) coordinate system \(x_i (i = 1, 2, 3)\) with respect to a point in the flow, velocity \(v_i (i = 1, 2, 3)\) around the point can be expressed as \(v_i = dx_i / dt = (\partial v_i / \partial x_j) x_j\), where the summation convention is applied. Then, the trajectory of the equation is specified by the eigenvalues and eigenvectors of \(\nabla \nu\), which classifies local flow geometry in terms of the Galilean invariants. If \(\nabla \nu\) has complex eigenvalues \(\psi_i \pm i \phi (i: \text{imaginary number})\) and eigenvectors \(\xi_i, \eta_i\) along with a real eigenvalue \(\epsilon\) and its eigenvectors \(\xi_\epsilon, \eta_\epsilon\), the local flow geometry around the point can be expressed as \(^{1, 9}\)

\[
x = 2 e^{\psi i} \left\{ \cos(\psi t) \xi_\psi + \sin(\psi t) \eta_\psi \right\} + e^{\phi i} \xi_\phi.
\]

(1)
Equation (1) indicates that the trajectory (flow) swirls with angular velocity $\psi$ in the plane defined by $\xi_p$ and $\eta_p$, hereafter referred to as swirl plane $I$. According to $\epsilon_R$, the flow then uniformly converges towards or diverges from the considered point, and proceeds along a vortical axis $\xi_p$. $\xi_p$ and $\eta_p$ can be orthogonal and, in contrast with the real eigenvectors, $\xi_p$ and $\eta_p$ are restricted in terms of the ratio of the norms (lengths) of these vectors to satisfy the eigenequation. A ratio $c$ of the lengths is then specified as an invariant, i.e.,

$$c = \frac{||\xi_p||}{||\eta_p||} \text{ or } \frac{||\eta_p||}{||\xi_p||} \quad (0 < c \leq 1).$$

(2)

Even though the complex eigenvalues are the same, the flow geometry differs according to $c$. The vortical flow is symmetric if $c = 1$, and skewness increases as $c$ approaches 0.

We construct the $x_i$ coordinate system so that the bases $e_i$ ($i = 1, 2, 3$) are orthonormal and $e_1$ and $e_r$ are parallel to $\xi_p$ and $\eta_p$, respectively. In this coordinate system, $\mathbf{V} = A = [\partial v_i/\partial x_j] (i, j = 1, 2, 3)$ is expressed as

$$A = \begin{bmatrix}
\epsilon_R & c \psi & a_{13} \\
-\psi/c & \epsilon_R & a_{23} \\
0 & 0 & \epsilon_a
\end{bmatrix}.$$  

(3)

$a_{13}$ and $a_{23}$ are associated with the vorticity components parallel to $I$. For the velocity $v'$ specified by $\mathbf{V} v'$, i.e., $v' = Ax$, we focus on its flow in the $x_1 - x_2$ plane ($I$). The radial component $v_r$ of $v'$ in $I(x_3 = 0)$ can be expressed as

$$v_r = \frac{1}{|x'|} (x'^T Q_{x'} x'),$$

(4)

$$Q_r = \begin{bmatrix}
a_{11} & (a_{12} + a_{21})/2 \\
(a_{12} + a_{21})/2 & a_{22}
\end{bmatrix},$$

(5)

where $x' = (x_1', x_2')$ denotes the point in the $x_1 - x_2$ plane and $Q_r$ is the quadratic form with respect to $v_r$. The condition of inflow from all directions of the point is that the two eigenvalues $\lambda_r$ ($i = 1, 2$) of $Q_r$ are negative. The symmetry of $v_r$ is specified by the ratio $c_r = \lambda_r / \lambda_1$ (or $\lambda_r / \lambda_2$), where $-1 \leq c_r \leq 1$. $\lambda_r$ ($i = 1, 2$) are given from Eqs. (3) and (5) as

$$\lambda_r, \lambda_2 = \epsilon_R \pm \sqrt{c - 1/c} \psi/2,$$

(6)

so they are invariant. The property representing the geometric average of the inflow (or outflow) intensity can be defined by the following equations.

$$\sigma = \text{sgn}(\lambda_r, \lambda_2) \sqrt{\epsilon_R^2 |\lambda_r\lambda_2|^2},$$

(7)

$$\lambda_1 \lambda_2 = \epsilon_R^2 - (c - 1/c)^2 \psi^2/4,$$

(8)

where $\sigma$ is the sourcity. If $\epsilon_R < 0$ and $0 < \sigma$, then $\lambda_r < 0$ ($i = 1, 2$) and the vortex has inflow from all directions. Equation (6) indicates the important fact that the real part of the complex eigenvalues of $\mathbf{V} v'$, $\epsilon_R$, represents the arithmetic average of the radial flow intensity, i.e., $\epsilon_R = (\lambda_r + \lambda_2)/2$.

On the other hand, $\psi$ is equal to the geometric average of eigenvalues $\lambda_0$ ($i = 1, 2$) of the quadratic form in terms of the azimuthal velocity $v_0$ of $v'$, and represents the intensity of swirling flow if $0 < \lambda_0$, $\lambda_0$. We call this property swirlinity. The symmetry of $v_0$ can be represented by the ratio of $\lambda_0$ ($i = 1, 2$), which is specified by $c^2$ (or $1/c^2$). $c$ is
associated with the symmetry of both \( v_r \) and \( v_\theta \). Figure 1 shows the flow geometries, including decomposed radial and azimuthal flows with the same complex eigenvalues but different \( c \) and \( \sigma \). This suggests that the complex eigenvalues are inadequate to specify the flow geometry uniquely or in detail.

3. NUMERICAL ANALYSIS

The vortices are analyzed in isotropic homogeneous decaying turbulence by the pseudo-spectral method in the region \( (2\pi)^3 \), composed of 256^3 nodes. For the wavenumber vector \( \mathbf{k} = (k_1, k_2, k_3) \), \( |k| < 121 \) where \( |k| = (k_1^2 + k_2^2 + k_3^2)^{1/2} \). The phase shifting method is used for dealiasing. The time step is 0.001 in the fourth-order Runge–Kutta method. An energy spectrum \( E(k) = (k/k_0)^6 \exp\{-2(k/k_0)^2\} \) \( (k = |k|, k_0 = 4) \) gives the initial velocity field with random phases of \( k \), where the Taylor Reynolds number \( Re_t = 311 \), Taylor microscale \( \lambda_t = 0.59 \), Kolmogorov length \( \eta = 0.015 \), and eddy turnover time \( t_{edd} = 1.14 \). The kinetic viscosity is 0.002. The swirl is nondimensionalized by its root mean square value at the corresponding time, and expressed as \( \psi, \varepsilon_R \) and \( \sigma \) are nondimensionalized in a similar way.

In the vortical region where \( \nabla \psi \) has complex eigenvalues \( (0 < \psi) \), we define a (sub) region \( V' \) where \( \varepsilon_R < 0 \). Figure 2 shows the joint probability density function (JPDF) of \( \varepsilon_R \) and \( \sigma \) in \( V' \). This reveals that the vortices with inflow are of two types, partial (average) inflow and inflow from all directions. Because \( \sigma \) depends on flow symmetry \( (c) \), we examine the symmetry of vortices in terms of positive \( \sigma \) (inflow from all directions) and negative \( \sigma \) (average inflow). Figure 3 shows JPDFs of \( \psi \) and \( c \) in \( V' \) in terms of negative/positive \( \sigma \). Comparing vortices in \( V' \) with negative/positive \( \sigma \), those with positive \( \sigma \) have greater symmetry. Slopes of \( \psi \) to \( c \) in negative/positive \( \sigma \) vortices specified by the least-squares approximation are 1.9 and 1.4, respectively.
Figure 4 shows a vortical region where $\varphi = 1$, and a zoomed sectional part of a tube-like inflow vortex with contours $\varepsilon_R = -0.3, c = 0.7$, and $\sigma = 0.05$ and $0.1$.

4. DISCUSSION

Figure 2 shows that vortices classified as inflow (convergent) have two types of inflow, and ~84% of such vortices have average inflow, or partial inflow in the swirl plane, as shown in Fig. 1 (case (1), top panels). This
Fig. 4 (1) Vortical region in the turbulence. Contours of $\varphi = 1$ in a region $133\eta \times 133\eta \times 42\eta$, and contours of zoomed and truncated vortical region where $\varphi = 1$ and (2) $c = 0.7$, (3) $e_R = -0.3$, and (4) $\sigma = 0.05$ and 0.1. Zoomed vortex is a very strong (swirling) vortex with a $4 < \varphi$ region. In (4), contour of $\sigma$ outside that of $\varphi = 1$ is outflow region.

derives the different feature in the vortex stretching associated with $P$. If $0 < \sigma$, it decreases the vorticity components parallel to $P$ and increases the vorticity component normal to $P$. Then it enhances both swirling and orthogonality of the vortical axis to $P$. On the other hand, if $\sigma < 0$, although it increases the component normal to $P$, it similarly increases one component parallel to $P$ and thereby decreases the orthogonality.

Figure 3 shows that vortices with inflow from all directions have greater symmetry, and the slope of $\varphi$ to $c$ is $\sim 0.75$ times that of vortices with average inflow ($\sigma < 0$) in $V$. It also shows that the vortical region where $0 < \sigma$ in $V$ is smaller in the high-$\varphi$ region than in the low-$\varphi$ region. This feature is shown in Fig. 4 (2)–(4) as a very strong vortical (swirling) region. The contours of $\varphi = 1$ and $e_R = -0.3$ almost overlap (Fig. 4 (3)), and their inner region is an inflow vortical region where $1 < \varphi$ and $e_R < -0.3$. $\varphi$ is $> 4$ in some regions. Although the symmetry is high, the $0 < \sigma$ region is fragmentary and very small. This indicates that a strong vortex does not always have strong inflow in all directions. When $\varphi$ is much greater than $|e_R|$, it is difficult to have a positive $\sigma$, even with high symmetry.

We examine relationships of the development between $c$ and $\sigma$. Figure 5 shows JPDFs of their partial derivatives with respect to time $t$, which was nondimensionalized by the Kolmogorov time. Because the correlation between $\varphi$ and $c$ appears different in low and high $\varphi$ regions (Fig. 3 (1)), the JPDFs in terms of $\varphi < 1$ and $1 < \varphi$ are shown
in Fig. 5. Both JPDFs show that $\sigma$ tends to statistically increase or decrease with $c$, and has a greater correlation in the high-$\varphi$ region where the positive $\sigma$ region is smaller.

$\sigma$ has the mathematical characteristic of being difficult to be positive in the high-$\varphi$ region, as shown by Eqs. (7) and (8). Nevertheless, if symmetry increases with the development of swirling ($\varphi$), $\sigma$ can maintain or increase its intensity. Figure 6 shows JPDFs of $\partial \varphi/\partial t$ and $\partial \sigma/\partial t$ in terms of $0 < \partial \varphi/\partial t$. This indicates that $\varphi$ and $\sigma$ tend to increase together statistically if $c$ increases, especially in the region of low $\varphi$ ($\varphi \leq 0.4$). In the higher-$\varphi$ region ($1 < \varphi$), this tendency is diffuse. This may be because there is less correlation between $\partial \varphi/\partial t$ and $\partial \sigma/\partial t$ ($\varphi$ and $c$) in that region (Fig. 3).

5. CONCLUSIONS
The characteristics of vortices with inflow were investigated in isotropic homogenous turbulence, using the properties of sourcity, swirility, and symmetry. Vortices with inflow from all directions in the swirl plane constituted only ~16% of all inflow vortices, and most inflow vortices included some outflow in the swirl plane. The region where inflow from all directions prevents vorticity diffusion and enhances orthogonality of the vortical axis to the swirl plane by vortex stretching is strongly limited.

A vortex with inflow from all directions (positive sourcity) has high symmetry. Although the development of both swirility and sourcity is contradictory in the mathematical characteristic of the sourcity, they increase together if the swirling intensity of the vortex develops with its symmetry. Weak vortices are more likely than strong vortices to have this feature, because weak vortices have greater correlation between swirility and symmetry.

REFERENCES