Paper

A discrete mechanics approach to gait generation for the compass-type biped robot

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Abstract: In this paper, we develop a new approach based on discrete mechanics to a gait generation problem for the compass-type biped robot. First, both continuous-time and discrete-time models of the compass-type biped robot are derived. We next formulate a discrete gait generation problem for the discrete compass-type biped robot as a finite dimensional nonlinear optimal control problem and show an algorithm to solve the problem based on the sequential quadratic programming. Then, we propose a transformation method that converts a discrete-time control input into a continuous-time zero-order hold one and apply it to gait generation for the continuous compass-type biped robot. Some simulations are also shown in order to verify the effectiveness of our new approach.

Key Words: discrete/continuous compass-type biped robot, discrete mechanics, gait generation, zero-order hold control input, sequential quadratic programming

1. Introduction

Recently, a lot of work on humanoid robots have been actively done by various approaches in research fields such as robotics, control theory, cognitive engineering and so on. Especially, the compass-type biped robot has been mainly studied as one of the simplest models of humanoid robots. For example, theoretical analysis of passive walking [1-3], researches associated with nonlinear dynamical theory such as Poincaré sections and limit cycles [4-8], gait pattern generation based on ZMP (zero-moment point) [9-11], and self-motivating acquirement of gaits by learning theory and evolutionary computing [12-15]. However, in general, it is quite difficult to realize stable gaits for humanoid robots in terms of nonlinear problems, and hence there are still many problems left to solve.

In almost every work on humanoid robots, models derived by continuous-time mechanics are utilized. On the other hand, discrete mechanics, which is a new discretizing and modeling tool for nonlinear mechanical systems and is derived by discretization of basic principles and equations of classical mechanics, has been focused on [16-21]. A discrete model (the discrete Euler-Lagrange
equations) in discrete mechanics has some interesting characteristics: (i) Discrete mechanics has the same characteristics as symplectic maps. (ii) It shows less numerical errors in comparison with other numerical solutions such as Euler method and Runge-Kutta method. (iii) It can describe energies for both conservative and dissipative systems with less numerical errors. (iv) Some laws of physics such as Noether’s theorem are still satisfied. (v) Simulations can be performed for large sampling times. Therefore, discrete mechanics has a possibility of analysis and controller synthesis with high compatibility with computers.

We have focused on discrete mechanics and considered its applications to control theory. In [22–24], we applied discrete mechanics to control problems for the cart-pendulum system, and confirmed the application potentiality to control theory. From these results, it can be expected that discrete mechanics has an applicability to humanoid robots, which are more complicated than the cart-pendulum system, and indicates a new methodology on control of mechanical systems.

In this paper, we develop a new approach to a gait generation problem for the compass-type biped robot based on discrete mechanics. The outline of this paper is as follows. First, in Section 2, some basic concepts on discrete mechanics are summarized. In Section 3, we next derive the continuous equations of motion in discrete mechanics. Consider a variation of the discrete action sum from the viewpoint of the sequential quadratic programming in Section 4. A solving algorithm of it from the continuous-time setting is also illustrated. After that, in Section 5, we consider a transformation method from a discrete control input obtained in Section 4 into a continuous-time zero-order hold control input, and apply it to the continuous compass-type biped robot. Finally, we show a numerical simulation on continuous gait generation for the continuous compass-type biped robot in order to confirm the effectiveness of our method.

2. Discrete mechanics

2.1 Discrete Hamilton’s principle and discrete Euler-Lagrange equations

This section sums up basic concepts of discrete mechanics. See [16–19] for further details on discrete mechanics. Let Q be an n-dimensional configuration manifold and $q \in \mathbb{R}^n$ be a generalized coordinate of Q. We also refer to $T_qQ$ as the tangent space of Q at a point $q \in Q$ and $\dot{q} \in T_qQ$ denotes a generalized velocity. Moreover, we consider a time-invariant Lagrangian as $L(q, \dot{q}) : TQ \to \mathbb{R}$.

We first explain about the discretization method of discrete mechanics. The time variable $t \in \mathbb{R}$ is discretized as $t = kh$ ($k = 0, 1, 2, \cdots$) by using a sampling time $h > 0$. We denote $q_k$ as a point of Q at the time step k, that is, a curve on Q in the continuous-time setting is represented as a sequence of points $q^d := \{q_k\}_{k=1}^N$ in the discrete-time setting. The transformation method of discrete mechanics is carried out by the replacement:

$$ q \approx (1 - \alpha)q_k + \alpha q_{k+1}, \quad \dot{q} \approx \frac{q_{k+1} - q_k}{h}, \quad (1) $$

where $\alpha$ is expressed as a internally dividing point of $q_k$ and $q_{k+1}$ with a ratio $\alpha$ ($0 < \alpha < 1$). We then define a discrete Lagrangian:

$$ L^d_\alpha(q_k, q_{k+1}) := hL((1 - \alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}), \quad (2) $$

and a discrete action sum:

$$ S^d_\alpha(q_0, q_1, \cdots, q_N) = \sum_{k=0}^{N-1} L^d_\alpha(q_k, q_{k+1}). \quad (3) $$

We next summarize the discrete equations of motion in discrete mechanics. Consider a variation of points on Q as $\delta q_k \in T_qQ$ ($k = 0, 1, \cdots, N$) with the fixed condition $\delta q_0 = \delta q_N = 0$ as shown in Fig. 1. In analogy with the continuous-time setting, we define a variation of the discrete action sum (3) as

$$ \delta S^d_\alpha(q_0, q_1, \cdots, q_N) = \sum_{k=0}^{N-1} \delta L^d_\alpha(q_k, q_{k+1}). \quad (4) $$

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The discrete Hamilton’s principle states that only a motion which makes the discrete action sum (3) stationary is realized. Calculating (4), we have

\[ \delta S_d^\alpha = \sum_{k=1}^{N-1} \left\{ D_1 L_d^\alpha(q_k, q_{k+1}) \delta q_k + D_2 L_d^\alpha(q_{k-1}, q_k) \right\} \delta q_k, \tag{5} \]

where \( D_1 \) and \( D_2 \) denotes the partial differential operators with respect to the first and second arguments, respectively. Consequently, from the discrete Hamilton’s principle and (5), we obtain the discrete Euler-Lagrange equations:

\[ D_1 L_d^\alpha(q_k, q_{k+1}) + D_2 L_d^\alpha(q_{k-1}, q_k) = 0, \quad k = 1, \ldots, N-1. \tag{6} \]

It turns out that (6) is represented as difference equations which contains three points \( q_{k-1}, q_k, q_{k+1} \), and we need \( q_0, q_1 \) as initial conditions when we simulate (6).

2.2 Discrete forces and discrete Lagrange-d’Alembert’s principle

In this subsection, we consider a method to add external forces to the discrete Euler-Lagrange equations. By considering an analogy of continuous mechanics, we denote discrete external forces by \( F_d^\alpha : Q \times Q \to T^* Q \), and discretize continuous Lagrange-d’Alembert’s principle as

\[ \delta \sum_{k=0}^{N-1} L_d^\alpha(q_k, q_{k+1}) + \sum_{k=0}^{N-1} F_d^\alpha(q_k, q_{k+1}) \cdot (\delta q_k, \delta q_{k+1}) = 0, \tag{7} \]

where we define right/left discrete external forces: \( F_{d+}^\alpha, F_{d-}^\alpha : Q \times Q \to T^* Q \) as

\[ F_{d+}^\alpha(q_k, q_{k+1}) \delta q_k = F_d^\alpha(q_k, q_{k+1}) \cdot (\delta q_k, 0), \]
\[ F_{d-}^\alpha(q_k, q_{k+1}) \delta q_{k+1} = F_d^\alpha(q_k, q_{k+1}) \cdot (0, \delta q_{k+1}), \tag{8} \]

respectively. By using right/left discrete external forces, a continuous external force \( F^c : TQ \to T^* Q \) can be discretized as

\[ F_{d+}^\alpha(q_k, q_{k+1}) = (1-\alpha) h F^c \left( (1-\alpha) q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h} \right), \]
\[ F_{d-}^\alpha(q_k, q_{k+1}) = \alpha h F^c \left( (1-\alpha) q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h} \right). \tag{9} \]

Therefore, by calculating variations for (7), we obtain the discrete Euler-Lagrange equations with discrete external forces:

\[ D_1 L_d^\alpha(q_k, q_{k+1}) + D_2 L_d^\alpha(q_{k-1}, q_k) + F_{d+}^\alpha(q_k, q_{k+1}) + F_{d-}^\alpha(q_{k-1}, q_k) = 0, \quad k = 1, \ldots, N-1. \tag{10} \]
3. Continuous/discrete compass-type biped robot

3.1 Setting of compass-type biped robot

In this subsection, we first give a problem setting of the compass-type biped robot. In this paper, we consider a simple compass-type biped robot which consists of two rigid bars (Leg 1 and 2) and a joint without rotational friction (Waist) as shown in Fig. 2. In Fig. 2, Leg 1 is called the supporting leg which connects to ground and Leg 2 is called the swing leg which is ungrounded. Moreover, for the sake of simplicity, we give the following assumptions:

(i) The supporting leg does not slip at the contact point with the ground.
(ii) The swing leg hits the ground with completely inelastic collision.
(iii) The compass-type biped robot is supported by two legs for just a moment.
(iv) The length of the swing leg gets smaller by infinitely small when the swing leg and the supporting leg pass each other.

Let $\theta$ and $\phi$ be the angles of Leg 1 and 2, respectively. We also use the notations: $m$: the mass of the legs, $M$: the mass of the waist, $I$: the inertia moment of the legs, $a$: the length between the waist and the center of gravity, $b$: the length between the center of gravity and the toe of the leg, $l (= a + b)$: the length between the waist and the toe of the leg. The Lagrangian of this system $L^c$ is given by

$$L^c(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{1}{2} (I + ma^2 + ml^2 + Ml^2) \dot{\theta}^2 + \frac{1}{2} (I + mb^2) \dot{\phi}^2 - mbl \cos (\theta - \phi) \dot{\theta} \dot{\phi} - (ma + mg + Ml)g \cos \phi + mgb \cos \phi. \quad (11)$$

![Fig. 2. Compass-type biped robot.](image)

In the walking process of the compass-type biped robot, there exist two modes: the swing phase and the impact phase. In the swing phase the swing leg is ungrounded, and in the impact phase the toe of the swing leg hit the ground. As shown in Fig. 3, it is noted that the swing phase and the impact phase occur alternately and the swing leg and the supporting leg switch positions with each other with respect to each collision. We denote the order of the swing phase and the impact phase by $i = 1, 2, \cdots, P$ and $i = 1, 2, \cdots, P - 1$, respectively. In addition, we assume that Leg 1 is the swing leg and Leg 2 is the supporting leg in odd-numbered swing phases, and Leg 1 is the supporting leg and Leg 2 is the swing leg in even-numbered swing phases.
3.2 Continuous compass-type biped robot (CCBR)

This subsection presents a continuous-time model of the compass-type biped robot which is called the continuous compass-type biped robot (CCBR) via normal continuous mechanics based on the problem setting shown in the previous subsection. Let $\theta^{(i)}$, $\phi^{(i)}$ be angles of Leg 1 and Leg 2 in the $i$-th swing phase, respectively. We derive both the swing phase and the impact phase for the case where Leg 1 is the swing leg and Leg 2 is the supporting leg as shown in Fig. 2. Hence, it is noted that for the case where Leg 1 is the supporting leg and Leg 2 is the swing leg, we can easily obtain the model by changing $\theta^{(i)}$ for $\phi^{(i)}$ in the both models.

First, we consider the model of the swing phase of the CCBR [6]. We substitute the continuous Lagrangian (11) into the continuous Euler-Lagrange equations. Then, we assume that a torque to the waist can be controlled and is denoted by $v^{(i)}$. Adding the torque $v^{(i)}$ to the right-hand side of the Euler-Lagrange equations, we obtain the model of the swing phase of the CCBR as

$$
\begin{align*}
&\begin{bmatrix}
ma^2 + ml^2 + I - mbl \cos(\theta^{(i)} - \phi^{(i)}) \\
-mbl \cos(\theta^{(i)} - \phi^{(i)}) & mb^2 + I
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta}^{(i)} \\
\ddot{\phi}^{(i)}
\end{bmatrix}

+ \begin{bmatrix}
-mbl \sin(\theta - \phi)(\dot{\theta}^{(i)})^2 - (ma + ml + Ma)g \sin \theta^{(i)} \\
mlb \sin(\theta^{(i)} - \phi^{(i)}) - mbg \sin \phi^{(i)}
\end{bmatrix}

= \begin{bmatrix}
v^{(i)} \\
v^{(i)}
\end{bmatrix}.
\end{align*}
$$

(12)

Next, we show the model of the impact phase of the CCBR. Considering the assumption that the
swing leg hits the ground with completely inelastic collision, we calculate the principle of conservation of angular momentum for the CCBR. Then, we have the model of the impact phase of the CCBR as (13).

\[
\begin{pmatrix}
-2mal + Ml^2 \cos (\theta^{(i)} - \phi^{(i)}) + mbl - I - mab - I \\
\end{pmatrix} \begin{pmatrix}
\dot{\theta}^{(i)}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-mbl^2 + mbl \cos (\theta^{(i+1)} - \phi^{(i+1)}) - I - (2ma^2 + Ml^2) + mbl \cos (\theta^{(i+1)} - \phi^{(i+1)}) - I \\
\end{pmatrix} \begin{pmatrix}
\dot{\phi}^{(i+1)}
\end{pmatrix}
\]

(13)

3.3 Discrete compass-type biped robot (DCBR)

Based on the problem setting shown in Subsection 3.1, we next derive the discrete-time model of the compass-type biped robot called the discrete compass-type biped robot (DCBR) via discrete mechanics.

We define the notations for the DCRB: a sampling time, \(\alpha\): a division ratio in discrete mechanics, \(k = 1, 2, \cdots, N\): a time step, \(i = 1, 2, \cdots, P\): the order of swing phases, \(\theta_k^{(i)}\), \(\phi_k^{(i)}\): the angles of Leg 1 and 2 at the time step \(k\) in the \(i\)-th swing phase, respectively, \(u_k^{(i)}\): the control input at the time step \(k\) in the \(i\)-th swing phase as a discrete torque for the swing leg.

We first derive the swing phase model of the DCRB for the case where Leg 1 is the swing leg and Leg 2 is the supporting leg as shown in Fig. 2. Note that for the case where Leg 1 is the supporting leg and Leg 2 is the swing leg, we can easily have the model by changing \(\theta_k^{(i)}\) for \(\phi_k^{(i)}\). Calculate the discrete Lagrangian \(L_k^{(i)}(\theta_k^{(i)}, \theta^{(i+1)}, \phi_k^{(i)}, \phi^{(i+1)})\) from (11) as

\[
L_k^{(i)}(\theta_k^{(i)}, \theta_k^{(i+1)}, \phi_k^{(i)}, \phi_k^{(i+1)}) =
\frac{1}{2}(I + ma^2 + Ml^2 + ml^2) \left( \frac{\theta_k^{(i)} - \theta_k^{(i+1)}}{h} \right)^2 + \frac{1}{2}(I + mbl^2) \left( \frac{\phi_k^{(i)} - \phi_k^{(i+1)}}{h} \right)^2
\]

\[
- mbl \cos((1 - \alpha)\theta_k^{(i)} + \alpha\theta_k^{(i+1)} - (1 - \alpha)\phi_k^{(i)} - \alpha\phi_k^{(i+1)}) \frac{\theta_k^{(i+1)} - \theta_k^{(i)}}{h} \frac{\phi_k^{(i+1)} - \phi_k^{(i)}}{h}
\]

\[
- (ma + mg + Ml) g \cos((1 - \alpha)\phi_k^{(i)} + \alpha\phi_k^{(i+1)}) + mgb \cos((1 - \alpha)\phi_k^{(i)} + \alpha\phi_k^{(i+1)}) \]

and substitute it into the discrete Euler-Lagrange equations (6). Moreover, adding the control input to the left-hand side of the discrete Euler-Lagrange equations, we obtain the swing phase model as

\[
-(ma^2 + Ml^2 + ml^2 + I)(\theta_k^{(i+1)} - \theta_k^{(i)}) + mbl \cos((1 - \alpha)(\theta_k^{(i)} - \phi_k^{(i)}) + \alpha(\theta_k^{(i+1)} - \phi_k^{(i+1)}))(\theta_k^{(i+1)} - \theta_k^{(i)})
\]

\[
+ mbl(1 - \alpha) \sin((1 - \alpha)(\theta_k^{(i)} - \phi_k^{(i)}) + \alpha(\theta_k^{(i+1)} - \phi_k^{(i+1)}))(\theta_k^{(i+1)} - \theta_k^{(i)})
\]

\[
+ (ma^2 + Ml^2 + ml^2 + I)(\theta_k^{(i)} - \theta_k^{(i-1)}) + (ma + ml + Ml)gh^2(1 - \alpha) \sin((1 - \alpha)\theta_k^{(i)} + \alpha\theta_k^{(i+1)})
\]

\[
- mbl \cos((1 - \alpha)(\theta_k^{(i)} - \phi_k^{(i)}) + \alpha(\theta_k^{(i+1)} - \phi_k^{(i+1)}))(\phi_k^{(i)} - \phi_k^{(i+1)})
\]

\[
+ (ma + ml + Ml)gh^2 \alpha \sin((1 - \alpha)\theta_k^{(i+1)} + \alpha\theta_k^{(i+1)})
\]

\[
+ mbl \alpha \sin((1 - \alpha)(\theta_k^{(i)} - \phi_k^{(i)})) + \alpha(\theta_k^{(i)} - \phi_k^{(i+1)}))(\theta_k^{(i)} - \theta_k^{(i+1)}) + hu_k^{(i)} = 0,
\]

(15)

\[
-(mb^2 + I)(\phi_k^{(i+1)} - \phi_k^{(i)}) + mbl \cos((1 - \alpha)(\theta_k^{(i)} - \phi_k^{(i)}) + \alpha(\theta_k^{(i+1)} - \phi_k^{(i+1)}))(\phi_k^{(i)} - \phi_k^{(i+1)})
\]

\[
- mbl(1 - \alpha) \sin((1 - \alpha)(\theta_k^{(i)} - \phi_k^{(i)}) + \alpha(\theta_k^{(i+1)} - \phi_k^{(i+1)}))(\theta_k^{(i+1)} - \theta_k^{(i)})
\]

\[
+ mglh^2(1 - \alpha) \sin((1 - \alpha)\theta_k^{(i)} + \alpha\theta_k^{(i+1)}) + mglh^2 \alpha \sin((1 - \alpha)\phi_k^{(i)} + \alpha\phi_k^{(i+1)})
\]

\[
- mbl \cos((1 - \alpha)(\theta_k^{(i+1)} - \phi_k^{(i+1)}) + \alpha(\theta_k^{(i)} - \phi_k^{(i)}))(\phi_k^{(i)} - \phi_k^{(i+1)}) + (mb^2 + I)(\phi_k^{(i)} - \phi_k^{(i+1)})
\]

\[
- mbl \alpha \sin((1 - \alpha)(\theta_k^{(i)} - \phi_k^{(i)})) + \alpha(\theta_k^{(i)} - \phi_k^{(i+1)}))(\theta_k^{(i)} - \theta_k^{(i+1)}) + hu_k^{(i)} = 0.
\]

(16)

We can find that (15) and (16) are represented as two difference equations which contains the variables at the three time steps \(k - 1\), \(k\), \(k + 1\). Moreover, (15) and (16) are also represented as implicit functions for the variables at the time step \(k + 1\): \(\theta_k^{(i+1)}, \phi_k^{(i+1)}\), and hence we have to use numerical solutions for nonlinear equations such as the Newton’s method in order to calculate \(\theta_k^{(i+1)}, \phi_k^{(i+1)}\).
Next, we consider the impact phase model of the DCBR. In this paper, we assume that the swing leg has a completely-elastic collision with the ground surface. Calculating the condition that discrete momenta before and after a collision are equivalent, that is,

\[ D_2 L_d^d(\theta_{N-1}^{(i)}, \phi_{N-1}^{(i)}, \phi_{N}^{(i)}) = D_1 L_d(\theta_{1}^{(i+1)}, \theta_{2}^{(i+1)}, \phi_{1}^{(i+1)}, \phi_{2}^{(i+1)}), \]

(17)

\[ D_4 L_d^d(\theta_{N-1}^{(i)}, \phi_{N-1}^{(i)}, \phi_{N}^{(i)}) = D_3 L_d(\theta_{1}^{(i+1)}, \theta_{2}^{(i+1)}, \phi_{1}^{(i+1)}, \phi_{2}^{(i+1)}), \]

we have

\[ (ma^2 + ml^2 + Ml^2 + I)(\dot{\theta}_{N}^{(i)} - \dot{\theta}_{N-1}^{(i)}) + (ma + ml + Ml)gh^2 \alpha \sin ((1 - \alpha)\theta_{N-1}^{(i)} + \alpha \theta_{N}^{(i)}) + mbl \sin ((1 - \alpha)(\theta_{N-1}^{(i)} - \phi_{N-1}^{(i)}) + \alpha(\theta_{N}^{(i)} - \phi_{N}^{(i)}))(\dot{\phi}_{N}^{(i)} - \dot{\phi}_{N-1}^{(i)}) - ml \cos ((1 - \alpha)(\theta_{N-1}^{(i)} - \phi_{N-1}^{(i)}) + \alpha(\theta_{N}^{(i)} - \phi_{N}^{(i)}))(\dot{\phi}_{N}^{(i)} - \dot{\phi}_{N-1}^{(i)}) \]

\[ = (ma^2 + ml^2 + Ml^2 + I)(\dot{\theta}_{1}^{(i+1)} - \dot{\theta}_{1}^{(i+1)}) + (ma + ml + Ml)gh^2 (1 - \alpha) \sin ((1 - \alpha)\theta_{1}^{(i+1)} + \alpha \theta_{2}^{(i+1)}) + mbl \sin ((1 - \alpha)(\theta_{1}^{(i+1)} - \phi_{1}^{(i+1)}) + \alpha(\theta_{2}^{(i+1)} - \phi_{2}^{(i+1)}))(\dot{\phi}_{2}^{(i+1)} - \dot{\phi}_{1}^{(i+1)}) + mbl \cos ((1 - \alpha)(\theta_{1}^{(i+1)} - \phi_{1}^{(i+1)}) + \alpha(\theta_{2}^{(i+1)} - \phi_{2}^{(i+1)}))(\dot{\phi}_{2}^{(i+1)} - \dot{\phi}_{1}^{(i+1)}) = 0, \]

(18)

\[ (mb^2 + I)(\theta_{1}^{(i)} - \theta_{N-1}^{(i)}) + mbl \sin ((1 - \alpha)(\theta_{N-1}^{(i)} - \phi_{N-1}^{(i)}) + \alpha(\theta_{N}^{(i)} - \phi_{N}^{(i)}))(\dot{\theta}_{N}^{(i)} - \dot{\theta}_{N-1}^{(i)}) + mbl \cos ((1 - \alpha)(\theta_{N-1}^{(i)} - \phi_{N-1}^{(i)}) + \alpha(\theta_{N}^{(i)} - \phi_{N}^{(i)}))(\dot{\phi}_{N}^{(i)} - \dot{\phi}_{N-1}^{(i)}) + mglh^2 \alpha \sin ((1 - \alpha)\theta_{N-1}^{(i)} + \alpha \theta_{N}^{(i)}) \]

\[ = (mb^2 + I)(\theta_{2}^{(i+1)} - \theta_{1}^{(i+1)}) + mbl \sin ((1 - \alpha)(\theta_{1}^{(i+1)} - \phi_{1}^{(i+1)}) + \alpha(\theta_{2}^{(i+1)} - \phi_{2}^{(i+1)}))(\dot{\theta}_{2}^{(i+1)} - \dot{\theta}_{1}^{(i+1)}) + mbl \cos ((1 - \alpha)(\theta_{1}^{(i+1)} - \phi_{1}^{(i+1)}) + \alpha(\theta_{2}^{(i+1)} - \phi_{2}^{(i+1)}))(\dot{\phi}_{2}^{(i+1)} - \dot{\phi}_{1}^{(i+1)}) \]

\[ = mglh^2 (1 - \alpha) \sin ((1 - \alpha)\theta_{1}^{(i+1)} + \alpha \theta_{2}^{(i+1)}) = 0. \]

(19)

Moreover, in the impact phase, the swing leg and the supporting leg replace each other, and this can be realized by the next equation:

\[ \theta_{1}^{(i+1)} = -\theta_{N}^{(i)}, \quad \phi_{1}^{(i+1)} = -\phi_{N}^{(i)}. \]

(20)

It turns out that (18) and (19) are represented as two difference equations which contains the variables at the four time steps \( N - 1, N, 1, 2 \). In addition, (15) and (16) are also represented as implicit functions for the variables at the time steps 1 and 2: \( \theta_{1}^{(i+1)}, \phi_{1}^{(i+1)}, \theta_{2}^{(i+1)}, \phi_{2}^{(i+1)} \), and then we also have to use the Newton’s method to calculate \( \theta_{k+1}^{(i)}, \phi_{k+1}^{(i)} \) numerically.

4. Discrete gait generation for DCBR

4.1 Discrete gait generation problem for DCBR

In this section, we consider a gait generation problem for the DCBR derived in the previous section and we propose the new concept “discrete gait.” We here deal with the following problem for the DCBR.

**Problem 1** : For the discrete compass-type biped robot (DCBR) (15)–(20), find a sequence of discrete-time control inputs: \( u_{1}^{(i)}, \cdots, u_{n}^{(i)}, \cdots, u_{1}^{(P)}, \cdots, u_{n}^{(P)} \) that generates a stable discrete gait.
\[
\begin{align*}
\min J &= \sum_{k=1}^{N-1} \{u_k\}^2, \\
\text{s.t.} & \ (15), (16) \\
& l \cos \theta_k^{(i)} - l \cos \phi_k^{(i)} > 0, \\
& \theta_1^{(i)} = -\theta_1^{(i)}, \ \phi_1^{(i)} = -\phi_N^{(i)}, \\
& \theta_2^{(i)}, \ \phi_2^{(i)}.
\end{align*}
\]  

(21)

In the formulation above, (23) represents a constraint on the vertical lengths of Leg 1 and 2, (24) is a boundary condition in order to generate a stable discrete gait, and (25) is an initial condition. In the impact phase between the \(i\)-th and \((i+1)\)-th swing phases, we can calculate initial states of the \((i+1)\)-th swing phases: \(\theta_1^{(i+1)}, \theta_2^{(i+1)}, \phi_1^{(i+1)}, \phi_2^{(i+1)}\) from (18)--(20).

The optimal control problem (21)--(25) can be considered as a finite dimensional constrained nonlinear optimization problem with respect to the \((3N - 1)\) variables: \(\theta_1^{(1)}, \ldots, \theta_N^{(1)}, \phi_1^{(1)}, \ldots, \phi_N^{(1)}, u_1^{(1)}, \ldots, u_{N-1}^{(1)}\). Therefore, we can solve this problem by using the sequential quadratic programming approach and so on [19, 25]. We summarize the control strategy for the DCBR as the following algorithm.

**Algorithm 1:**

**Step 0:** Determine the number of steps: \(N\) and the number of swing phases: \(L\). Set initial conditions: \(\theta_1^{(1)}, \theta_2^{(1)}, \phi_1^{(1)}, \phi_2^{(1)}\).

**Step 1:** Solve the optimal control problem (21)--(25) for the first swing phase and compute a sequence of a sequence of discrete-time control inputs: \(u_1^{(1)}, \ldots, u_{N-1}^{(1)}\). From the impact phase model of the DCBR (18)--(20) with \(\theta_1^{(N-1)}, \theta_N^{(1)}, \phi_1^{(N-1)}, \phi_N^{(1)}\), calculate the initial conditions of the second swing phase: \(\theta_1^{(2)}, \theta_2^{(2)}, \phi_1^{(2)}, \phi_2^{(2)}\). In a repetitive manner, obtain a sequence of discrete-time control inputs for all the swing phases: \(u_1^{(1)}, \ldots, u_{N-1}^{(1)}, \ldots, u_1^{(P)}, \ldots, u_{N-1}^{(P)}\).

**Step 2:** Apply \(u_1^{(1)}, \ldots, u_{N-1}^{(1)}\) to the swing phase model of the DCBR (15), (16) and calculate \(\theta_1^{(1)}, \ldots, \theta_N^{(1)}, \phi_1^{(1)}, \ldots, \phi_N^{(1)}\) by solving the closed-loop systems with the Newton’s method. From the impact phase model of the DCBR (18)--(20) with \(\theta_1^{(N-1)}, \theta_N^{(1)}, \phi_1^{(N-1)}, \phi_N^{(1)}\), calculate the initial conditions of the second swing phase: \(\theta_1^{(2)}, \theta_2^{(2)}, \phi_1^{(2)}, \phi_2^{(2)}\). In a repetitive manner, obtain a sequence of the angles of Leg 1 and 2 for all the swing phases: \(\theta_1^{(1)}, \ldots, \theta_N^{(1)}\ldots, \theta_1^{(P)}, \ldots, \theta_N^{(P)}, \phi_1^{(1)}, \ldots, \phi_N^{(1)}\ldots, \phi_1^{(P)}, \ldots, \phi_N^{(P)}\).

It must be noted that the period of a gait can be estimated by \((N - 1) \times h \times 2\), that is, (the number of intervals) \(\times\) (the sampling time) \(\times\) 2. Therefore, we can determine the period of a gait by setting \(N\) and \(h\). Moreover, the length of stride can be also designed by setting the initial conditions of the angles of Leg 1 and 2: \(\theta_1^{(1)}, \theta_2^{(1)}, \phi_1^{(1)}, \phi_2^{(1)}\).

### 4.2 Simulation results

This subsection performs a numerical simulation on discrete gait generation for the DCBR based on the method proposed in the previous subsection (Algorithm 1), and check the availability of our method.

First, we set parameters as follows; parameters on gait generation: \(N = 21\), \(P = 15\), parameters of the DCBR: \(m = 2.0\, \text{kg}, M = 10.0\, \text{kg}, I = 1.0\, \text{kgm}^2\), \(a = 0.5\, \text{m}, b = 0.5\, \text{m}, l = 1.0\, \text{m}, \alpha = 1/2, h = 0.05\, \text{s}\), initial states of the DCBR: \(\theta_1^{(1)} = -0.1\, \text{rad}, \phi_1^{(1)} = 0.1\, \text{rad}, \theta_2^{(1)} = -0.09\, \text{rad}, \phi_2^{(1)} = 0.09\, \text{rad}\). The estimated period of the gait is \((21 - 1) \times 0.05\, \text{s} \times 2 = 2\, \text{s}\). In order to solve the optimal control problem (21)--(24), we use the sequential quadratic programming method [25]. In addition, under the assumption that the sampling time \(h\) is sufficiently small, the changes of \(\theta\) and \(\phi\) can be estimated to
be small. Hence, when we use the Newton’s method to compute the angles of Leg 1 and 2 at the next step from the model of the DCBR (15)–(20) in Step 1 and 2 of Algorithm 1, we choose the previous ones as an initial condition of the Newton’s method.

Figures 4–7 show the simulation results. Figure 4 illustrates the time series of the obtained discrete input. In Fig. 5, the time series plots of Leg 1 and 2 ($\theta$ and $\phi$) are illustrated. Figure 6 shows the plot of solution trajectory in the phase space of $\theta - \phi$. In Fig. 7, a snapshot of the discrete gait is depicted, where the $z$-axis indicates the vertical direction. From these results, it can be confirmed that our approach can generate a stable gait for the DCBR. We also confirm stable gaits for a large number of $L$ (steps of walking) by simulations. Furthermore, the period of the gait is about 2 s, and it gives close agreement with the estimated one.

![Fig. 4. Discrete-time control input.](image)

![Fig. 5. Time series of $\theta$ and $\phi$ of DCBR (solid line: $\theta$, broken line: $\phi$).](image)
5. Continuous gait generation for CCBR

5.1 Transformation to continuous-time inputs

In the previous section, we have considered the discrete gait generation problem for the DCBR and formulated it by a finite dimensional nonlinear optimal control problem. As a result, we can obtain a sequence of discrete-time control inputs which realize a desired stable discrete gait. However, when we consider a continuous gait generation for the CCBR, the discrete-time control inputs cannot be utilized and a continuous-time control input is needed. Hence, in this subsection, we consider transformation from a discrete control input obtained via discrete mechanics into a continuous-time one.

First, we define a continuous gait problem for the CCBR as follows.

Problem 2: For the continuous compass-type biped robot (CCBR) (12), (13), find a continuous-time control input that generates a stable continuous gait.

There exist infinite methods to generate a continuous-time control input from a given discrete-time one, and a continuous-time control input generated from a given discrete-time one has to be consistent with laws of physics. Hence, in this paper, we deal with a zero-order hold control input which is one of simplest continuous-time inputs in the form:

\[ v^{(i)}(t) = u_k^{(i)}, \quad kh \leq t < (k+1)h. \]  

We have to derive a relationship between a discrete input \( u_k^{(i)} \) \( (k = 1, 2, \cdots, N - 1) \) and a zero-order
hold input (26). By using the discrete Lagrange-d’Alembert’s principle which is explained in Section 2, we can derive the following.

**Theorem 1**: A zero-order hold control input (26) which satisfies discrete Lagrange-d’Alembert’s principle is given by

\[ u_k^{(i)} = \left( \frac{-\alpha}{1-\alpha} \right)^{k-1} v_1^{(i)} + \frac{1}{(1-\alpha)h} \sum_{l=1}^{k} \left( \frac{-\alpha}{1-\alpha} \right)^{k-l+1} u_l^{(i)}, \quad k = 2, 3, \ldots, N - 1, \]  

(27)

where

\[ v_1^{(i)} = \frac{-1}{(1-\alpha)h} \{ D_1 L_{\alpha}^d(\theta_1^{(i)}, \theta_2^{(i)}, \phi_1^{(i)}, \phi_2^{(i)}) + D_2 L_c(\theta_1^{(i)}, \theta_2^{(i)}, \phi_1^{(i)}, \phi_2^{(i)}) \}. \]  

(28)

(Proof) Denote the right/left discrete external forces by \( f_k^{+} \) and \( f_k^{-} \). From the definition of the discrete inputs \( u_k^{(i)} \), we have the next relationship:

\[ u_k^{(i)} = f_k^{+} + f_k^{-}. \]  

(29)

On the other hand, from (8) the zero-order hold input (26) and \( f_k^{+}, f_k^{-} \) satisfy

\[ f_k^{-} = ahv_{k-1}^{(i)}, \quad f_k^{+} = (1-\alpha)hv_k^{(i)}. \]  

(30)

Therefore, substituting (30) into (29), we have

\[ u_k^{(i)} = ahv_{k-1}^{(i)} + (1-\alpha)hv_k^{(i)}. \]  

(31)

By solving the difference equation (31), we obtain (27). Moreover, since the left discrete external force \( f_k^{-} \) satisfies

\[ D_1 L_{\alpha}^d(\theta_1^{(i)}, \theta_2^{(i)}, \phi_1^{(i)}, \phi_2^{(i)}) + D_2 L_c(\theta_1^{(i)}, \theta_2^{(i)}, \phi_1^{(i)}, \phi_2^{(i)}) + f_1^{-} = 0, \]  

(32)

as a boundary condition [16–18], we have (28) from (30)–(32).

By (27) in Theorem 1, we can obtain a zero-order hold control input from \( u_k^{(i)} \), \( i = 1, \ldots, N - 1 \) which is obtained by the method via discrete mechanics shown in Subsection 4.1 and an initial input \( v_1^{(i)} \) derived from initial conditions.

5.2 Simulation results

In this subsection, we will perform a numerical simulation on continuous gait generation for the CCBR by using our new method proposed in the previous subsection, and confirm the effectiveness of our method.

First, we set parameters as follows; parameters on gait generation: \( N = 21, P = 15 \), parameters of the CCBR: \( m = 2.0 \) kg, \( M = 10.0 \) kg, \( I = 1.0 \) kgm\(^2\), \( a = 0.5 \) m, \( b = 0.5 \) m, \( l = 1.0 \) m, \( \alpha = 1/2 \), \( h = 0.05 \) s. In addition, initial states of the CCBR are set as \( \theta_1^{(i)} = -0.1 \) rad, \( \phi_1^{(i)} = 0.1 \) rad, \( \dot{\theta}_1 = 0.4382 \) rad/s, \( \dot{\phi}_1 = 0.3413 \) rad/s.

The simulation results are shown in Figs. 8–11. Figure 8 illustrates the time series of the continuous-time control input derived by Theorem 1. In Fig. 9, the time series plots of Leg 1 and 2 (\( \theta \) and \( \phi \)) are illustrated, and from this figure we can see that a periodic gait is realized by changing the roles of the supporting leg and the swing in alternate shifts. Figure 10 shows the plot of solution trajectory in the phase space of \( \theta - \phi \), and from this result it turns out that the S-shaped trajectory can be found and hence a periodic gait is generated. In Fig. 11, a snapshot of the continuous gait is depicted and from this snapshot we confirm a stable and natural gait. From these results, we can see that a stable gait for the CCBR can be generated by the proposed approach and the estimated period of the gait.
Compared to previous researches on control of the compass-type biped robots [4–11], our new method based on discrete mechanics that developed in this paper has some advantages as follows:

(i) Since (21)–(25) can be formulated a finite dimensional optimal control problem, we can easily solve it by optimization methods such as the sequential quadratic programming. If we formulate similar optimization problem for the continuous-time model, it is represented by a infinite dimensional optimal control problem and hence it is quite difficult to solve it. (ii) We can easily treat additional constraints such as input saturation and state constraints by adding them to (21)–(25). (iii) A zero-order hold control input obtained by our transformation method (Theorem 1) are quite compatible with computers and actuators. (iv) In some of previous studies on control of the compass-type biped robots, only stable gaits of a few steps are realized. However, our new method can generate a stable gaits for a large number of $P$ (steps of walking).
Fig. 10. Solution trajectory of CCBR on $\theta$-$\phi$ space.

Fig. 11. Snapshot of gait of CCBR.

gait that consists of an endless number of steps.

6. Conclusions
In this paper, we have considered a gait generation problem for the compass-type biped robot via discrete mechanics. We have developed a solving method for the problem from the viewpoint of the reduction to a finite dimensional optimization problem and the sequential quadratic programming method for the discrete compass-type biped robot (DCBR). We also have introduced a transformation method from a sequence of discrete-time control inputs to a zero-order hold control input based on discrete Lagrange-d’Alembert principle. Simulation results have confirmed stable gaits and indicated the effectiveness of our approach. Through this paper, we have developed a new method for gait generation for the compass-type biped robot that is totally different from the previous methods and has some advantages in comparison with them.

Our future work on control humanoid robots via discrete mechanics are as follows: (i) theoretical analysis of the model of the DCBR, (ii) gait generation of the CCBR in various environments such as slopes, stairs and irregular grounds, (iii) experimental evaluation of the proposed control method, (iv) applications of discrete mechanics to more human-like robots.

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References


