Performance Evaluation of the Selective Smoothed Finite Element Methods Using Tetrahedral Elements with Deviatoric/Hydrostatic Split in Large Deformation Analysis

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The performance of the selective smoothed finite element methods (selective S-FEMs) using 4-node tetrahedral (T4) elements with deviatoric/hydrostatic split is evaluated. The selective S-FEMs discussed in this study are called FS/NS-FEM-T4 and ES/NS-FEM-T4 for short, which are known as locking-free formulations with no increase in the degrees of freedom. This study reveals that both the selective S-FEMs have three major issues: limitation of material constitutive model to handle; pressure oscillation in case of nearly incompressible material; and corner locking, through some examples of analysis. At the same time, the examples imply that the performance of the selective S-FEMs is comparable to an advanced hybrid T4 finite element formulation in the majority of practical problems.

1. INTRODUCTION

Smoothed finite element methods (S-FEMs) are promising for accurate numerical approach in various PDE problems. A variety of S-FEMs has been proposed: the node-based S-FEM (NS-FEM), edge-based S-FEM (ES-FEM), face-based S-FEM (FS-FEM), and so on. The selective S-FEMs are especially expected to be one of the best S-FEM formulations for solid mechanics problems in terms of locking-free property without increasing any additional degrees of freedom (DOF). At the same time, the selective S-FEMs preserve the property even with the use of 4-node tetrahedral elements (T4) or triangular elements (T3); thus, they can be easily applied to practical problems with bodies in complex shapes.

Recently, we have developed modified selective S-FEMs (selective FS/NS-FEM-T4 and selective ES/NS-FEM-T3) for severely large deformation problems of nearly incompressible materials. The modified methods adopted the deviatoric/hydrostatic (dev/typ) split in the selective integration, which was a simple extension to the conventional selective S-FEMs with the \( \mu/\lambda \) split. This extension made selective S-FEMs applicable to most of material constitutive models with keeping the locking-free property. Some examples of analysis with severely large deformation and mesh rezoning were presented to show the accuracy and the potential of the modified selective S-FEMs.

However, the performance of the selective S-FEMs in large deformation analysis has not been thoroughly evaluated so far. Previous studies evaluated the accuracy of the modified selective S-FEMs only in the point of view of load, displacement, and Mises stress distribution. In the point of view of formulation, pressure distribution, local deformation and so on, the modified selective S-FEMs are not supposed to be flawless.

Therefore, in this paper, we present some results of performance evaluation of the modified S-FEM-T4s in various points of view. Several limitations and issues in the modified selective S-FEMs are revealed.
through some examples of analysis. In addition to the modified selective FS/NS-FEM-T4, another formulation of selective S-FEM-T4 called selective ES/NS-FEM-T4\(^1\) with the dev/hyd split is also evaluated. A quick review of the formulations of the modified S-FEM-T4s is presented in Section 2. Some results and discussion on the example analyses are shown in Section 3 to reveal the limitations or issues in the modified selective S-FEMs. At last in Section 4, we summarize the properties of some S-FEMs in various points of view.

2. MATERIALS AND METHODS

In this section, a quick review of the formulation of the two types of modified selective S-FEMs with T4 elements, the modified selective FS/NS-FEM-T4 and ES/NS-FEM-T4, are presented. See references\(^6,20\) for the detail.

Hereafter, the modified selective S-FEMs discussed in this paper are called FS/NS-FEM-T4 and ES/NS-FEM-T4 for short.

2.1 FS/NS-FEM-T4

In FS/NS-FEM-T4, the deviatoric stress part is calculated at each element face whereas the hydrostatic stress part is calculated at each node. The strain smoothing and domain integration of S-FEMs are performed over discretized domains without overlaps nor gaps, which are called “smoothing domains”. Figure 1(a) and (c) show the smoothing domains in each T4 element for the deviatoric and hydrostatic stress calculation, respectively, in the FS/NS formulation. In each smoothing domains

The smoothed strain–displacement matrices (\(B\) matrices) at the face \(i\) (\([\text{Face}_i B_L]\)) and node \(j\) (\([\text{Node}_j B_L]\)) are given by

\[
[\text{Face}_i B_L] = \frac{1}{\text{Face}_i V} \sum_{k \in \text{Face}_i} \begin{bmatrix} \text{Elem}_k B_L \end{bmatrix} \text{Elem}_k V/4,
\]

\[
[\text{Node}_j B_L] = \frac{1}{\text{Node}_j V} \sum_{k \in \text{Node}_j} \begin{bmatrix} \text{Elem}_k B_L \end{bmatrix} \text{Elem}_k V/4,
\]

where \([\text{Elem}_k B_L]\) is the standard \(B\) matrix of the element \(k\), \([\text{Face}_i V]\) is the set of elements attached to the face \(i\), \([\text{Node}_j V]\) is the set of elements around the node \(j\), \([\text{Elem}_k V]\) is the volume of the element \(k\), \([\text{Face}_i V]\) is the volume of the smoothing domain of the face \(i\) (= \(\sum_{k \in \text{Face}_i} \text{Elem}_k V/4\)), \([\text{Node}_j V]\) is the volume of the smoothing domain of the

![Fig.1 Three types of smoothing domains in S-FEMs with 4-node tetrahedral elements. The volume of the smoothing domain (a) and (c) is just 1/4 of the element volume, whereas that of (b) is just 1/6 of the element volume.](image-url)
node \( j (= \sum_{k \in \text{Node}}^\text{Elem} v_k / 4) \), and \( A \) is the finite element assembly operator. The nodal internal force vector \((f^{\text{int}})\) is then calculated as

\[
f^{\text{int}} = \sum_{i \in \mathcal{I}} \left[ A_{i,j}^{\text{Face}(i)} + A_{i,j}^{\text{Node}(j)} \right]
= \sum_{i \in \mathcal{I}} \left[ A_{i,j}^{\text{Face}(i)} \text{Dev}(\text{Face}(i)) \text{Face}(i) + A_{i,j}^{\text{Node}(j)} \text{Hyd}(\text{Node}(j)) \text{Node}(j) \right],
\]

where \( \mathcal{I} \) is the set of faces in the analysis domain, \( J \) is the set of nodes in the analysis domain, \( T \) is the Cauchy stress, \( \text{Dev}(\mathcal{E}) \) is the operator to extract the deviatoric part of the stress, and \( \text{Hyd}(\mathcal{E}) \) is the operator to extract the hydrostatic part of the stress. FS/NS-FEM-T4 adopts selective integration and thus has limitations of material constitutive models to handle (the details are provided later).

2.2 ES/NS-FEM-T4

In ES/NS-FEM-T4, on the other hand, the deviatoric stress part is calculated at each element edge. The hydrostatic stress part is calculated at each node in the same fashion as FS/NS-FEM-T4. Figure I(b) and (c) show the smoothing domains in each T4 element for the deviatoric and hydrostatic stress calculation, respectively, in the ES/NS formulation.

The smoothed \( B \) matrix at the edge \( i \left( B_{i}^{\text{Edge}} \right) \) is given by

\[
B_{i}^{\text{Edge}} = \frac{1}{V_{i}^{\text{Edge}}} \sum_{k \in \text{Elem}} \frac{1}{V_{k}^{\text{Elem}}} \left[ A_{i,k}^{\text{Edge}(i)} \text{Edge}(i) \text{Edge}(i) \right],
\]

where \( V_{i}^{\text{Edge}} \) is the set of elements attached to the edge \( i \), and \( V_{k}^{\text{Elem}} \) is the volume of the smoothing domain of the edge \( i (= \sum_{k \in \text{Elem}} V_{k}^{\text{Elem}} / 6) \). The calculation of \((f^{\text{int}})\) is about the same way as Eq. (3):

\[
f^{\text{int}} = \sum_{i \in \mathcal{I}} \left[ A_{i,j}^{\text{Edge}(i)} + A_{i,j}^{\text{Node}(j)} \right]
= \sum_{i \in \mathcal{I}} \left[ A_{i,j}^{\text{Edge}(i)} \text{Dev}(\text{Edge}(i)) \text{Edge}(i) + A_{i,j}^{\text{Node}(j)} \text{Hyd}(\text{Node}(j)) \text{Node}(j) \right],
\]

where \( \mathcal{I} \) in this equation is the set of edges in the analysis domain and \( J \) is the set of nodes in the analysis domain. ES/NS-FEM-T4 also has the same limitations of material constitutive models to handle as FS/NS-FEM-T4.

3. RESULTS AND DISCUSSION

3.1 ISSUE #1: LIMITATION OF MATERIAL CONSTITUTIVE MODEL TO HANDLE

As noted in Section 1, the modified selective S-FEMs has a limitation of material constitutive model to handle as is the case in selective integration elements\(^4\). The modified selective S-FEMs split the Cauchy stress tensor into the deviatoric and hydrostatic parts (Eq. (3) and (5)), whereas the B-bar or F-bar method in FEM\(^4\) splits the deformation gradient tensor into the deviatoric and volumetric parts. Therefore, the deviatoric stress part at nodes and the hydrostatic stress part at faces or edges are inaccurate in the modified selective S-FEMs. This lack of accuracy is not problematic when we handle the commonly-used material constitutive models such as neo-Hookean, Mooney–Rivlin, Ogden, Hencky\(^4\), etc. However, this lack of accuracy may induce enormous error when we handle a constitutive model that has a coupling term between the deviatoric and volumetric parts. For example, the Steinberg–Cochran–Guirana–Lund (SCGL) model\(^2\), whose shear modulus depends on the pressure, is a typical constitutive model that has dev/vol coupling terms. Although the analysis with such a model is quite rare in practice, it should be noted that the modified selective S-FEMs cannot handle material constitutive models that have dev/vol coupling terms.
The dev/vol coupling terms cannot be treated properly as long as the selective stress integration is adopted. Therefore, it would be impossible to improve the selective S-FEMs to resolve this issue in the framework of the selective stress integration.

3.2 ISSUE #2: PRESSURE OSCILLATION IN CASE OF NEARLY INCOMPRESSIBLE MATERIAL

For the sake of evaluating the accuracy of pressure distribution, a bending analysis of a cantilever is performed. Figure 2 shows the outline of the analysis. The material of the cantilever is neo-Hookean hyperelastic material:

\[ T = 2C_{10} \frac{\text{Dev}(\bar{B})}{J} + \frac{2}{D_1 (J - 1)} I, \]

where \( J \) is the determinant of the deformation gradient (i.e., \( \det(F) \)) and \( \bar{B} \) is defined as \( F : F^T / J^{2/3} \). The material constant \( C_{10} \) is fixed at \( 1 \times 10^9 \) Pa and \( D_1 \) is varied between \( 2.143 \times 10^{-10} \) and \( 2.000 \times 10^{-15} \) Pa\(^{-1}\) so that the initial Poisson’s ratio \( \nu_0 = \frac{3-2C_{10}D_1}{6+2C_{10}D_1} \) is varied between 0.4 and 0.499999. The global mesh size in the domain is 0.2 m. The final vertical displacement at the cantilever tip (○ in Fig. 2) is about 6.5 m, and thus the analysis is a large deflection analysis. Some analyses with a commercial software, ABAQUS/Standard (Dassault Systèmes Simulia Corp., Providence, RI, USA)\(^{21}\), are also performed for comparison.

The comparison of the final vertical displacements at the cantilever tip is shown in Fig. 3(a). Only the result with the standard constant strain tetrahedral elements (ABAQUS C3D4) exhibits the locking effect, whereas the others are free from locking. The relative displacement error is also shown in Fig. 3(b). In the

Fig. 2 Outline of the bending analysis of a cantilever. The final vertical displacement is about 6.5 m and the initial Poisson’s ratio of the material is between 0.4 and 0.499999; thus, this is a large deflection analysis of nearly incompressible material.

Fig. 3 Comparison of the vertical displacement at the final state of the bending analysis of a cantilever (a). In the calculation of the relative displacement error (b), the result of ABAQUS C3D20H is regarced as the reference solution.
calculation of the error, the result with the second-order hybrid hexahedral elements (ABAQUS C3D20H) is regarded as the reference solution. In descending order of the displacement accuracy, the formulations with T4 element are arranged as ES/NS-FEM-T4, FS/NS-FEM-T4, and the first-order tetrahedral hybrid element (ABAQUS C3D4H). Therefore, the modified selective S-FEMs are considered to be accurate among the conventional T4 formulations.

However, the selective S-FEMs have an issue in large deflection analysis of nearly incompressible

![Fig.4 Distribution of the sign of pressure at the final state of the bending analysis of a cantilever with the initial Poisson’s ratio of 0.49. Pressure oscillation is observed in every case.](image)

![Fig.5 Comparison of the total internal energy with the total work done by the external force in the bending analysis of a cantilever using the FS/NS-FEM-T4. There is no difference between them; therefore, the pressure oscillation is not because of energy divergence.](image)
material, which is pressure oscillation. Figure 4(a), (b), (c) and (d) show the distributions of the sign of pressure in the results of FS/NS-FEM-T4, ES/NS-FEM-T4, ABAQUS C3D4H and NS-FEM-T4, respectively. They all present the typical checkerboard patterns; thus, the pressure oscillation in selective S-FEM-T4s is attributable to that in NS-FEM-T4. The comparison of the total internal energy with the total work done by the external force in case of FS/NS-FEM-T4 is plotted in Fig. 5. There is no difference between them; therefore, the pressure oscillation in selective S-FEM-T4s is not because of energy divergence.

With the same aim in a large strain analysis, a partial compression analysis of a block is also performed. Figure 6 shows the outline of the analysis. The material of the block is Arruda–Boyce hyperelastic material:

\[
T = 2C_{10}(I_1) \frac{\text{Dev}(\mathbf{B})}{J} + \frac{1}{D}(J - \frac{1}{J})I,
\]

\[
C_{10}(I_1) = \mu(k_1 + k_2 I_1 + k_3 I_3^2 + k_4 I_4 + k_5 I_5),
\]

\[
k_1 = \frac{1}{2}, \quad k_2 = \frac{1}{20} \lambda_m^2, \quad k_3 = \frac{11}{1050} \lambda_m^4, \quad k_4 = \frac{19}{7000} \lambda_m^6, \quad k_5 = \frac{519}{673750} \lambda_m^8,
\]

Fig.6 Outline of the partial compression analysis of a block. The final vertical displacement is more than 0.9 m and the initial Poisson’s ratio of the material is 0.4999; thus, this is a large strain analysis of nearly incompressible material.

Fig.7 Comparison of the vertical displacements as a function of applied pressure in the partial compression analysis of a block.
where \( \tilde{I}_1 \) is the first invariant (trace) of \( \tilde{B} \). The material constants \( \mu, \lambda_m \), and \( D \) are \( 0.079 \times 10^9 \) Pa, 7.0, and \( 5.0 \times 10^{-12} \) Pa\(^{-1}\), respectively, and thus the initial Poisson’s ratio is 0.4999. The global mesh size in the domain is 0.1 m.

A comparison of the vertical displacements at the corner as a function of applied pressure is shown in Fig. 7. The displacement results except ABAQUS C3D4 are similar each other although there are some differences in the limit of convergence. In this example, ES/NS-FEM-T4 has a shorter limit of convergence in comparison to the other locking-free methods. However, the final vertical displacement are around \(-0.95 \) m in the locking-free cases and thus their differences in convergence performance are minor\(^1\).

Figure 8(a), (b) and (c) show the distributions of the sign of pressure in the results of FS/NS-FEM-T4, ES/NS-FEM-T4 and ABAQUS C3D4H, respectively. By the same token, the modified selective S-FEMs present typical pressure oscillation. Therefore, in case of large deformation analysis of nearly incompressible material, the selective S-FEMs are considered to have enough accuracy in displacement and force but have insufficient accuracy in pressure.

The pressure oscillation issue seems difficult to be resolved as long as NS-FEM-T4 is adopted to calculate the hydrostatic part. It is necessary to change the way of hydrostatic part calculation to suppress the pressure oscillation. Expanding the domain for the strain smoothing, appending additional DOF to the element, and so on would resolve this issue, which are our future works.

### 3.3 ISSUE #3: CORNER LOCKING

For the sake of evaluating the accuracy at corner elements, a compression analysis of a cylinder is performed. Figure 9 shows the outline of the analysis. The material of the cylinder is neo-Hookean hyperelastic material as defined in Eq. (6). The material constants \( C_{10} \) and \( D_1 \) are \( 1 \times 10^9 \) Pa and \( 2.143 \times 10^{-10} \) Pa\(^{-1}\), respectively, and thus the initial Poisson’s ratio is 0.4. The global mesh size in the domain is 0.05 m in a coarse case, whereas that in the fine case is 0.025 m.

Figure 10(a), (b) and (c) show the pressure distributions of the results of FS/NS-FEM-T4, ES/NS-FEM-T4 and ABAQUS C3D4H, respectively, with the coarse mesh. Also, Fig. 10(d), (e) and (f) show them with the fine mesh. Their element deformations are similar each other. However, many locked elements are found around the corner (including the rim) in every case although distorted elements are present right next to them. This locking is thought to be due to the lack of adjoined elements at the corner faces and edges. The corner faces and edges adjoin only one or few elements and thus their strains are not smoothed sufficiently, resulting in shear locking as the constant strain elements. The corner locking

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\(^1\)In our experience, the differences highly depend on the mesh structure. It is difficult to tell which formulation has better convergence performance.
Fig. 9 Outline of the compression analysis of a cylinder. The initial Poisson's ratio of the material is 0.4; thus, this is a large strain analysis of a compressible material.

(a) FS/NS-FEM-T4 (Coarse)  (b) ES/NS-FEM-T4 (Coarse)  (c) ABAQUS C3D4H (Coarse)

(d) FS/NS-FEM-T4 (Fine)  (e) ES/NS-FEM-T4 (Fine)  (f) ABAQUS C3D4H (Fine)

Fig. 10 Distribution of pressure at the final state of the compression analysis of a cylinder. Many locked elements are observed around the corner and rim in every case.
arises only right at the corner elements as shown in Fig. 10; hence, the results with the finer mesh has less influence of the corner locking.

The corner locking issue appears to originate in the setting of the smoothing domain of FS-FEM-T4 or ES-FEM-T4. Therefore, as well as ISSUE #2, expanding the domain for the strain smoothing, appending additional DOF to the element, and so on would resolve this issue.

4. CONCLUSION

Three major issues on the modified selective smoothed finite element methods (selective S-FEMs) in static analysis of solids are clarified through the comparison among various FEMs with 4-node tetrahedral (T4) elements. The results of the performance evaluation are summarized in Table 1. The modified selective S-FEMs resolve the locking issues and the spurious zero-energy modes with no increase in the degrees of freedom (DOF). However, the formulation of the selective stress integration brings the limitation of applicable material constitutive models. In addition, the pressure oscillation issue in case of nearly incompressible materials and the corner locking issue are left unsolved.

In a practical point of view, analysis with dev/vol coupled materials is quite rare. Local averaging of the pressure among nodes may minimize the negative impact of the pressure oscillation on pressure estimation. Finer meshing around corners can reduce the bad influence of the corner locking to some extent. Therefore, these issues would not be critical in the majority of cases in practice. Nevertheless, further improvement of S-FEMs is preferable to resolve these issues.

The greatest characteristic of S-FEMs is the accuracy improvement with no increase in DOF. In contrast to the hybrid formulations, S-FEMs have neither pressure nor volumetric strain as unknown parameters, thereby eliminating the needs for Lagrange multipliers and static condensations. The stiffness matrix of S-FEMs with only displacement unknowns inherits the good characteristic of diagonally dominant sparse band matrices from the standard FEM. In that sense, the development of hybrid S-FEMs by appending additional DOF will not an adequate concept even though it may resolve the three unresolved issues.

Recently, a new type of ES-FEM called “bubble-enhanced ES-FEM (bES-FEM)” for triangular and tetrahedral elements was proposed. The bES-FEM satisfies the discrete inf–sup condition in exchange for increase in displacement DOF by adding bubble nodes in the middle of elements. In addition, another new type of ES-FEM called “F-bar aided ES-FEM (F-barES-FEM)” was also proposed lately. The F-barES-FEM reduces the pressure oscillation with increase in bandwidth of stiffness matrices but without any increase in DOF. Further development of these non-hybrid S-FEMs are anticipated to resolve all the issues listed in Table 1.

REFERENCES

Table 1 Performance comparison of 4-node tetrahedral (T4) elements in the standard FEM, various S-FEMs, and ABAQUS hybrid formulation. Each “✓” and “✗” denotes benefit and drawback, respectively. The left half data of this table is found on the reference⁷.

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