Micromechanical Analysis of the Effect of Temperature on Toughness of Ceramic-Based Fiber-Reinforced Composites

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In the present research, by using the extended modified equivalent inclusion method, micromechanical analysis is performed on a discontinuous fiber-reinforced ceramic-based composites showing interfacial sliding behavior between a fiber and a matrix, and analytical expressions for energy release rate and the dissipation energy release rate due to sliding are derived. In order to investigate the effect of temperature on the toughness of such a ceramic-based composite, the relationship between temperature and the interfacial friction stress is derived. By using this relationship, two energy release rates, and the surface energy of the matrix, the criteria for the propagation of the matrix crack is derived. In addition, fracture toughness of such a composite is evaluated in terms of the critical size of the crack which propagates across a composite in an unstable manner and the change in toughness with temperature can be obtained successfully. The results obtained are consistent with the Brennan's experimental results.

1. INTRODUCTION

Fracture behavior of the fiber-reinforced composites is strongly influenced by difference in rupture strain between a matrix and a fiber as well as volume fraction of a fiber. For instance, suppose that the fiber-reinforced composite material whose matrix rupture strain is smaller than that of a fiber is subjected to a tensile applied load parallel to the fiber direction. When the magnitude of the strain induced in the composite reaches that of the matrix rupture strain, cracking may occur in the matrix. This matrix cracking immediately leads to the overall rupture of the composite if volume fraction of a fiber is not enough to be able to bear the load otherwise sustained by a matrix alone. However, when volume fraction of a fiber is adequate, matrix cracking does not directly go to the final rupture of the composite and then there may appear in the composites many fibers bridging over the surfaces of the matrix cracks. These crack bridging fibers may play an important role in suppressing the opening of the crack, and this is followed by a sequence of the microscopic events such as interfacial sliding between the crack bridging fiber and a matrix and pull-out of fibers. Thus, even if all the constituents of the composite are brittle, such composite material will not fail in a brittle manner as a whole but rather in a ductile one even if all the constituents of the composites are brittle as was experimentally observed to the SiC fiber-reinforced LAS glass composite. Therefore, it is too much to say that the fracture behavior is controlled by the interfacial phenomena relevant to the mechanical behavior of the crack bridging fiber.

The interfacial bonding between a matrix and a fiber in the ceramic-based composites is said to be not a chemical bonding but a mechanical one due to the friction stress over the interface. The interfacial sliding between a fiber and a matrix may be influenced by the interfacial friction stress.
Since the friction stress is caused by the residual stress in the composites which stems from difference in the coefficient of linear thermal expansion and the elastic modulus difference between a fiber and a matrix, the magnitude of the friction stress may depend upon temperature. Therefore, the fracture behavior of such composites is closely related to the temperature. For example, Prewo and Brennan\(^5\) and Brennan and Prewo\(^5\) investigated experimentally the influence of temperature on both strength and fracture toughness of SiC fiber-reinforced glass composites and they reported that the value of fracture toughness of a SiC/LAS composite became more than 17 MPa m\(^{1/2}\) over the range from the room temperature to 1100\(^\circ\)C although fracture toughness of the monolithic LAS was 1.5 MPa m\(^{1/2}\).

In the present study, the authors propose the method to analyze such a phenomenon as prominent increase in fracture toughness based on the micromechanical modeling of a SiC/LAS composite as one which contains many matrix cracks and crack bridging fibers as well as interfacial sliding between a fiber and a matrix. By analyzing the micromechanical model constructed for a SiC/LAS composites, analytical expressions for energy release rate of a matrix crack and dissipation energy release rate due to interfacial sliding are derived. In order to investigate effect of temperature on the toughness of such a ceramic-based composite, the relationship between temperature and the friction stress over the interface is derived by using changes in both elastic moduli and coefficients of linear thermal expansion of a matrix and a fiber with temperature. By using this relationship, two energy release rates, and the surface energy of the matrix, the criteria for the propagation of the matrix crack will be derived. In addition, fracture toughness of such a composite is evaluated in terms of the critical size of the crack which propagates across a composite in an unstable manner and the change in toughness of such composites with temperature will be examined and compared with Brennan's results.

2. MODELING OF DISCONTINUOUS FIBER-REINFORCED COMPOSITES WITH CRACK BRIDGING FIBER

2.1 Geometrical Modeling

Figure 1 shows the composites containing many crack bridging fibers. Generally, the matrix crack in the real fiber-reinforced composites is various in size. In the present modeling, the discontinuous fibers of the same length are assumed to be arranged in the same orientation and the length of matrix cracks be the same size.

![Fig.1 Crack bridging fiber in the composites.](image-url)
The geometrical modeling of crack bridging fibers in a matrix is shown in Fig.2. The fiber whose longitudinal axis is parallel to the loading axis is a prolate ellipsoid with a longitudinal length $2l$ and a length of a minor axis $2r$. The fiber region is denoted by $\Omega_f$. The matrix crack whose surfaces are perpendicular to the loading axis is a penny shape with a radius $a$ and a thickness $2c$. This region is denoted by $\Omega_c$. Moreover, the crack bridging fiber region $\Omega_e$ is assumed to be a penny shape with a radius $r$ and a thickness $2c$. Now, the overall volume fraction of fibers in the composite material is defined as $f$. The volume fractions of cracks and crack bridging regions are defined as $f_c$ and $f_e$, respectively. Referring to the definition by Budiansky and O'Connell, in the present study the crack density $\rho$ for penny shape matrix crack is defined as follows:

$$\rho = \frac{1}{V_p} \frac{4}{3} \pi a^3 n_c = f_c$$  \hspace{1cm} (1)

By using this definition, the volume fraction of crack bridging regions is given as $f_e = \rho f^7$.

2.2 Inclusion Modeling of Discontinuous Fiber-Reinforced Composites

Let the extended modified equivalent inclusion method be applied to the geometrical model shown in Fig.2. The fiber region $\Omega_f$ whose elastic modulus $C_{ijkl}^{f}$ differs from that of matrix is regarded as the inhomogeneity in a sense defined in the field of micromechanics. The crack region $\Omega_c$ is replaced by the inclusion having the eigenstrain. Moreover, the crack bridging region $\Omega_e$ which belongs to both crack and fiber regions is considered as the inhomogeneous inclusion. Since the distortion of the crack bridging region may not be equal to that of the crack region, the eigenstrains in both regions must be different each other. These eigenstrains are defined as follows in the same way as the analysis performed by Mori et al.\(^9\).
in \( \Omega_{e} \) : \( \varepsilon_{ij}^{e} = \varepsilon_{ij}^{ce} \), 
(2) 

in \( \Omega_{o} \) : \( \varepsilon_{ij}^{o} = -(1-\alpha) \varepsilon_{ij}^{ce} \), \( 0 \leq \alpha < 1 \)  
(3) 

The total eigenstrain in the crack bridging region is equal to \( \alpha \varepsilon_{ij}^{ce} \) from Eqs.(2) and (3). The modes of distortion in the crack bridging region in accordance with \( \alpha \) are shown in Fig.3.

Under the uniform external stress \( \sigma_{ij}^{o} \) whose direction is parallel to the fiber one, the modified equivalent inclusion expression in \( \Omega_{f} \) is

\[
\sigma_{ij}^{o} + \sigma_{ij}^{o} - \sigma_{ij}^{o} = C_{ijkl} (\varepsilon_{kl}^{o} - \varepsilon_{kl}^{e} - \text{\textsf{S}}_{klnn} \varepsilon_{mn}^{e} - \varepsilon_{kl}^{e} ) ,
\]

where

- \( C_{ijkl} \) : elastic constants of a matrix and a fiber,
- \( \sigma_{ij}^{o} \), \( \sigma_{ij}^{o} \) : interaction field, \( \sigma_{ij}^{o} = C_{ijkl} \varepsilon_{kl}^{e} \),
- \( \sigma_{ij}^{o} \), \( \varepsilon_{ij}^{o} \) : applied stress and strain, \( \sigma_{ij}^{o} = C_{ijkl} \varepsilon_{ij}^{o} \),
- \( \sigma_{ij}^{o} (\Omega_{f}) \) : eigenstress caused by an inclusion \( \Omega_{f} \),
- \( \varepsilon_{ij}^{f} \) : \( \text{\textsf{S}}_{ijkl} \) : equivalent eigenstrain in \( \Omega_{f} \) and Eshelby's tensor for a fiber.

Substituting Eshelby tensor\(^{10}\) for a fiber into Eq.(4) and rearranging, we have

\[
\varepsilon_{11}^{f} = \frac{1}{A} \left\{ B_{11} \varepsilon_{11}^{o} + B_{12} \varepsilon_{33}^{o} + (B_{12} - \nu B_{11}) \frac{\sigma_{33}^{o}}{E} \right\} , 
\]

(5-a)

\[
\varepsilon_{33}^{f} = \frac{1}{A} \left\{ B_{31} \varepsilon_{11}^{o} + B_{32} \varepsilon_{33}^{o} + (B_{32} - \nu B_{31}) \frac{\sigma_{33}^{o}}{E} \right\} , 
\]

(5-b)
for the equivalent eigenstrain of the fiber and

\[
\frac{\sigma_{11}^e (\Omega_f)}{\mu} = F_{11} \bar{\epsilon}_{11}^e + F_{12} \bar{\epsilon}_{12}^e + \frac{F_{13}}{E} \bar{\epsilon}_{13}^e ,
\]

\[
\frac{\sigma_{23}^e (\Omega_f)}{\mu} = F_{23} \bar{\epsilon}_{23}^e + \frac{F_{33}}{E} \bar{\epsilon}_{33}^e ,
\]

(6-a)

(6-b)

for the eigenstress caused by a fiber, where \( \mu, \nu \) and \( E \) are shear modulus, Poisson's ratio and elastic modulus of the matrix, respectively. \( A, B_{ij} \) and \( F_{ij} \) are the constants containing only the elastic constant of a matrix.

The eigenstresses caused by the eigenstrain \( \epsilon_{33}^{\text{ec}} \) in the crack region are given as \(^{11}\)

\[
\sigma_{11}^e (\Omega_c) = -\frac{(1+2 \nu)}{4 (1-\nu)} \frac{\pi \mu}{a} \epsilon_{33}^{\text{ec}} ,
\]

\[
\sigma_{23}^e (\Omega_c) = -\frac{\pi \mu}{2 (1-\nu)} \frac{c}{a} \epsilon_{33}^{\text{ec}} .
\]

(7-a)

(7-b)

Since the configuration of the crack bridging region is a penny shape as well as that of the crack region, by referring to Eq.(7) the eigenstresses caused by the eigenstrain \( -(1-\alpha) \epsilon_{33}^{\text{ec}} \) in the crack bridging region \( \Omega_o \) are obtained as follows:

\[
\sigma_{11}^e (\Omega_o) = \frac{(1+2 \nu)}{4 (1-\nu)} \frac{\pi \mu}{r} (1-\alpha) \epsilon_{33}^{\text{ec}} ,
\]

\[
\sigma_{23}^e (\Omega_o) = \frac{\pi \mu}{2 (1-\nu)} \frac{c}{r} (1-\alpha) \epsilon_{33}^{\text{ec}} .
\]

(8-a)

(8-b)

The integral of internal stress over the whole body is equal to zero irrespective of the location of the stress source and the elastic modulus of the matrix, namely,

\[
\int_D \sigma_{ij} \, dD = 0 .
\]

(9)

When the total internal stress in \( \Omega \) is expressed as \( \sigma_{ij}^{\text{total}} (\Omega) \), the internal stress in each region is given as follows:

\[
\sigma_{ij}^{\text{total}} (\Omega_f - \Omega_o) = \sigma_{ij}^e (\Omega_f) + \bar{\sigma}_{ij} ,
\]

\[
\sigma_{ij}^{\text{total}} (\Omega_c - \Omega_o) = \sigma_{ij}^e (\Omega_c) + \bar{\sigma}_{ij} ,
\]

\[
\sigma_{ij}^{\text{total}} (\Omega_o) = \sigma_{ij}^e (\Omega_o) + \sigma_{ij}^c (\Omega_c) + \sigma_{ij}^c (\Omega_f) + \bar{\sigma}_{ij} ,
\]

\[
\sigma_{ij}^{\text{total}} (D - \sum \Omega_o) = \sigma_{ij} .
\]

(10)

Substituting Eq.(10) into Eq.(9) and performing the integration by considering the volume fraction of each region, we have

\[
\bar{\sigma}_{ij} = -[f \sigma_{ij}^c (\Omega_f) + \rho \{ \sigma_{ij}^c (\Omega_c) + f \sigma_{ij}^c (\Omega_o) \}]
\]

(11)
Moreover, since the stress on the crack surface is free,
\[
s_{33}^{\text{free}}(\Omega_c - \Omega_t) + s_{33}^p - s_{33}^p(\Omega_c) + s_{33}^E = 0
\]  
(12)

must be satisfied. By substituting Eqs. (6), (7), (8) and (11) into Eq. (12), we have the following expression for the eigenstrain in the crack region \( \epsilon_3^c \),
\[
\epsilon_3^c = \frac{2(1 - \nu)}{\pi\mu} \frac{a}{c} \xi s_{33}^E,
\]  
(13)

where \( \xi \) is the function containing the parameter \( \alpha \) and crack length \( a \).

2.3 Modeling of Interfacial Sliding between a Fiber and a Matrix

The modeling of the ceramic-based fiber-reinforced composites taking into consideration of the interfacial sliding behavior has been performed by Marshall, Cox and Evans\(^{(12)}\) and Mori et al.\(^{(9)}\) In the present research, the same model as Mori's one will be used. The model to the interfacial sliding between a fiber and a matrix is shown in Fig.4. This figure shows only upper half from the center line between crack surfaces. The hatched region in the figure is the fiber one. The sliding region of the interface between a fiber and a matrix is the region from the crack surface to the border \( A \cdot A' \) in the figure. The length of the sliding region is defined as \( \delta \). Under the constant temperature, the friction stress \( k \) over the interface is assumed to be constant through the sliding region. The interface above the border \( A \cdot A' \) in the figure is assumed to be bonded perfectly. The internal stress in the perfectly-bonding fiber region is \( s^{\text{int}}_{33}(\Omega_c - \Omega_t) \) and the total internal stress in the crack bridging region is \( s^{\text{int}}_{33}(\Omega_c) \). Thus, considering the equilibrium of force in the fiber region and using Eq. (10), we have
\[
s_{33}^{\text{int}}(\Omega_c) + s_{33}^E(\Omega_c) = 2\delta \frac{k}{r}
\]  
(14)

By substituting Eqs. (7) and (8) into Eq. (14) and rearranging, the parameter \( \alpha \) which controls the distortion of the crack bridging region can be defined as follows:
\[
\alpha = 1 - \frac{1 - \nu}{f} \left( \frac{r}{a} \right)
\]  
(15)

Generally, ceramic materials are fabricated by the sintering process under the high temperature and pressure. When the thermal expansion coefficient of a matrix differs from that of a fiber, the contact residual pressure may occur on the interface between a fiber and a matrix after the sintering process. The contact residual pressure can be hardly estimated exactly because the configuration of the fiber is a prolate ellipsoid. Thus, in the present research, the contact residual pressure \( p_c \) over the interface is assumed to be approximately equal to the contact pressure which occurs by the shrink fit of a circular cylindrical rod into an infinite body having a circular cylindrical hole. Namely,
\[
p = \left\{ \frac{1}{2} \frac{b}{r} - \left( \alpha_k - \alpha_k' \right) \Delta T \right\} \left( \frac{1 + \nu}{E} + \frac{1 - \nu'}{E'} \right)
\]  
(16)

where \( b \) is the magnitude of shrinkage, \( \alpha_k \) and \( \alpha_k' \) are the thermal expansion coefficient of a matrix and a fiber, respectively, and \( E' \) and \( \nu' \) are elastic modulus and Poisson's ratio of the fiber. \( \Delta T \) is the increment in the temperature from the reference one (20°C). At the reference temperature, elastic moduli of a matrix and a fiber are defined as \( E \) and \( E' \), respectively, and the contact residual pressure as \( p_c \). Rewriting Eq. (16) by using these definitions, we have finally
Fig. 4 Interfacial sliding model.

\[ p = \frac{\left( \frac{1 + v}{E_r} + \frac{1 - v'}{E'_r} \right) p_r - \left( \alpha_k - \alpha'_k \right) \Delta T}{(1 + v) E' + (1 - v') E} EE' \]  

(17)

The friction stress \( k \) over the interface can be calculated from \( k = \mu_k \cdot p \), where \( \mu_k \) is the friction coefficient between a fiber and a matrix. \( \mu_k = 0.29 \) is used irrespective of the temperature.

2.4 Energy Release Rates for Matrix Crack and Dissipation Energy Release Rate

The total potential energy \( F \) caused only by the variation of eigenstrain is defined as

\[ F = -\frac{1}{2} \int_{\Omega} \sigma_0^e \varepsilon_0^e dD - \frac{1}{2} \int_{\Omega} \alpha_0^e \varepsilon_0^e dD - \frac{1}{2} \int_{\Omega} \sigma_0^e (1 - \alpha) \varepsilon_0^e dD \]

\[ -\frac{1}{2} \int_{\Omega} \left( \sigma_0^e (\Omega_x) + \sigma_0^e (\Omega_y) \right) \alpha \varepsilon_0^e dD \]  

(18)

The energy release rate of a matrix crack \( G \) can be obtained by using \( F \) as follows:

\[ G = -\frac{\delta F}{\delta (\pi a^2)} \frac{1}{n_c} \left( \frac{1 - v}{1 - v_r} \frac{\mu_r}{\mu} \right) \frac{a}{r} G_0 \zeta \]  

(19)

where \( G_r \) is the energy release rate for the case that there is only a penny shape crack with radius \( r \) in the matrix at the reference temperature,

\[ G_r = \frac{2 (1 - v)}{\pi \mu_r} (\sigma_0^e)^2 r \]  

(20)
where \( \mu_s \) and \( \nu_s \) are the shear modulus and Poisson's ratio of the matrix at the reference temperature, respectively.

The energy dissipation over the interface between a fiber and a matrix may occur due to the sliding each other. Since moving distance of the fiber to the original crack surface shown in Fig.4 is equal to \( 2 \alpha \epsilon_{ij}^R \) and the length of the sliding region is \( \delta \), the dissipation energy \( W_d \) caused by movement of the fiber can be estimated as

\[
W_d = (2 \pi r \delta k) \cdot (2 \alpha \epsilon_{ij}^R) n_p
\]

The change in the dissipation energy \( W_d \) with the increase in the area of matrix crack can be expressed as follow:

\[
\Delta G = \frac{\partial W_d}{\partial (\pi a^2)} = \left( \frac{1}{1-v} \right) \left( \frac{\mu_s}{\mu} \right) \frac{a}{r} G_e \kappa
\]

\( \Delta G \) is called as the dissipation energy release rate.

3. NUMERICAL RESULTS AND DISCUSSIONS

3.1 Elastic Property of the Model and Definition of Parameter \( \rho_s \)

In the present research, the discontinuous SiC ceramic fiber-reinforced LAS grass is used for a sample material of the composites under consideration. The aspect ratio of the fiber is taken 100. The change in elastic moduli of these constitutive materials with the increment in the temperature \( \Delta T \) are shown in Fig.5. The approximate equations expressed in the figure are obtained by the regression to the data represented in the references\(^1\) by means of the least square method. Figure 6 shows the change in the thermal expansion coefficients of a matrix and a fiber with \( \Delta T \). Both Poisson's ratios of a fiber and a matrix are assumed to be constant irrespective of temperature and their values are listed in Table 1.

![Elastic Moduli vs. Temperature](image)

**Fig.5** Change in elastic moduli with temperature.
3.2 Energy Release Rate for Matrix Crack and Dissipation Energy Release Rate

Figure 7 shows the relations of the energy release rate $G / G_s$ for a matrix crack and the dissipation energy release rate $\Delta G / G_s$ to the crack length normalized by the fiber radius $a / r$ to the various volume fraction of fibers $f$ at the reference temperature. The solid line corresponds to $G / G_s$, and the dotted line $\Delta G / G_s$. It can be seen from the figure that $G$ increases with increase in the crack length, while decreases with increase in $f$ to any crack length. According to fracture mechanics, the energy release rate for a matrix crack increases as the crack length increases. The former is consistent with this. The latter can be explained by considering that the stress in the matrix decreases with increase in the fiber content. Furthermore, $\Delta G$ has the same tendency as that of $G$. 

Table 1 Poisson's ratios of fiber and matrix.

<table>
<thead>
<tr>
<th></th>
<th>Poisson's ratio</th>
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</thead>
<tbody>
<tr>
<td>Matrix (LAS)</td>
<td>0.20</td>
</tr>
<tr>
<td>Fiber (SiC)</td>
<td>0.25</td>
</tr>
</tbody>
</table>

The crack density $\rho_s$ is the function of the length of a crack $a$. Thus, in the numerical analysis, the parameter $\rho_s$

$$\rho_s = \frac{1}{V_p} \cdot \frac{4}{3} \pi r^3 n_c = \left( \frac{r}{a} \right)^3 \rho$$

is used. It can be seen from Eq.(23) that $\rho_s$ is one-to-one correspondence to the number of crack $n_c$. 

Fig.6 Change in thermal expansion coefficients with temperature.
$\Delta G$ represents the change in the energy dissipated by the interfacial sliding between a fiber and a matrix in accordance with the propagation of a matrix crack. In other words, $\Delta G$ implies the necessary energy for the interface to slide in advance with the unit area of the matrix crack. The number of crack bridging regions where the interfacial sliding may occur increases with increasing the crack length $a/r$. Thus, the necessary energy for the interfacial sliding may increase rapidly with increase in $a/r$. Since the opening displacement of crack bridging region may be suppressed by the fiber, $\Delta G$ may be considered to decrease with increase in $f$.

The solid circle in the figure represents the point that $G = \Delta G$. When the length of the matrix crack reaches to this point, the matrix crack will be arrested because that the released energy $G$ with the propagation of a matrix crack becomes smaller than the dissipated energy by the interfacial sliding. From this result, the critical length of the matrix crack will be examined in the following section.

3.3 Critical Crack Length

By adopting Griffith's concept of the brittle fracture to the present model, the criterion for the propagation of the matrix crack can be expressed by using the energy release rate $G$ and the dissipation energy release rate $\Delta G$ as follows:

$$G - (\Delta G + 2\gamma_c) = G' \geq 0,$$  \hspace{1cm} (24)

where $\gamma_c$ is the surface energy of the matrix. $G'$ in Eq.(24) is called as a total energy release rate. Substituting Eqs.(19) and (22) into Eq.(24) and using the condition that Poisson's ratio of the matrix is constant irrespective of the temperature as listed in Table 1, we have

$$G' = \frac{\mu}{\mu} \frac{a}{r} G_r (\zeta - \kappa) - 2\gamma_c$$  \hspace{1cm} (25)
Furthermore, by using the experimental result obtained by Brennan et al.\textsuperscript{5)} that the fracture toughness $K_{fc}$ of monolithic LAS glass is constant irrespective of temperature, the surface energy of the matrix $\gamma_c$ can be expressed as follows:

$$K_{fc} = \frac{2\mu}{1-v} G_{lc} = \frac{2\mu}{1-v} \cdot 2\gamma_c \quad \therefore \gamma_c = \frac{(1-v^2)}{2E} K_{fc}^2 = \frac{E_r}{E} \gamma_{cr} \quad (26)$$

where $\gamma_{cr}$ is the surface energy of the matrix at the reference temperature. By substituting Eq.(26) into Eq.(25), the total energy release rate which implies the criterion for the propagation of the matrix crack is obtained finally as follows:

$$\frac{G'}{G_r} = \left( \frac{E_r}{E} \right) \left( \frac{a}{r} \right) (\xi - \kappa) - \left( \frac{2\gamma_{cr}}{G_r} \right) \geq 0 \quad (27)$$

By using Eq.(27), $\partial(G'/G_r)/\partial(a/r)$ can be calculated numerically. Figure 8 shows the relations of $\partial(G'/G_r)/\partial(a/r)$ to the crack length $a/r$. A solid circle for the case of $f=0.3$ shown in the figure shows the point that the total energy release rate $G'$ is equal to zero. Namely, the values of $\partial(G'/G_r)/\partial(a/r)$ and $G'$ are always positive within the range shown in the figure irrespective of the crack length and the volume fraction of fibers. Therefore, the matrix crack will continue to propagate across the composite in an unstable manner as far as Eq.(27) is satisfied, and will arrest if not. Thus, the critical matrix crack length at which the matrix crack will be arrested by the energy dissipation due to the interfacial sliding can be obtained by using the condition of Eq.(27). When the critical matrix crack length is too large, the matrix cracking may immediately lead to the overall rupture of the composites. In other words, smaller critical crack length implies that the length of the matrix crack existing in the composites is shorter after the propagation and arrest of the matrix crack, and hence the composites are considered to be more tough. Therefore, the fracture toughness of the composite can be evaluated in terms of the critical length of the crack. The length is named as a critical crack length $a_c$. 

![Fig.8 $\partial(G'/G_r)/\partial(a/r)$ versus $a/r$.](image-url)
3.4 Effect of Temperature on Critical Crack Length

The effect of the friction stress $k$ over the interface on the critical crack length $a_c$ at the reference temperature is shown in Fig. 9. It can be seen from the figure that in the lower range of the value of $k$, i.e., $k / \sigma_{33}^0 < 0.3$, $a_c$ decreases with increase in the friction stress $k$, while in the range of $k / \sigma_{33}^0 > 0.3$, $a_c$ has a tendency to increase with increase in $k$. For the former case, the crack shielding effect due to the crack bridging fiber is considered to become more weak because the interface between a matrix and a fiber may slide more easily, and hence the value of the dissipation energy release rate $\Delta G$ becomes also smaller under the prescribed condition that the length of the sliding region shown in Fig. 4 is constant. On the contrary, for the latter case the dissipation energy due to sliding may decrease since the sliding becomes hard to occur. Therefore, the dissipation energy may decrease whenever the value of $k$ becomes too large and too small. Since the propagation of the matrix crack may become more hard to be arrested as the dissipation energy decreases, the value of $a_c$ is considered to show a concave change with the increase in the friction stress. As discussed before, shorter $a_c$ implies that the composites become more tough. Therefore, the result shown in Fig. 9 gives a suggestion that there exists the optimal value of the friction stress for obtaining the maximum toughness of SiC fiber-reinforced LAS glass composite material.

The changes in the critical crack length $a_c$ with the temperature increment to the various contact residual pressures $p_f / \sigma_{33}^0$ prescribed at the reference temperature (20°C) and normalized by the applied stress are shown in Fig. 10. The result indicated by open circles and the solid line shows that of the fracture toughness $K_f$ experimentally obtained by Brennan et al. for the unidirectional SiC fiber-reinforced LAS glass$^5$. The curve of the change in $K_f$ with $\Delta T$ is reported to be convex as is shown in the figure. We can see from the figure that all the curves of the change in $a_c$ with $\Delta T$ are concave irrespective of $p_f / \sigma_{33}^0$. Hence, the curve of the change in the toughness of the composite with $\Delta T$ is interpreted to be convex and this tendency is agreement with that of Brennan’s result. Moreover, it can be seen from the figure that as the contact residual pressure $p_f$ prescribed at the reference temperature increases, the optimum temperature at which the value of the toughness of the composite becomes largest shifts to lower one.

![Fig. 9](image.png)

Fig. 9 Effect of friction stress $k$ on the critical crack length $a_c$. 
This may be caused by the change in $k$ with $p_r$. The optimal temperature is $\Delta T = 960K$ in the Brennan's results and about $\Delta T = 920K$ for the case of $p_r/\sigma^{\infty}_{33} = 0.25$ in the present results.

4. CONCLUSIONS

In the present study, by using the extended modified equivalent inclusion method, micromechanical analysis is performed on a discontinuous fiber-reinforced ceramic-based composites which contains many cracks and crack bridging regions showing interfacial sliding behavior between a fiber and a matrix, analytical expressions for energy release rate and the dissipation energy release rate due to interfacial sliding can be derived. Moreover, in order to investigate the effect of temperature on the toughness of such a ceramic-based composites, the relationship between temperature and the friction stress over the interface is derived. By using this relation, two energy release rates and the surface energy of the matrix, the criteria for the propagation of the matrix crack can be derived successfully. In addition, fracture toughness of such a composite is evaluated in terms of the critical size of the crack which propagates across a composite in an unstable manner and the change in toughness of such ceramic based composites with temperature can be estimated. We have the following conclusions:

(1) The existence of the optimal value of the friction stress over the interface between a fiber and a matrix, which makes the critical crack length $a_c$ to minimize, i.e., the toughness of the composites to maximize, can be demonstrated analytically.

(2) The present tendency of the change in the toughness of the composite with temperature increment is good agreement with that of the result experimentally observed by Brennan et al.
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10) pp.74-84 in the reference 8.