A Design Method for Low-Sensitivity Control with Robust Stability for Minimum Phase Systems with Uncertain Relative Degree

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In this paper, we examine a design method for low-sensitivity control with robust stability for minimum phase single-input/single-output systems. Yamada clarified that low-sensitivity control systems with robust stability can be designed under the assumption that the relative degree of the plant is equal to that of the nominal plant. In some cases, it is difficult to obtain an accurate relative degree of the plant. We expand Yamada’s result and propose a design method for low-sensitivity control systems with robust stability for systems with uncertain relative degree. Our method adapts a parallel compensation technique and a design method for a parallel compensator is given. A design procedure for low-sensitivity control systems with robust stability using a parallel compensation technique is presented.

1 Introduction

In this paper, we examine a design method for low-sensitivity control with robust stability (in the following, we denote low-sensitivity control by LSC). It is well known that LSC systems have good characteristics; for example, disturbance rejection and high rates of convergence to the reference input. Therefore, designing high-performance control systems is equivalent to designing LSC systems. That is, to attenuate disturbances rapidly, an LSC system is required. Since LSC systems have large complementary sensitivity functions, they tend to compromise robust stability for large uncertainty.

However, controllers that achieve low-sensitivity characteristics do not always make the control system unstable. Maeda considered the LSC system with robust stability based on the infinite gain margin problem. References 1, 2, 3 clarified that, for a certain class of uncertainty, an LSC system can be designed. Yamada gave the robust stability condition for LSC in systems with an uncertain number of unstable poles 4. Yamada 4 extended that result in reference 5, and clarified that when the relative degree of the plant is equal to that of the nominal plant or one smaller, if the control system satisfies conditions similar to those in reference 6, then the control system is robustly stable.

The condition that the relative degree of the plant is equal to that of the nominal plant or one larger than that of the nominal plant is very difficult. It is conjectured that the number of applicable plants is small. Therefore, to design LSC system with robust stability, it is important to precisely estimate the relative degree of the plant. If the following procedure is adopted, then the relative degree of the plant is equal to that of the nominal plant. When the compensator is juxtaposed to the original plant, let the system with the original plant and the parallel compensator be the plant. If the parallel compensator is biproper, then the system with the original plant and the parallel compensator is biproper, independent of the dynamics of the original plant. By requiring the nominal plant to be biproper, the relative degree of the system with the original plant and the parallel compensator is equal to that of the nominal plant. Therefore, using the parallel compensation technique, it is expected that we can design LSC systems with robust stability for systems with uncertain relative degree. However, this case has not previously been considered.

In this paper, we propose a design method for LSC systems with robust stability for the minimum phase single-input/single-output system with uncertain relative degree by using the idea of parallel compensation. We first summarize the result in reference 7 briefly. Next, we propose a design method for LSC systems with robust...
stability for the minimum phase system by using the idea of parallel compensation technique. We present a design method for the parallel compensator to maintain certain conditions, and a design procedure for LSC systems.

Notation

- $\mathbb{R}$: the set of real numbers
- $\mathbb{R}_{++}$: $\mathbb{R} \cup \{\infty\}$
- $\mathbb{C}$: the set of complex numbers
- $R(s)$: the set of real rational functions with $s$
- $RH_{\infty}$: the set of asymptotically stable real rational transfer functions
- $\Re\{\cdot\}$: the real part of $\{\cdot\} \in \mathbb{C}$
- $\lambda(\{\cdot\})$: relative degree of a rational function $\{\cdot\}$

2 Problem formulation

In this section, we will introduce the result in reference [13] and describe the problem considered in this paper.

Let us consider the following unity feedback control system:

$$\begin{cases} 
y &= G(s)u \\
u &= C(s)(r - y)
\end{cases} \tag{1}$$

Here, $G(s) \in R(s)$ is a single-input/single-output continuous-time strictly proper minimum phase system. That is:

$$G(s) \neq 0 \quad \forall \{s\} \geq 0 \quad (s \in \mathbb{C}) \tag{2}$$

holds. $G(s)$ is assumed to be stabilizable and detectable. Without loss of generality, the coefficients of the highest degree of both numerator of $G(s)$ and denominator of $G(s)$ are positive numbers. $C(s) \in R(s)$ is the controller, $r \in R(s)$ is the reference input, $u \in R(s)$ is the control input and $y \in R(s)$ is the output.

The nominal plant of $G(s)$ is denoted by $G_m(s)$. $G_m(s)$ is assumed to be stabilizable and detectable. The uncertainty between $G(s)$ and $G_m(s)$ is denoted by $\Delta(s)$. $G(s)$ is written as:

$$G(s) = G_m(s)(1 + \Delta(s)) \tag{3}$$

Yamada [13] considered the robust stability condition for the following class $\Omega_1(G_m(s), W(s))$ of plants defined by Definition 1 and gave Theorem 1.

**Definition 1** If the following expressions hold, then $G(s)$ is called the elementary of $\Omega_1(G_m(s), W(s))$.

- the number of unstable zeroes is equivalent to that of $G_m(s)$:

$$\left| \frac{\Delta(j\omega)}{1 + \Delta(j\omega)} \right| < |W(j\omega)| \quad \forall \omega \in \mathbb{R}_{++} \tag{4}$$

where $W(s) \in RH_{\infty}$.

**Theorem 1** The controller $C(s)$ is assumed to stabilize $G_m(s)$. $C(s)$ is a robustly stabilizing controller for all plants included in the class $\Omega_1(G_m(s), W(s))$ if and only if:

$$\left\| \frac{1}{1 + G_m(s)C(s)} W(s) \right\|_{\infty} = \|S(s)W(s)\|_{\infty} \leq 1 \tag{5}$$

is satisfied. Here $S(s)$ is the sensitivity function written by:

$$S(s) = \frac{1}{1 + G_m(s)C(s)}$$

According to Theorem 1, by adapting the class of plants $\Omega_1(G_m(s), W(s))$, we can design LSC systems with robust stability. To satisfy (5), the relative degree of $G(s)$ must be equal to that of $G_m(s)$, and the number of zeroes of $G(s)$ in the closed right-half plane must be equal to that of $G_m(s)$ [13].
Remarks 1 Note that to satisfy (5), the relative degree of \( G(s) \) must be equal to that of \( G_m(s) \) and the number of zeroes of \( G(s) \) in the closed right-half plane must be equal to that of \( G_m(s) \). According to \(^1\)\(^3\), if the relative degree of \( G(s) \) is not equal to that of \( G_m(s) \), then no controller exists that will stabilize all plants included in \( \mathcal{G}_i(G_m(s), W(s)) \). That is, if the relative degree of \( G(s) \) is not equal to that of \( G_m(s) \), we cannot design LSC system robust stability. In addition, if the number of zeroes of \( G(s) \) differ from that of \( G_m(s) \), we cannot apply the method in \(^1\)\(^3\) to systems with an uncertain number of unstable poles. Thus when we design LSC system with uncertain number of poles in the closed right half plane, the relative degree of \( G(s) \) and the number of zero of \( G(s) \) are required.

In this paper, we consider a design method for LSC systems with robust stability for systems with uncertain relative degree.

3 Low-sensitivity control using parallel compensation

In this section, we consider a design method for LSC systems for systems of uncertain relative degree using parallel compensation.

From Theorem 1, in order to design an LSC system with robust stability, the relative degree of the plant \( G(s) \) must be equal to that of the nominal plant \( G_m(s) \), and the number of zeroes of \( G(s) \) in the closed right-half plane must be equal to that of the nominal plant \( G_m(s) \). Therefore, to design an LSC system with robust stability, the way in which the relative degree of the plant \( G(s) \) is adjusted to that of the nominal plant \( G_m(s) \) is important. The way in which the number of zeroes of the plant \( G(s) \) in the closed right-half plane is adjusted to that of the nominal plant \( G_m(s) \) is also important. First, we consider a method to adjust the relative degree of the plant \( G(s) \) to that of the nominal plant \( G_m(s) \). Next, we give a method to adjust the number of the zeroes of the plant \( G(s) \) in the closed right-half plane to that of the nominal plant \( G_m(s) \).

The main idea of the method to adjust the relative degree of the plant to that of the nominal plant is as follows. From the assumption that \( G(s) \) is strictly proper, if \( K(s) \) is biproper, the relative degree of the system \( G(s) \) written as:

\[
G(s) = G(s) + K(s)
\]

is always 0-degree. If the relative degree of \( K(s) \) is one, the relative degree of the system \( G(s) + K(s) \) is always one. Let the nominal plant of \( G(s) \) be \( G_m(s) \). If \( G_m(s) = G_m(s) + K(s) \) is found to be biproper, then the relative degree of \( G(s) + K(s) \) is always equal to that of \( G_m(s) = G_m(s) + K(s) \). That is, both the relative degree of \( G(s) = G(s) + K(s) \) and that of \( G_m(s) \) are 0-degree.

Considering \( G(s) \) and \( G_m(s) \), the plant and the nominal plant, respectively, the relative degree of the plant is equal to that of the nominal plant. Thus, by using the parallel compensation technique, the relative degree of the plant \( G(s) \) is equal to that of the nominal plant \( G_m(s) \).

Next, we consider a method to adjust the number of zeroes of the plant \( G(s) \) in the closed right-half plane to that of the nominal plant \( G_m(s) \). From the assumption that \( G(s) \) is of minimum phase, there always exists \( K(s) \) such that \( G(s) + K(s) \) is of minimum phase \(^1\)\(^3\). In designing \( K(s) \) so that \( G(s) + K(s) \) will be of minimum phase, the following theorem is satisfied.

Theorem 2 \( G(s) \) is assumed to be of minimum phase. \( G(s) \) and \( K(s) \) are denoted by:

\[
G(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}
\]

and:

\[
K(s) = \frac{B(s)}{A(s)} = \frac{b_{n_k} s^{n_k} + b_{n_k-1} s^{n_k-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}
\]

respectively. Without loss of generality, \( m < n \) and \( n_k \geq n - m \) are satisfied. It is also assumed that \( K(s) \in \text{RH}_\infty \).

We denote \( G_0(s) \) and \( G_i(s)(i = 1, \ldots, n-m) \):

\[
G_i(s) = \sum_{i=0}^{n_i-m} b_i s^i
\]

\[
G_i(s) = \frac{G_0(s)}{A(s)}
\]
and:

\[ G_i(s) = \frac{\hat{h}(n-k-n+m+i)}{A(s)} \]

respectively. If:

\[ \left\| \frac{G(s)}{G_0(s)} \right\|_\infty < 1 \]  \hspace{1cm} (11)

is satisfied, and for an arbitrary sufficiently small positive number \( \alpha_i (i = 1, \ldots, n-m) \):

\[ \left\| \frac{\alpha_i s}{1 + \alpha_i s} - \frac{G_i(s)}{G(s) + \sum_{j=0}^{i-1} G_j(s)} \right\|_\infty < 1 \quad (i = 1, \ldots, n-m) \]  \hspace{1cm} (12)

is satisfied, then \( G(s) + K(s) \) is of minimum phase.

(Proof) From (9) and (10), \( G(s) + K(s) \) is rewritten as:

\[ G(s) + K(s) = G(s) + \sum_{i=0}^{n-m} G_i(s). \]  \hspace{1cm} (13)

The proof consists of showing that the following expressions hold:

- If (11) and (12) hold, then \( G(s) + G_0(s) \) is of minimum phase.
- We use induction on \( j \). If \( G(s) + \sum_{i=0}^{j-1} G_i(s) \) is of minimum phase, then \( G(s) + \sum_{i=0}^{j+1} G_i(s) \) is of minimum phase.

First, we show that \( G(s) + G_0(s) \) is of minimum phase. \( G(s) + G_0(s) \) is rewritten by:

\[ G(s) + G_0(s) = G(s) \left( 1 + \frac{G_0(s)}{G(s)} \right). \]  \hspace{1cm} (14)

From the assumption that \( G(s) \) is of minimum phase and (11), \( 1 + G_0(s)/G(s) \) is of minimum phase. Thus, \( G(s) + G_0(s) \) is of minimum phase.

Next we show that if \( G(s) + \sum_{i=0}^{j} G_i(s) \) is of minimum phase, then \( G(s) + \sum_{i=0}^{j+1} G_i(s) \) is of minimum phase. \( G(s) + \sum_{i=0}^{j+1} G_i(s) \) is rewritten by:

\[ G(s) + \sum_{i=0}^{j+1} G_i(s) = \left( G(s) + \sum_{i=0}^{j} G_i(s) \right) + G_{j+1}(s) \]

\[ = (1 + \alpha_i s) \left( G(s) + \sum_{i=0}^{j} G_i(s) \right) \left( 1 - \frac{\alpha_i s}{1 + \alpha_i s} + \frac{G_{j+1}(s)}{1 + \alpha_i s} \right) \]  \hspace{1cm} (15)

From the assumption that \( G(s) + \sum_{i=0}^{j} G_i(s) \) is of minimum phase and (12):

\[ \left( 1 - \frac{\alpha_i s}{1 + \alpha_i s} + \frac{G_{j+1}(s)}{G(s) + \sum_{i=0}^{j} G_i(s)} \right) \]

is of minimum phase. From the assumption that both \( 1 + \alpha_i s \) and \( G(s) + \sum_{i=0}^{j} G_i(s) \) are of minimum phase, \( G(s) + \sum_{i=0}^{j+1} G_i(s) \) is of minimum phase.

We have thus proved this theorem.
Remarks 2 From (11) and (9), if all the coefficients of the numerator of \( G_0(s) \) are sufficiently small, then (11) will generally be satisfied. In addition, from (12) and (10), if all the coefficients of the numerator of \( G_1(s) \) are sufficiently small, then (12) will generally be satisfied. Therefore, if all the coefficients of the numerator of \( K(s) \) are sufficiently small, then \( G(s) + K(s) \) is of minimum phase.

When the relative degree of \( G(s) \) satisfies:

\[
0 \leq \lambda(G(s)) \leq 1,
\]  

(10)

the following theorem is satisfied.

Theorem 3 Assume that (16) is satisfied. When \( K(s) = k \in R, k_{\min} \) exists such that \( G(s) + K(s) \) is of minimum phase for all \( k \) to hold \( 0 < k \leq k_{\min} \).

(Proof) We will prove this theorem using the information of the root-locus method \(^{10}\). When \( k \) is gradually decreased, from (16), the zeroes of \( 1 + G(s)/k \) approach the zeroes of \( G(s) \) and \( -\infty \) monotonically \(^{10}\). This implies that, from the assumption that \( G(s) \) is of minimum phase, when \( k \) is gradually decreased, \( k_{\min} \) exists such that \( 1 + G(s)/k \) has no zeroes in the closed right-half plane for all \( k \leq k_{\min} \). The zeroes of \( 1 + G(s)/k \) are obviously equal to those of \( G(s) + k \).

We have thus proved the theorem.

Using Theorem 2 and Theorem 1, the LSC system is designed using the following procedure.

1. Let the system in Fig. 1 be the plant. That is, we consider \( G(s) + K(s) \) to be the plant. Where \( K(s) \in RH_{\infty} \)

![Figure 1: Parallel compensation](image)

Figure 1: Parallel compensation

is made to conform to the following conditions.

(a) The relative degree of \( K(s) \) is greater than \( G(s) \).
(b) \( G(s) + K(s) \) is of minimum phase. From Theorem 2 and Remarks 2, \( G(s) + K(s) \) is made minimum phase by making all the coefficients of the numerator of \( K(s) \) sufficiently small.

2. Applying system identification methods, determine the nominal plant \( \hat{G}_m(s) \):

\[
G(s) = \hat{G}_m(s) (1 + \hat{\Delta}(s))
\]  

(17)

and \( \hat{W}(j\omega) \) satisfying:

\[
\left| \frac{\hat{\Delta}(j\omega)}{1 + \hat{\Delta}(j\omega)} \right| < |\hat{W}(j\omega)|, \quad \forall \omega \in RH_{\infty}
\]  

(18)

Here, \( \hat{G}_m(s) \) is defined to be biproper and of minimum phase.

3. Find the controller \( \hat{C}(s) \) satisfying:

\[
\left\| \frac{1}{1 + \hat{G}_m(s)\hat{C}(s)} \hat{W}(s) \right\|_{\infty} = \left\| \hat{S}(s)\hat{W}(s) \right\|_{\infty} \leq 1.
\]  

(19)

where \( \hat{S}(s) \) is the sensitivity function written by:

\[
\hat{S}(s) = \frac{1}{1 + \hat{G}_m(s)\hat{C}(s)}.
\]  

(20)
The control structure of the control system designed by the above procedure is shown in Fig. 2. The sensitivity function $S(s)$ of the control system in Fig. 2 is given by:

$$S(s) = \frac{1 + K(s)C(s)}{1 + G_m(s)C(s)}$$  \hfill (21)

Generally, even if the sensitivity function $S(s)$ is small, the sensitivity function $S(s)$ does not always have a small value. Next, we will show that if $S(s)$ is small, then the sensitivity function $S(s)$ has a small value.

Assume that $|\hat{S}(j\omega)|$ given in (20) has a small value and $|G_m(j\omega)|$ is not sufficiently smaller than $|K(j\omega)|$. When $|\hat{S}(j\omega)|$ has a small value, from (20), $|\hat{C}(j\omega)|$ has a large value. The gain of the sensitivity function $S(s)$ in (21) is given by:

$$|S(j\omega)| = \frac{|1 + K(j\omega)C(j\omega)|}{|1 + G_m(j\omega)C(j\omega)|}$$  \hfill (22)

When $1/|C(j\omega)|$ is sufficiently smaller than $|K(j\omega)|$, we have:

$$|S(j\omega)| \simeq \frac{|K(j\omega)|}{|G_m(j\omega) + K(j\omega)|}$$  \hfill (23)

From the assumption that all of coefficients of the numerator of $K(s)$ are sufficiently small, $|K(j\omega)|$ has a small value. This implies that $|S(j\omega)|$ also has a small value. Conversely, when $|K(j\omega)|$ is sufficiently smaller than $1/|C(j\omega)|$, we have:

$$|S(j\omega)| \simeq \frac{1}{|1 + G_m(j\omega)C(j\omega)|}$$

$$= \frac{|1 + G_m(j\omega)C(j\omega)|}{|S(j\omega)|}.$$  \hfill (24)

From the assumption that $|\hat{S}(j\omega)|$ has a small value, $|S(j\omega)|$ is also small. Therefore, using the proposed method, if $|\hat{S}(j\omega)|$ has a small value, then $|S(j\omega)|$ also has a small value. Thus we have shown the validity of the proposed method.

4 Numerical example

In this section, we show a numerical example.

Let the nominal plant $G_m(s)$ and the plant $G(s)$ be:

$$G_m(s) = \frac{0.001s + 1}{s^2 + 2s + 1}$$  \hfill (25)

and:

$$G(s) = \frac{1}{s^2 + 3s + 1}$$  \hfill (26)

respectively. Both $G_m(s)$ and $G(s)$ are minimum phase. However, as the relative degree of $G(s)$ is not equal to that of $G_m(s)$, the method in [19] cannot apply to this case. Using the proposed method, we design a low-sensitivity control system with robust stability.
Using the result in Theorem 2, when $K(s)$ is defined by:

$$K(s) = \frac{0.01s + 0.009}{s^2 + 2s + 1},$$  \hspace{1cm} (27)

$G(s) + K(s)$ is minimum phase and the relative degree of $G(s) + K(s)$ is one. $G_m(s)$ and $G(s)$ are given by (17). Let $W(s)$ be:

$$W(s) = \frac{0.59(s + 15.2)}{s + 5}.$$ \hspace{1cm} (28)

Bode plots of $\Delta(s)/(1 + \Delta(s))$ and $W(s)$ are shown in Fig. 3. Here the solid line shows the bode plot of

![Bode plot of $\Delta(s)/(1 + \Delta(s))$ and $W(s)$](image)

Figure 3: Bode plots of $\Delta(s)/(1 + \Delta(s))$ and $W(s)$

$\Delta(s)/(1 + \Delta(s))$ and the dashed line shows that of $W(s)$. Fig. 3 shows that (18) holds. Using the LMI control toolbox, we obtain the controller for $G_m(s)$ satisfying (19). The sensitivity function $S(s)$ of the control system in Fig. 2 and the sensitivity function $\bar{S}(s)$ written by (20) are shown in Fig. 4. Here the solid line shows the

![Bode plot of $S(s)$ and $\bar{S}(s)$](image)

Figure 4: Bode plots of $S(s)$ and $\bar{S}(s)$

bode plot of $S(s)$ and the dashed line shows that of $\bar{S}(s)$. Thus we obtain a controller that achieves low-sensitivity characteristics with robust stability.

5 Conclusion

In this paper, we proposed a design method for LSC systems with robust stability for systems with uncertain relative degree by using the idea of parallel compensation. We provided a design method for a parallel compensator to meet certain conditions. In addition, we showed a design procedure for LSC systems as well as the validity of the proposed method.
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