Invited Paper

Detecting generalized synchronization

Ulrich Parlitz

\textit{Max Planck Institute for Dynamics and Self-Organization, Am Faßberg 17, 37077 Göttingen, Germany, and Institute for Nonlinear Dynamics, Georg-August-Universität Göttingen, Am Faßberg 17, 37077 Göttingen, Germany}

\textit{ulrich.parlitz@ds.mpg.de}

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Abstract: Different cases of generalized synchronization are discussed with emphasis on methods for detecting them from time series. In particular, we shall focus on synchronization resulting in complex (non-smooth and/or non-invertible) functions or relations between drive and response.

Key Words: synchronization, time series analysis, conditional Lyapunov exponents, auxiliary system method, uni-directional coupling

1. Introduction

Interacting dynamical systems may exhibit different kinds of synchronization phenomena \cite{1-3}. In particular, uni-directionally coupled chaotic systems have been investigated by many authors because of their potential applications in technical devices (e.g., communication systems). If the coupled systems are (almost) identical their temporal evolution may converge to (almost) the same (chaotic) trajectory. This phenomenon is called \textit{identical synchronization} of chaotic systems (also called \textit{complete synchronization}). However, pairs of identical systems are an idealization and very often (structurally) different systems are coupled and may show interesting dynamics of interactions. In this case more sophisticated types of synchronization occur such as (chaotic) \textit{phase synchronization} or \textit{generalized synchronization}. In the following we shall discuss different notions or definitions of generalized synchronization and present methods for detecting them from time series.

2. Notions and definitions of generalized synchronization

Generalized synchronization of uni-directionally coupled systems has first been investigated by Afraimovich et al. \cite{4} and Rulkov et al. \cite{5} who coined the notion of \textit{generalized synchronization}. In the following we consider two \textit{different} uni-directionally coupled continuous systems given by vector fields $f$ and $g$, where $g$ includes the coupling

\begin{align*}
\dot{x} &= f(x) \\
\dot{y} &= g(y, x)
\end{align*}

or two (different) uni-directionally coupled maps
\[
\begin{align*}
x(t + 1) &= f(x(t)) \\
y(t + 1) &= g(y(t), x(t))
\end{align*}
\]

where \(x \in \mathbb{R}^n\) and \(y \in \mathbb{R}^m\) denote the states of the drive and the response system, respectively. Here we assume the two coupled systems to be different, but generalized synchronization may also occur with identical systems that show no identical synchronization [6, 7].

To cope with the case of nonidentical systems two generalizations of the notion of identical synchronization have been suggested:

(i) the existence of an asymptotic function or relation between states \(x\) and \(y\) [4, 5] and/or

(ii) asymptotic stability of the response system (i.e. reproducible response for a given drive signal) [8].

Both definitions are useful, but not equivalent as will be discussed in the following sections. Note that we shall consider here generalized synchronization in uni-directionally coupled systems, only. Possible generalizations for bi-directionally coupled systems shall be mentioned in the conclusions.

2.1 Functions and relations between states of coupled systems

2.1.1 Functions of states

The idea to characterize generalized synchronization by the existence of a function between the states \(x\) and \(y\) of drive and response, respectively, was first stated by Afraimovich et al. [4] and later adopted by Rulkov et al. [5]. Formally, it means that there exists a (continuous) function \(h\) mapping states of the drive into the state space of the response system

\[
h : \mathbb{R}^n \rightarrow \mathbb{R}^m \\
x \mapsto h(x)
\]

with:

\[
\lim_{t \to \infty} \|y(t) - h(x(t))\| = 0.
\]

In this case, the response state \(y\) is asymptotically given by the state \(x\) of the driving systems in terms of \(y = h(x)\) and there exists an attracting synchronization manifold \(M\) (given by \(h\)) in the full state space of the coupled systems (see Fig. 1). Necessary conditions for this type of generalized synchronization are invertible drive dynamics and asymptotic stability of the response system [9]. Methods for computing or approximating this manifold are discussed in [10].

Fig. 1. Attracting synchronization manifold \(M\) with basin \(B\) in the full state space \((x, y)\) of the coupled system.

To illustrate this kind of generalized synchronization we consider as an example a baker map

\[
\begin{align*}
f_1(x_1, x_2) &= \begin{cases} 
\alpha x_1 & \text{if } x_2 < a_1 \\
\alpha + \beta x_1 & \text{if } x_2 \geq a_1
\end{cases} \\
f_2(x_1, x_2) &= \begin{cases} 
x_2/a_1 & \text{if } x_2 < a_1 \\
(x_2 - a_1)/a_2 & \text{if } x_2 \geq a_1
\end{cases}
\end{align*}
\]
with $\alpha = \beta = 0.5$, $a_1 = 0.49$, $a_2 = 0.51$ driving a linear system [11]

$$
\begin{align*}
x(n+1) &= f[x(n)] \\
y(n+1) &= ay(n) + \cos[2\pi x_1(n)].
\end{align*}
$$

(2)

For $\alpha = \beta = b$ the response state $y$ can be explicitly written as a function of the driving variable $x_1$

$$
y = \frac{1}{a} \sum_{j=1}^{\infty} a^j \cos(2\pi b^{-j} x_1 + \Phi).
$$

The resulting function $y = h(x_1)$ is for $b < a$ continuous but nowhere differentiable (i.e., a Weierstrass function) [12]. The transition from a smooth function to a nowhere differentiable function is illustrated in Fig. 2 where the response state $y$ is plotted versus the drive variable $x_1$.

![Fig. 2. Baker map (1) drives linear system (2) with different contraction rates given by the parameter $a$.](image)

Generalized synchronization like this where (due to weak coupling) complicated or even fractal functions occur is also called \textit{weak synchronization} [13] and can arise due to different bifurcation scenarios [14, 15]. Furthermore, close to the synchronization threshold intermittency phenomena may occur [16].

2.1.2 Relations of states

The relationship between drive and response cannot only be very complex (non-smooth) but even multivalued [17], i.e. given by a surjective and injective relation. This is the case if the driving dynamics is noninvertible [18] or if subharmonic response (for example, due to a period doubling bifurcation) occurs.

As an example for multivaluedness due to noninvertible drive dynamics we consider the Bernoulli map (3) driving a linear system:

$$
\begin{align*}
x_{n+1} &= \begin{cases} 
2x_n & x_n < 0.5 \\
2(x_n - 0.5) & x_n \geq 0.5
\end{cases} \\
y_{n+1} &= ay_n + x_n.
\end{align*}
$$

Here typical states of the driving system possess a tree of different prehistories. These different driving histories result in different response states and a multivalued structure independent of details of the coupling. As a result, a multivalued relation occurs as illustrated in Fig. 3.

Multivalued relations are not only a result of noninvertible driving dynamics but occur already for periodic oscillations. Figure 4 shows a typical example consisting of a 1:2 frequency locking. A given state (red dot) of the drive corresponds to two states of the response. The ‘phase’ in which both response states occur simultaneously with the drive depends on the initial conditions of the response system (i.e., there are different basins of entrainment). In contrast to the previous example (Bernoulli map drives linear system) this kind of multivaluedness depends on the coupling strength and is often the result of a (period doubling) bifurcation. It also occurs with unstable periodic orbits embedded in chaotic attractors [19, 20] and induces a relation between drive and response where each state of the drive is mapped to a finite number of states of the response system.
Bernoulli map (3) drives a linear system (4) resulting in a multivalued relation between states of drive and response. Left: $y_n$ vs. $x_n$. Right: enlarged section visualizing self-similar structure.

**Fig. 4.** Multivalued relation due to 1:2 frequency locking.

2.2 Asymptotic stability of the response system

As outlined in the preceding section the existence of a functional relation depends on stability features of the response system. This aspect can be used to formulate another definition of generalized synchronization in terms of asymptotic stability (entrainment) of the response system. There are basically two approaches to test for synchronization in this sense: conditional Lyapunov exponents and the auxiliary systems approach.

2.2.1 Conditional Lyapunov exponents

Conditional Lyapunov exponents (also called transversal Lyapunov exponents) characterize the dynamics of small perturbations $z = y + e$ of the response state $y$ providing for continuous systems an error dynamics

$$\hat{e} = g(x, y) - g(x, z) = g(x, y) - g(x, y - e)$$

where linearization results in

$$\hat{e} = Dg_y(x, y) \cdot e.$$  \hspace{1cm} (5)

For discrete systems (iterated maps) the dynamics of the perturbation $e$ is governed by the update rule

$$e(t + 1) = Dg_y(x(t), y(t)) \cdot e(t).$$  \hspace{1cm} (6)

The conditional Lyapunov exponents (CLEs) are computed using the Jacobian matrices $Dg_y(x(t), y(t))$ along the orbit and characterize the asymptotic local stability of the response system. If all CLEs are negative the response system is stable and shows generalized synchronization according to this second definition (not implying the existence of a unique function).

As an example we consider a Hénon map

$$x_1(t + 1) = 1 - 1.4x_1^2(t) - 0.3x_2(t)$$
$$x_2(t + 1) = x_1(t)$$  \hspace{1cm} (7)

driving a two-dimensional iterated Gaussian map

$$y_1(t + 1) = \exp \left[ -a^2(y_1(t) - b)^2 \right] + dy_2(t) + cx_1(t)$$
$$y_2(t + 1) = y_1(t)$$  \hspace{1cm} (8)
where \( a = 3.5 \), \( b = 0.5 \), \( d = 0.2 \), and \( c \) denotes the coupling constant. Figure 5 shows the largest conditional Lyapunov exponent \( \lambda_{CLE} \) of the auxiliary system method (right) vs. coupling strength \( c \) (solid curve: identical response systems; dashed curve: slightly mismatched response systems).

To investigate the question whether a (unique) function or just a relation of states is established, Figure 6 shows the values of the \( y_1 \) variable of the Gaussian map (8) versus the state variables \( x_1 \) and \( x_2 \) of the driving Hénon map for \( c = 0.8 \). As can be seen the relationship between the response variable \( y_1 \) and the state \((x_1, x_2)\) of the driving system is rather complex. To visualize the underlying function or relation, in Fig. 7 the same data are plotted but now the values of the response variable \( y_1 \) are given by the color coding of the dots. The enlarged section of this “bird’s view” (right diagram in Fig. 7) indicates that in some locations on the Hénon attractor points \((x_1, x_2)\) with very different corresponding \( y_1 \)-values (i.e., very different colors) are very close together (and appear intermingled). Whether this is due to a strongly fluctuating function or some (multivalued) relation between drive and response is very difficult to decide using numerical simulations and finite data sets, only.

2.2.2 The auxiliary system method
Another direct test for asymptotic stability is the auxiliary system method introduced by Abarbanel et al. [8] using an identical copy of the response system that is driven by the same driving signal. For
Fig. 7. Hénon map (7) drives Gaussian map (8) with coupling constant $c = 0.8$. States $(x_1, x_2)$ of the driving Hénon map are colored by the values of the $y_1$ variable of the corresponding states of the response system (8). Left: Full attractor. Right: Enlarged section showing a part on the drive attractor where corresponding $y_1$ values strongly fluctuate as can be seen with red ($\approx 1.7$) and green ($\approx 0.8$) dots being very close to each other.

Continuous systems this set-up is given by

\[
\begin{align*}
\dot{x} &= f(x) \\
\dot{y} &= g(x, y) \\
\dot{z} &= g(x, z)
\end{align*}
\]

whereas with discrete systems (iterated maps) the auxiliary systems approach reads

\[
\begin{align*}
x(t+1) &= f(x(t)) \\
y(t+1) &= g(x(t), y(t)) \\
z(t+1) &= g(x(t), z(t)).
\end{align*}
\]

With this characterization generalized synchronization occurs if $\lim_{t \to \infty} \|y(t) - z(t)\| = 0$, i.e. if the response system and the auxiliary system show identical synchronization. The diagram in the right hand side of Fig. 5 shows the averaged synchronization error

\[
E_{\text{AUX}} = \frac{1}{N} \sum_{t=1}^{N} \|x(t) - y(t)\|^2
\]

for two cases: The blue curve represents the error of two identical systems and the dashed black curve shows the averaged error of two slightly detuned response systems where the auxiliary system has a parameter value of $d = 0.202$ instead of $d = 0.2$. Whereas with identical copies of response systems the onset of generalized synchronization is clearly visible and is consistent with the result of the conditional Lyapunov exponent, this is not the case if both response systems slightly differ.

To illustrate the auxiliary system approach and the other methods for detecting generalized synchronization with unidirectionally coupled continuous systems we consider now an example [13] where a Rössler system

\[
\begin{align*}
\dot{x}_1 &= \alpha (-x_2 - x_3) \\
\dot{x}_2 &= \alpha (x_1 + 0.2x_2) \\
\dot{x}_3 &= \alpha (0.2 + x_3(x_1 - 5.7))
\end{align*}
\]

drives a Lorenz system

\[
\begin{align*}
\dot{y}_1 &= 10(-y_1 + y_2) \\
\dot{y}_2 &= 28y_1 - y_2 - y_1y_3 + cx_2 \\
\dot{y}_3 &= y_1y_2 - 2.666y_3
\end{align*}
\]
with coupling term $\alpha x_2$. The parameter $\alpha = 6$ is used to adjust the time scale of both systems. The influence of the coupling and the onset of synchronization are illustrated in Fig. 8. Figure 8(a) shows the state space of the driving Rössler system where states in some particular location (neighborhood) are marked in red. In Fig. 8(b) the projection of the trajectories in the subspace corresponding to the driven Lorenz system is plotted for $c = 0$. The red dots in this figure correspond to the red dots in Fig. 8(a), i.e. they denote states of the Lorenz systems that occur simultaneously with the marked states of the Rössler system. Since there is no coupling between both systems these states are scattered on the whole Lorenz attractor. As soon as the coupling is switched on, the cloud of marked states concentrates as can be seen in Fig. 8(c). Although the coupling constant of $c = 5$ is still below the threshold of synchronization (here: $c_s \approx 6.66$) the distribution of red dots is more localized compared to the uncoupled case (Fig. 8(b)). If the coupling strength is increased further, generalized synchronization occurs and all neighboring points in the Rössler state space (Fig. 8(a)) are mapped to a small neighborhood in the Lorenz subspace as shown in Fig. 8(d) for $c = 100$. A more detailed analysis (not presented here) shows that this relationship possesses a complicated structure on small scales which makes it difficult to decide whether a (complicated) function or (just) a (multivalued) relation exists (another test is suggested in [21]).

![Fig. 8. Rössler system (10) drives Lorenz system (11). (a) Rössler attractor ($\alpha = 6$). (b) Lorenz attractor $c = 0$ (no coupling) no synchronization. (c) $c = 5$ (weak coupling) no synchronization. (d) $c = 100$ (strong coupling) synchronization.](image)

Figure 9 (left) shows the largest conditional Lyapunov exponent of the driven Lorenz system (11) vs. coupling strength $c$. For $c > 6.66$ negative CLEs occur indicating generalized synchronization. This result is confirmed by the auxiliary system test where the Rössler model drives two identical copies of the Lorenz system. Figure 9 (right) shows the averaged synchronization error of the response system and the auxiliary system that drops down to zero as soon as the coupling constant $c$ exceeds...
the synchronization threshold \( c_S \approx 6.66 \).

An important issue is the complexity and smoothness of the function \( h \) and the corresponding synchronization manifold \( M \). If \( M \) is normally hyperbolic (i.e. contractions of the flow normal to \( M \) are stronger (faster) than tangential contractions in \( M \)) then \( M \) is robust with respect to perturbations of the dynamical system and remains smooth [12]. Practically this means that strong coupling results in smooth synchronization manifolds (or functions) but weak coupling (close to the onset of synchronization) may lead to complicated and even non-smooth functional relations.

In general, the auxiliary system method is mainly used with simulations, because experimentally it is often difficult to have access to an identical pair of response systems (but no impossible, see for example [22, 23]).

3. Detecting generalized synchronization from time series

Until now we assumed that we know the dynamical equations to compute all quantities of interest like conditional Lyapunov exponents or synchronization errors. For many experimental investigations, however, the underlying dynamics is not or only partly known. In this case only (scalar) time series \( \{x_t\} \) and \( \{y_t\} \) are available and have to be analyzed for synchronization in some reconstructed state space.

3.1 Detecting functional relationships

The majority of time series based synchronization tests aims at detecting functional relationships between delay embedding spaces [5, 24–31]. Common to most of these methods is the goal to show that close neighbors in the reconstructed state space of the drive signal are mapped to nearest neighbors in the reconstructed state space of the response system. If this is the case then there is evidence that a continuous function exists that is considered as a manifestation of generalized synchronization. Of course there are many different ways to implement such a data based test for a functional relation. Often these techniques are also used as measures of interdependence (e.g. for EEG analysis [29]) combined with surrogate data tests [32]. To detect and describe nonlinear (functional) relations between drive and response one can also use canonical correlation analysis based on kernel functions [33].

In the following we shall use a simple algorithm [26] to illustrate the basic idea of this class of nearest neighbors tests and its limitations. Firstly, delay embeddings [34, 35] are introduced for both time series \( \{x_t\} \) and \( \{y_t\} \)

\[
x_t = (x_t, x_{t-1}, \ldots, x_{t-m+1}) \in \mathbb{R}^m
\]

\[
y_t = (y_t, y_{t-1}, \ldots, y_{t-n+1}) \in \mathbb{R}^n
\]

To check whether nearest neighbors in \( x \)-space are mapped to close neighbors in \( y \)-space we search for \( x_{nnx(t)} \) which is the nearest neighbor of \( x_t \) in \( x \)-space \( \mathbb{R}^m \) and compute the averaged distance of corresponding points in \( y \)-space.
\[ d_{xy} = \frac{1}{N \delta_y} \sum_{t=1}^{N} \| y_t - y_{nnx(t)} \|. \]

The normalization factor \( \delta_y \) is the mean distance between randomly chosen \( y \)-vectors. Analogously, we look in the opposite direction whether nearest neighbors in \( y \)-space are mapped to nearest neighbors in \( x \)-space

\[ d_{yx} = \frac{1}{N \delta_x} \sum_{t=1}^{N} \| x_t - x_{nny(t)} \|. \]

Figure 10 shows results of this kind of test for the existence of a functional relationship applied to time series from the Rössler system (10) driving the Lorenz model (11). The upper row shows \( d_{xy} \) vs. embedding dimensions \( m \) and \( n \) with large values for weak coupling (first and second column) and values close to zero in the case of generalized synchronization (third column). The lower row of Fig. 10 shows \( d_{yx} \) vs. \( m \) and \( n \) for the three cases investigated. Without coupling (left column, \( c = 0 \)) all values are close to one indicating that there exists no neighborhood relation. But in contrast to \( d_{xy} \), already weak coupling is sufficient to reduce \( d_{yx} \) (second column, \( c = 5 \)). This is due to the fact that the response time series is an observable of the full coupled system (in the sense of Taken’s reconstruction theorem) that enables in particular forecasting the drive signal (even without any synchronization).

This is an example where generalized synchronization is associated with a relation between the states of drive and response that can be approximated by a function and can be identified in the data. However, as discussed above, such a function may be very complicated or there may even be instead of a function a multivalued relation. In such a case all methods checking whether nearest neighbors are mapped to nearest neighbors fail [36, 37] and have to be revised [38].

To illustrate this problem and the difference between different notions of generalized synchronization we shall consider now an example where the iterated tent map drives an iterated Gaussian map with coupling strength \( c \).

\[ x_{t+1} = 1 - 2|x_t - 0.5| \]  \hspace{1cm} (12)

\[ y_{t+1} = \exp(-a^2 [y_t - b]^2) + cx_t. \]  \hspace{1cm} (13)
For $a = 3.5$ and $b = 0.5$ the free running ($c = 0$) Gaussian map exhibits chaotic dynamics. Figures 11(a) and 11(b) show the conditional Lyapunov exponent $\lambda_{CLE}$ and the averaged synchronization error of the auxiliary system method $E_{AUX}$ vs. $c$ indicating a clear transition to synchronization at $c > 0.4$ (and a small window of synchronization for $0.25 < c < 0.28$).

Figure 12 shows the relation between states $x_t$ and $y_t$ of both systems in the state space of the full coupled system for synchronization ($c = 0.6$, right diagram) and below the threshold for synchronization ($c = 0.35$, left diagram). The main difference is the additional structure visible in the case of synchronization (right figure) that is due to the multivalued relation between $x_t$ and $y_t$.

Since there is no functional relation between $x_t$ and $y_t$, nearest neighbors statistics fail to clearly indicate synchronization in this case. This is illustrated in Fig. 13 where $d_{xy}$ and $d_{yx}$ are plotted vs. the embedding dimensions $m$ and $n$. The statistics $d_{yx}$ attains relatively small values as soon as the reconstructed states contain at least two samples of the response time series ($n \geq 2$), because this is sufficient to reconstruct the dynamics of the full (drive & response) system. The values of $d_{xy}$ decrease slightly when going from $c = 0.35$ (no synchronization) to $c = 0.6$ (synchronization) but there difference is not clear, because neighboring points in $x$-space are actually not mapped to neighboring points in $y$-space (see Fig. 12).

Note that for this example there exists a smooth and simple functional relation $(x_t, y_t) \rightarrow y_{t+1}$ that may be exploited for cross prediction or detection of interrelations between time series [39]. This relation, however, always exists independently of the issue of synchronization.

Instead of testing for functions or relations we shall now try to apply the second definition of generalized synchronization in terms of asymptotic stability.
3.2 Conditional Lyapunov exponents from time series

Pyragas introduced a method for estimating conditional Lyapunov exponents from time series [40] using delay embeddings

\[ x_t = (x_t, x_{t-1}, \ldots, x_{t-m+1}) \in \mathbb{R}^m \]
\[ y_t = (y_t, y_{t-1}, \ldots, y_{t-n+1}) \in \mathbb{R}^n \]

and local linear approximations of the flow \( \phi = (\phi_x, \phi_y) \) in the full (reconstructed) state space

\[ A_r(x_i - x_r) + B_r(y_i - y_r) \approx \phi_y(y_i) - \phi_y(y_r) = y_{i+1} - y_{r+1} \]  \quad (14)

at reference points \((x_r, y_r)\) using \(K\) nearest neighbours \((x_i, y_i)\). The matrices \(B_r\) at different reference points along the trajectory provide (approximations of) the Jacobian matrices \(D\phi_y(x_r, y_r)\) required to compute the conditional Lyapunov exponents \(\lambda_{xy}\) by means of the well know QR-algorithm. If the role of drive and response is not known a priori one may also exchange both time series \(\{x_t\}\) and \(\{y_t\}\) and compute also \(\lambda_{yx}\) and compare it with \(\lambda_{xy}\). Figure 14 shows as an example results using time series from the tent map (12) driving the Gaussian map (13). Note that the direction of coupling can be inferred from the difference between \(\lambda_{xy}\) and \(\lambda_{yx}\) even in the synchronized state (occurring here for \(c > 0.4\)). If higher dimensional embeddings are used spurious (conditional) Lyapunov exponents occur that have to be distinguished from the true CLEs (for example, using additional regularization when estimating the Jacobian matrices).

3.2.1 Auxiliary System Test for Time Series

As an alternative we suggest to extend the concept of comparison with an auxiliary system to the case where only data are given. In this case we use as copy of the response systems a (black-box) model that is generated from the data. To generate a copy of the response system any method for modeling input-output dynamics can be used, including neural networks, NARMA-models, radial basis functions etc. In discrete time the response system may formally be described by a driven dynamical system of the form
Fig. 14. Conditional Lyapunov exponents $\lambda_{xy}$ (red curve) and $\lambda_{yx}$ (green curve) estimated from time series generated by the tent map (12) drives Gaussian map (13). The CLE $\lambda_{xy}$ decreases and indicates generalized synchronization for coupling constants $c > 0.4$ while $\lambda_{yx}$ remains constant because the $x$-dynamics (tent map) is not influenced by the $y$-dynamics. For comparison the blue curve shows the true conditional Lyapunov exponent $\lambda_{CLE}$ (computed with full model equations (12) and (13)).

$$y_t = g(x_{t-1}, \ldots, x_{t-m}; y_{t-1}, \ldots, y_{t-n}).$$

Once we have approximated the function $g$ using the given data we may use this model as auxiliary system. We drive it by the same $x$ time series

$$z_t = g(x_{t-1}, \ldots, x_{t-m}; z_{t-1}, \ldots, z_{t-n})$$

and compare for some test data (from the same source but not used for learning the model) the output $z_t$ with the measured response $y_t$ in terms of the average synchronization error between the measured response time series $\{y_t\}$ and the output $\{z_t\}$ of the model auxiliary system.

Fig. 15. Tent map (12) drives Gaussian map (13). Synchronization errors of auxiliary system test $E_{xy}$ (top row) and $E_{yx}$ (bottom row) for $c = 0.35$ (left column, no synchronization) and $c = 0.6$ (right column, synchronization).
Here $\delta_y$ denotes again a normalization factor computed with random forecasts.

Figure 15 shows results obtained for time series from the tent map (12) and the Gaussian map (13). 16000 samples of the time series were used for modeling the auxiliary system using a simple local nearest neighbor approach and 4000 samples for computing the auxiliary system synchronization error. The top row of Fig. 15 shows $E_{xy}$ for $c = 0.35$ (left, no synchronization) and $c = 0.6$ (right, synchronization). In contrast to the nearest neighbors statistics shown in Fig. 13 there is a clear difference between both cases and synchronization is reliably detected in terms of small values of $E_{xy}$. The bottom row of Fig. 15 shows $E_{yx}$ vs. $m$ and $n$ where the roles of both time series are exchanged. There are no differences between no synchronization (left) and synchronization (right). The low values of $E_{yx}$ for $n = 0$ reflect the fact that the driving time series $\{x_t\}$ can be very well predicted when using only previous $x$-samples as input. Furthermore, $E_{xy}$ and $E_{yx}$ are different even in the case of synchronization ($c = 0.6$). This feature that might be exploited to detect directions of coupling of synchronized systems.

4. Conclusion

Different notions of generalized synchronization were presented, illustrated and compared. While generalized synchronization in terms of asymptotic stability of the response system can be reliably detected using conditional Lyapunov exponents or the auxiliary system method, any verification of the existence of a functional relation is limited by the available resolution. In particular, if the response system is only weakly stable, very complicated or even fractal functions may occur, whose graphs cannot be resolved with a finite number of data points (that may in addition be spoiled by noise). Furthermore, in many cases although the response system is (asymptotically) stable, the states of drive and response, however, are not related by a function but a relation (mapping a state of the drive to different states of the response system). These cases, fractal functions and/or multivalued relations, also limit effective representations or approximations of the synchronization manifold in the full state space of drive and response.

A topic that has not been addressed in this article are suitable notions of generalized synchronization in bi-directionally coupled systems. Some authors suggested straightforward generalizations (e.g., the existence of an invertible function $y = h(x)$, connecting the issue of synchronization to the topic of inertial manifolds), but in particular characterizations of generalized synchronization in terms of asymptotic stability still have to be explored for bi-directionally coupled systems.

Current research on generalized synchronization is in particular focussed on generalized synchronization between and in dynamical networks [41–44] where both types of coupling, uni-directional and bi-directional may occur resulting in new complex synchronization phenomena.

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