Paper

Memristor based-elements for chaotic circuits

Fernando Corinto^{1a)} and Alon Ascoli¹

¹ *Department of Electronics, Politecnico di Torino
Corso Duca degli Abruzzi, 24, Torino, Italy*

^{a)} *fernando.corinto@polito.it*

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Abstract: The peculiar features of the memristor, a fundamental passive two-terminal element characterized by nonlinear relationship between charge and flux, promise to revolutionize integrated circuit design in the next few decades. Besides its most popular potential applications, ultra-dense non-volatile memories and brain-simulating systems, much research has been lately devoted to their use in chaotic circuits. Although the physical memristor is inherently-asymmetric, complementary behaviors arise in devices with opposite polarity. In this work we demonstrate that this makes it feasible to devise a number of practical realizations of a monotone-increasing odd-symmetric charge-flux nonlinearity suitable for chaos-based applications through the sole use of physical memristors of the kind recently fabricated at Hewlett-Packard Labs. Confirmation for such claim is obtained through comparison of chaotic behavior of two modified Chua's oscillators, in which the nonlinear diode is replaced in one case with an artificial memristor with symmetric nonlinearity and in the other case with one of the proposed symmetric memristor combinations.

Key Words: memristor, Chua's nonlinear circuit, chaotic circuits

1. Introduction

The voltage- or current-controlled memristor is a fundamental passive two-terminal element with resistance depending on the time history of voltage across or current through it. It was theoretically introduced in 1971 [1] and then classified within the larger class of memristive systems five years later [2]. However, it was only after the 2008 conscious discovery of memristor behavior at the nanoscale [3] that it started to receive significant attention from the research community for the large spectrum of opportunities it opens up in integrated circuit (IC) design, mostly for the realization of large-capacity non-volatile memories [4–7] and neuromorphic systems [8–12].

The practical nano-device fabricated under the supervision of R.S. Williams at Hewlett-Packard (HP) Labs three years ago [13] is a thin film of titanium dioxide (TiO_2) sandwiched between two platinum (Pt) contacts. The film is made up of a left-hand conductive layer of oxygen-deficient titanium dioxide (TiO_{2-x} , $x = 5\%$) and of a right-hand insulating layer of stoichiometric titanium dioxide (TiO_2 , see Fig. 3(b) as reference). The oxygen vacancies act as mobile positively-charged dopants and may be repelled (attracted) by the electrical field generated by applying a positive

(negative) voltage between left-hand and right-hand contact. This causes the layer boundary to shift rightwards (leftwards), with a consequent increase (decrease) in device conductivity, which, accordingly, facilitates (obstructs) electronic transport. Therefore the device behavior depends on the coupled migration of positively-charged ions and negatively-charged electrons under an external controlling source.

Power switch-off determines a halt in ionic (and electronic) motion. The resulting freeze of layer boundary position causes the device conductivity to keep itself unaltered until source is turned on once again. This is at the origin of the device memory capability.

Swapping the two layers, i.e. considering a memristor with opposite polarity (see Fig. 3(a) as reference), complementary dynamics may be observed. In fact, in this case a positive (negative) voltage between left-hand and right-hand contact determines a strengthening (weakening) in device capability to obstruct electronic motion.

Besides its main fields of application, the memristor is extensively studied for its potential use in chaos-based electronic systems. A number of theoretical studies [14,15] have reported the occurrence of chaos in circuits employing an artificially-conceived memristive element characterized by a monotone-increasing odd-symmetric charge-flux relationship.

In Section 2 of this manuscript one of these chaotic circuits, whose architecture is based on Chua's oscillator, is reviewed in some detail. The natural question arising from these works is whether or not such a symmetric charge-flux characteristic may be practically realized through the sole use of real memristors of the kind fabricated at HP Labs. It is the aim of this work to address this issue. In Section 3 of this manuscript we gain a deep insight into the most important models describing the dynamics of the HP memristor.

In Section 4 we first propose a new simple, analytically-integrable and accurate model for this device. The proposed Boundary Condition-based model (BCM) is based upon the linear dopant drift model [3] augmented with suitable boundary conditions [16]. Section 4 is then devoted to the analytical integration to the proposed BCM with the aim to highlight the inherent asymmetry of the HP physical device and the complementarity of asymmetric characteristics of memristors with opposite polarity. Section 5 exploits such complementary behaviors to devise a number of practical realizations of a symmetric charge-flux characteristic suitable for chaos-based applications through the only use of appropriate combinations of HP memristors. Then, in Section 6 we validate our theoretical findings by showing agreement between the chaotic dynamics of the circuit reviewed in Section 2 and of its counterpart obtained by replacing the artificially-conceived memristor with one of the memristor combinations proposed in Section 5. Finally, Section 7 outlines conclusions and future research developments.

2. Chaos in memristor circuits

The authors from [14] investigated the chaotic dynamics of various memristor-based circuits, among which one, depicted in Fig. 1, was derived from Chua's oscillator by replacing Chua's diode with an active device (within a dotted rectangular box in Fig. 1) made up of the parallel between a negative resistance and a memristor with the following piece-wise linear (PWL) monotone-increasing odd-symmetric charge q -flux φ characteristic:

$$q(\varphi) = b\varphi + \frac{a-b}{2} (|\varphi + 1| - |\varphi - 1|), \quad (1)$$

where coefficients a and b are positive since the device is passive.

This passivity implies the necessity to add a suitably-negative resistance of magnitude R_2 in parallel with the memristor so as to obtain a locally-active device, i.e. a device characterized by a q - φ nonlinearity with a negative slope either on the central segment or on the two outer segments. This is an essential requirement for the occurrence of chaotic behavior in an autonomous dynamical system [17]. For clarity's sake, note that single-valuedness and odd-symmetry in the charge-flux characteristic of the locally-active device are not mandatory properties.

The circuit equations of the oscillator of Fig. 1 are:

Fig. 1. Chua's oscillator with Chua's diode replaced with the parallel between a memristor and a negative resistance.

$$\begin{aligned}
 C_1 \frac{dv}{dt} &= \frac{v_2 - v}{R_1} + \frac{v}{R_2} - i, \\
 C_2 \frac{dv_2}{dt} &= -\tilde{i} - \frac{v_2 - v}{R_1}, \\
 L \frac{d\tilde{i}}{dt} &= v_2 - r\tilde{i}, \\
 \frac{d\varphi}{dt} &= v,
 \end{aligned} \tag{2}$$

where the significance of each symbol is shown in Fig. 1. The memductance of the memristor is defined as

$$W(\varphi) = \frac{dq(\varphi)}{d\varphi}.$$

Using the chain rule, memristor current i may be expressed as

$$i = \frac{dq}{dt} = \frac{dq(\varphi)}{d\varphi} \frac{d\varphi}{dt} = W(\varphi)v.$$

Charge differentiation of Eq. (1) yields the following expression for $W(\varphi)$:

$$W(\varphi) = b + \frac{a-b}{2} (\text{sgn}(\varphi+1) - \text{sgn}(\varphi-1)). \tag{3}$$

Defining state variables as $x_1 = v$, $x_2 = v_2$, $x_3 = -R_1\tilde{i}$ and $x_4 = \varphi(R_1C_2)^{-1}$ (each state variable has the dimensions of a voltage), letting $\alpha = C_2(C_1)^{-1}$, $\beta = R_1^2C_2(L)^{-1}$, $\gamma = R_1rC_2(L)^{-1}$ and $\xi = R_1(R_2)^{-1}$, and considering normalized time variable $\tau = t(\bar{t})^{-1}$, where $\bar{t} = R_1C_2$ is the system time scale, Eq. (2) may be recast through the following state equations:

$$\begin{aligned}
 \frac{dx_1}{d\tau} &= \alpha[x_2 + (\xi - 1)x_1 - \tilde{W}(x_4)x_1], \\
 \frac{dx_2}{d\tau} &= x_3 - x_2 + x_1, \\
 \frac{dx_3}{d\tau} &= -\beta x_2 - \gamma x_3, \\
 \frac{dx_4}{d\tau} &= x_1,
 \end{aligned} \tag{4}$$

where

$$\tilde{W}(x_4) = R_1W(x_4) = \tilde{b} + \frac{\tilde{a} - \tilde{b}}{2} (\text{sgn}(\varphi+1) - \text{sgn}(\varphi-1)), \tag{5}$$

with

Fig. 2. Projection of chaotic attractor of dynamical system (4) on the $x_1 - x_4 - x_2$ state space for $\alpha = 10$, $\beta = 13$, $\gamma = 0.35$, $\xi = 1.5$, $\tilde{a} = 0.3$ and $\tilde{b} = 0.8$.

$$\tilde{a} = R_1 a, \quad (6)$$

$$\tilde{b} = R_1 b. \quad (7)$$

Defining state vector as $\mathbf{x} = [x_1, x_2, x_3, x_4] \in \mathbb{R}^4$, initial conditions are chosen as $[0.1, -0.1, 0.1, 0]$. Setting circuit element values to $R_1 = 1.5297 \text{ k}\Omega$, $R_2 = 1 \text{ k}\Omega$, $r = 41.185 \Omega$, $C_1 = 10 \text{ nF}$, $C_2 = 100 \text{ nF}$ and $L = 18 \text{ mH}$, system parameters are numerically given by $\alpha = 10$, $\beta = 13$, $\gamma = 0.35$, $\xi = 1.5$, $\tilde{a} = 0.3$ and $\tilde{b} = 0.8$. Under this parameter setting, dynamical system (4) experiences chaotic behavior, as it is shown in Fig. 2, where the chaotic attractor is projected on state space $x_1 - x_4 - x_2$.

A number of other works have recently investigated memristor-based chaotic circuits:

- In [15] two identical oscillators, each using a memristor characterized by the charge-flux relationship given in Eq. (1), were found to experience unsynchronized chaotic oscillations when the active device of Fig. 1 was used as coupling element.
- A smooth charge-flux relationship of the active device of Fig. 1 characterizes the active element implemented through analog operational amplifiers in [18], where the oscillator of Fig. 1 was fabricated on a breadboard.

A question arises from these preliminaries: is it possible to devise a practical realization of a monotone-increasing odd-symmetric charge-flux characteristic based upon the sole use of memristor nano-structures, i.e. without using discrete analog components? In this work we shall prove that, in spite of the inherent asymmetry of the HP nano-scale device, making use of the complementarity of dynamical behaviors of physical elements with opposite polarity, under particular conditions some appropriate memristor combinations exhibit such symmetry property.

In the next Section 3 we briefly summarize memristor models of the HP nano-scale device.

3. Insight into existing memristor models

The first mathematical model of the HP memristor [3] consists of a dynamic equation governing the rate of motion of ions under application of a controlling source, and of an Ohm's Law-based input-output equation.

Figure 3 shows physical structure (plots (a) and (b)), equivalent circuit (plots (c) and (d)) and circuit symbol (plots (e) and (f)) of a practical memristor for the two possible device polarities. Polarity coefficient is denoted as $\eta = \pm 1$.

Let D be the entire length of the thin oxide film, w the length of the left-hand layer ($D - w$ is then the length of the right-hand layer), and v and i the voltage across and the current through the memristor respectively. The reference direction for the y-axis (along which the thin film lies and the layer boundary moves under application of an external input), v and i are also given in Fig. 3.

The linear model originally proposed in [3] is given by¹

$$\frac{dx(t)}{dt} = \eta \frac{i(t)}{Q_0}, \quad (8)$$

$$v(t) = (R \circ x)(t) i(t), \quad (9)$$

where the state variable is defined as $x = l(D)^{-1}$, with l , representing the length of the conductive layer, equals to w for the memristor with $\eta = +1$ (Fig. 3(b)) and to $D - w$ for the memristor with $\eta = -1$ (Fig. 3(a)), $(R \circ x)(t)$ denotes² the memristance or memory-resistance, expressed by (see Figs. 3(c)–3(d))

$$(R \circ x)(t) = R_{on}x(t) + R_{off}(1 - x(t)) = R_{off} - \Delta R x(t), \quad (10)$$

with $\Delta R = R_{off} - R_{on}$, where R_{on} and R_{off} indicate the memristance of the fully-conductive and of the fully-insulating memristor respectively, while

$$Q_0 = \frac{D^2}{\mu R_{on}}$$

is the amount of charge required to flow through the device with polarity coefficient η for x to increase from 0 to 1 under an external input, with μ standing for the average dopant mobility. The model, assuming a constant dopant drift rate throughout the entire device length, qualitatively reproduces the dynamics of the memristor with a good degree of accuracy. However, it does not specify boundary conditions. Therefore it is valid only as long as the controlling source is such that the layer boundary does never reach any end ([19, 20]).

Nonlinear dopant drift models ([3, 16, 21]), later introduced to ameliorate the linear model, fall into the following class:

$$\frac{dx(t)}{dt} = \eta \frac{i(t)}{Q_0} F(x(t), \eta i(t), p), \quad (11)$$

$$v(t) = (R \circ x)(t) i(t), \quad (12)$$

where $F(x, \eta i, p)$ is a non-negative parameterized (p is a positive integer) window function imposing boundary conditions and possibly accounting for nonlinear effects on ionic transport due to the large electric field developing across the conductive nano-layer under application of an external input.

Joglekar [21] proposed a window function F independent on ηi (let us call it F_J):

$$F_J(x, p) = 1 - (2x - 1)^{2p}, \quad (13)$$

where p controls the window decrease rate (from the maximum unitary value at $x = 1/2$), occurring as x approaches either 0 or 1 (see Fig. 4(a)). For $p = 1$ Eq. (13) is identical (for less than a proportionality

¹For simplicity we shall hereafter redefine the dimensionless time $\tau = t\bar{t}^{-1}$ as t , i.e. $\bar{t} = 1\mu s$.

²Throughout the manuscript we adopt the following notation. For a function $f(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$, its argument denotes time t , unless the argument itself is explicitly shown. In addition, given $f_1(\cdot)$ and $f_2(\cdot)$, $(f_1 \circ f_2)(\cdot)$ stands for the function composition. Finally, time dependence of such composite function is not explicitly shown if is not needed, i.e. $f_1(f_2)$ is also used to designate it.

Fig. 3. Practical memristors with opposite polarity (polarity coefficient $\eta = \pm 1$). Plots (a)–(b): physical structure with green- and white-filled areas respectively denoting conductive TiO_{2-x} ($x = 5\%$) and insulating TiO_2 layers; Plots (c)–(d): equivalent circuit; Plots (e)–(f): circuit symbol.

factor) to another window previously proposed by Williams himself in [3]. Furthermore, Eq. (13) is responsible for the existence of two equilibria, specifically $x = 0$ and $x = 1$, in Eq. (11). Whenever the layer boundary hits one of the device ends, it stays there for all subsequent times. As p gets bigger and bigger, the two x -values at which the window decreases to 90% of its unitary value get farther and farther away from $x = \frac{1}{2}$ and the window decrease rate becomes larger and larger until for very large p the function tends to 1 $\forall x \in (0, 1)$, while it exhibits a sudden transition to 0 at $x = 0$ and at $x = 1$.

Due to its independence on ηi , Joglekar's window does not specify realistic boundary conditions. In fact, in a real case scenario, reversing the controlling source sign when layer boundary is at one end causes it to leave that position and move towards the other end after some time delay. Further, depending on the particular memristor realization under consideration, such time delay may be distinct for each device end. In [16] Biolek introduced a window function F (let us call it F_B) capturing this behavior for no time delay at each device end. Its expression is:

$$F_B(x, \eta i, p) = 1 - (x - 1)^{2p} \text{ if } \eta i \leq 0, \quad (14)$$

$$F_B(x, \eta i, p) = 1 - x^{2p} \text{ if } \eta i > 0. \quad (15)$$

Let us consider a simple numerical example in order to clarify the time evolution of x and F_B with variation in ηi . Here we choose $\eta = 1$ and $p = 2$. Figure 4(b) shows (14)–(15) for this case. Setting $x(0) = 0$ and assuming an initially positive current, i.e. $i(0) > 0$, F_B is expressed by (15) and starts from 1. The positive current causes x to increase towards 1 according to (11). Correspondingly window function decreases with p -dependent rate towards 0 along curve number 1_+ . Assuming current sign is kept positive, at some point x and F_B become respectively equal to 1 and 0. Thereafter they shall maintain their respective values as long as current sign is kept unaltered. Note that for a negative or zero current it is (14) which describes F_B . Therefore occurrence of a reversal in current direction causes the window to experience a sudden $0 \rightarrow 1$ transition (solid line number 2_0). Obviously, current sign could be altered before that x gets equal to 1. If this occurs when x equals 0.9, F_B jumps upwards along dashed line 2_0^* .

Due to current negative sign in Eq. (11), state variable shall then decrease towards 0. Correspond-

(a) : F_J , $p = \{1, 3, 10\}$ (b) : F_B , $p = 2$, $\eta = 1$ (c) : $\tilde{F}$, $\eta = 1$

Fig. 4. Window functions. Plot (a): Joglekar's window for $p = 1, 3, 10$. Plot (b): Biolek's window for $p = 2$ and $\eta = 1$ (subscripts $+$, $-$ and 0 on the curve numbers denote positive sign, negative sign and zero value for the control source). Plot (c): Proposed window function for $\eta = +1$ (subscripts $+$, $-$, $+$, 0 and $-, 1$ on the curve numbers denote positive sign, negative sign, positive value larger than $i_{th,0}$, negative value smaller than $-i_{th,1}$ for the control source). For $\eta = -1$ in plots (b)–(c) arrows hold the same direction but curve number subscripts $+$, $-$, $+$, 0 and $-, 1$ need to be replaced with $-, +$, $-, 0$ and $+, 1$ respectively.

ingly the window, expressed by (14), decays towards 0 with p -dependent rate along curve number 3_- . As long as current sign is unaltered, at some time instant both x and F_B shall equal 0. Provided that current sign is kept negative, no variation in the values of x and F_B occurs thereafter. Only a new change in current sign produces a sudden $0 \rightarrow 1$ transition in the window (see solid line number 4_+), since for a positive current F_B is defined by (15) once again. Current sign could be altered before that x gets equal to 0. If this occurs when x equals 0.1, F_B jumps upwards on dashed line 4_+^* .

Unlike Joglekar's window, whose unitary maximum is at $x = 1/2$, Biolek's has unitary maxima at $x = 0$ and at $x = 1$. Keeping $\eta = 1$, as p gets bigger and bigger, the x -value at which the window decreases to 90% of its unitary value with $i > 0$ ($i < 0$) gets farther and farther away from end $x = 0$ ($x = 1$) and the window decrease rate becomes larger and larger, until for very large p the window assumes unitary value $\forall x \in (0, 1)$, while it exhibits sudden 1-to-0 and 0-to-1 transitions at end $x = 1$ for $i > 0$ and $i < 0$ (at end $x = 0$ for $i < 0$ and $i > 0$) respectively. This is also clear from Fig. 4(c).

One of the main drawback of memristor models (11)–(12) with windows (13) or (14)–(15), is that their analytical integration is far from trivial. In addition, a closed-form expression of the charge q -flux φ characteristic cannot be obtained. It is worth to point out that q - φ , or equivalently memductance-flux characteristics, permits one to define all memristor properties (see [22]).

In the next section we introduce the simplest possible memristor model which is able to characterize memductance-flux characteristics. The simplicity of the proposed model, guaranteed by the linear dopant drift assumption, allow it to possess closed-form solutions.

4. Boundary condition-based memristor model

In this section we present a novel Boundary Condition-based model (BCM) for memristive nano-structures. We assume that ions drift rate under an external input is constant throughout the film extension. This provides a simple set of differential-algebraic equations which allows for closed-form solutions. Further, dynamics at boundaries may be suitably tuned so as to capture a wide range of nonlinear behaviors. This yields a general model that is able to reproduce asymmetric charge q -flux φ characteristic of HP memristive nano-films.

The proposed boundary condition-based model for memristor nano-structures consists of the set of differential-algebraic Eqs. (11)–(12) employing the following window (let us call it $\tilde{F}$), displayed in Fig. 4(c) for $\eta = 1$ (a similar window with discontinuity at boundaries was recently proposed in [23] to model the dynamical behavior of memristive filaments):

$$\tilde{F}(x(\tau), \eta i(\tau)) = \begin{cases} 1 & \forall \tau : C_1 \text{ holds,} \\ 0 & \forall \tau : C_2 \text{ or } C_3 \text{ holds,} \end{cases} \quad (16)$$

where tunable conditions C_m $m = \{1, 2, 3\}$ are defined as

$$\begin{aligned} C_1 &= \{x(\tau) \in (0, 1) \mid (x(\tau) = 0 \ \& \ \eta i(\tau) > i_{th,0}) \\ &\quad \mid (x(\tau) = 1 \ \& \ \eta i(\tau) < -i_{th,1})\}, \\ C_2 &= \{x(\tau) = 0 \ \& \ \eta i(\tau) \leq i_{th,0}\}, \\ C_3 &= \{x(\tau) = 1 \ \& \ \eta i(\tau) \geq -i_{th,1}\}, \end{aligned}$$

where, referring to the case $\eta = +1$ for simplicity, $i_{th,0}$ ($i_{th,1}$) denotes the threshold current the magnitude of the external input needs to pass over after it gets positive (negative) while C_2 (C_3) holds, before Eq. C_1 may be met.

As in Biolek's case, the proposed window depends on the control source. Under sign-varying input, for $x \in (0, 1)$ it may evolve on a single curve only (similarly to Joglekar's case), while for x at boundary $x = 1$ ($x = 0$), it experiences $1 \rightarrow 0$ and $0 \rightarrow 1$ vertical transitions for $\eta i \geq -i_{th,1}$ and $\eta i < -i_{th,1}$ (for $\eta i \leq i_{th,0}$ and $\eta i < -i_{th,1}$). Taking Fig. 4(c) as reference, let us clarify these dynamics through a numerical example, where η is set to $+1$. Setting $x(0) = 0$ and $i(0) > i_{th,0}$, condition C_1 is fulfilled, thus Eq. (16) implies an initial unitary value for $\tilde{F}$. Assuming that afterwards current sign is positive, x increases towards 1 according to Eq. (11), while the window keeps its unitary value (see line 1₊). Provided current sign is kept positive, at some point x gets equal to 1, implying fulfillment to condition C_3 and causing window (16) to experience a sudden $1 \rightarrow 0$ transition (see line 2₊). After that, state variable and window function keep their respective values unless current sign is reversed and current magnitude gets larger than $i_{th,1}$. In fact, in case this happens, condition C_1 holds once again and window (16) experiences a sudden $0 \rightarrow 1$ transition (see line 3_{-,1}). Later, provided current sign is kept negative, solution x to Eq. (11) decreases towards 0 while the window keeps its unitary value (see line 4₋). Provided current sign is kept negative, at some point x gets equal to 0, condition C_2 is satisfied, and a sudden $1 \rightarrow 0$ transition, (see line 5₋), is triggered in window (16). Then both x and $\tilde{F}$ keep equal to 0 unless current sign is reversed and current magnitude gets larger than $i_{th,0}$ once again. If this occurs, Eq. C_1 is met yet again and the window experiences a sudden $0 \rightarrow 1$ transition (see line 6_{+,0}).

The main difference with Biolek's window (see Fig. 4(b)) lies in the fact that ours may only experience vertical transitions when x equals 0 or 1³. In fact, in case current sign is reversed while $x \in (0, 1)$, this simply causes state variable to vary in the opposite direction according to Eq. (11) (with Eq. (16) keeping its unitary value due to the fulfillment to condition C_1) thereafter. In this case, for $\eta = 1$, the window time evolution occurs only along one path, defined by lines 1₊ and 4₋ for positive and negative i respectively (see Fig. 4(c)).

Remark 1 The position of the layer boundary, defined by $x(t)$, is constrained to lie at any time instant either somewhere between the two device ends. This is expressed by the condition C_1 in Eq. (16). Conditions C_2 and C_3 make it possible that $x(t)$ gets stuck at physical boundaries (in actual devices $x(t)$ gets close to the ends) for a time interval the depends on the controlling source $i(t)$. In addition, $i(t)$ drives the layer boundary $x(t)$ in such a way that it moves towards the other end under suitable conditions, i.e. boundary conditions defining the behavior of the layer boundary at the ends depends on $i(t)$. In the next section the relationship between boundary conditions and

³In one of its forms, particularly in the matched case with $i_{th,0} = i_{th,1} = i_{th} = 0$ A, Eq. (16) is an approximation to Biolek's window for very large p .

current $i(t)$ is derived. When $x(t)$ is at the end 0 (1), charge and flux are linearly related and the device acts as a resistor with resistance R_{off} (R_{on}) for fully-undoped (fully-doped) film.

4.1 Boundary conditions

Assumption (16) implies that Eqs. (11)–(12) hold with $\tilde{F}(x, \eta i) = 1$. Time integrating both sides of Eq. (11), state variable $x(t)$ is found to be a function of time through charge $q(t)$. Its expression and the corresponding inverse function assume the following forms:

$$(x \circ q)(t) = x(0) + \frac{\eta [q(t) - q(0)]}{Q_0}, \quad (17)$$

$$(q \circ x)(t) = q(0) + \frac{Q_0 [x(t) - x(0)]}{\eta}, \quad (18)$$

where $q(0) = \int_{-\infty}^0 i(t) dt$ designates the entire current history.

From Eq. (17) it is evident that the normalized length of conductive layer is a linear function of charge. Thereby, using Eq. (18), limit values of the charge $q(t)$ when $x(t) = 0$ or $x(t) = 1$, let us call them q_{th_0} and q_{th_1} respectively, are expressed by:

$$q_{th_0} = (q \circ x)(t_0) = q(0) - \frac{Q_0}{\eta} x(0) \quad (19)$$

$$q_{th_1} = (q \circ x)(t_1) = q(0) + \frac{Q_0}{\eta} [1 - x(0)] = q_{th_0} + \frac{Q_0}{\eta}. \quad (20)$$

where t_0 and t_1 are such

$$t_0 \in \{t : x(t) = 0\} = \bigcup_k [\alpha_0^{(k)}, \beta_0^{(k)}], \quad (21)$$

$$t_1 \in \{t : x(t) = 1\} = \bigcup_k [\alpha_1^{(k)}, \beta_1^{(k)}]. \quad (22)$$

with $\alpha_0^{(k)}$ ($\alpha_1^{(k)}$) and $\beta_0^{(k)}$ ($\beta_1^{(k)}$) defined as the first and the last instants at which $x(t) = 0$ ($x(t) = 1$), respectively⁴.

It follows that on each time interval

- for all $t_0 \in [\alpha_0^{(k)}, \beta_0^{(k)}]$ then $x(t) = 0$ and $\tilde{F}(x, \eta i) = 0$. It implies Eq. (11) is fulfilled for any $i(t)$ and the φ - q characteristic is obtained from Eq. (12) written as $v(t) = R_{off} i(t)$, i.e. $\forall t = t_0 \in [\alpha_0^{(k)}, \beta_0^{(k)}]$ with $k = 1, 2, \dots$

$$\varphi(t) = \varphi(\alpha_0^{(k)}) + R_{off} [q(t) - q(\alpha_0^{(k)})] = \varphi(\alpha_0^{(k)}) + R_{off} \int_{\alpha_0^{(k)}}^t i(t') dt' \quad (23)$$

being

$$q(t) = q(\alpha_0^{(k)}) + \int_{\alpha_0^{(k)}}^t i(t') dt' = q_{th_0} + \int_{\alpha_0^{(k)}}^t i(t') dt' \quad (24)$$

- for all $t_1 \in [\alpha_1^{(k)}, \beta_1^{(k)}]$ then $x(t) = 1$ and $\tilde{F}(x, \eta i) = 0$. It implies Eq. (11) is fulfilled for any $i(t)$ and the φ - q characteristic is obtained from Eq. (12) written as $v(t) = R_{on} i(t)$, i.e. $\forall t = t_1 \in [\alpha_1^{(k)}, \beta_1^{(k)}]$ with $k = 1, 2, \dots$

$$\varphi(t) = \varphi(\alpha_1^{(k)}) + R_{on} [q(t) - q(\alpha_1^{(k)})] = \varphi(\alpha_1^{(k)}) + R_{on} \int_{\alpha_1^{(k)}}^t i(t') dt' \quad (25)$$

⁴Memristor usually operates under periodic controlling source. As a consequence $q(t)$, $\varphi(t)$ and $x(t)$ are periodic with the same period T . In such a case, t_0 (t_1) belongs to time intervals $[\alpha_0 + kT, \beta_0 + kT]$ ($[\alpha_1 + kT, \beta_1 + kT]$) for $k = 1, 2, \dots$

being

$$q(t) = q(\alpha_1^{(k)}) + \int_{\alpha_1^{(k)}}^t i(t') dt' = q_{th_1} + \int_{\alpha_1^{(k)}}^t i(t') dt' \quad (26)$$

For the sake of simplicity we assume through the paper that for all $k = 1, 2, \dots$ the following conditions hold⁵

$$q(\beta_0^{(k)}) = q(\alpha_0^{(k)}) = q_{th_0} \quad \Rightarrow \quad \int_{\alpha_0^{(k)}}^{\beta_0^{(k)}} i(t) dt = 0 \quad (27)$$

$$q(\beta_1^{(k)}) = q(\alpha_1^{(k)}) = q_{th_1} \quad \Rightarrow \quad \int_{\alpha_1^{(k)}}^{\beta_1^{(k)}} i(t) dt = 0 \quad (28)$$

The conditions (27) and (28) also imply that (see Eqs. (23) and (25))

$$\varphi(\beta_0^{(k)}) = \varphi(\alpha_0^{(k)}) = \varphi_{th_0} \quad \Rightarrow \quad \int_{\alpha_0^{(k)}}^{\beta_0^{(k)}} v(t) dt = 0 \quad (29)$$

$$\varphi(\beta_1^{(k)}) = \varphi(\alpha_1^{(k)}) = \varphi_{th_1} \quad \Rightarrow \quad \int_{\alpha_1^{(k)}}^{\beta_1^{(k)}} v(t) dt = 0 \quad (30)$$

It turns out Eqs. (27) and (28) impose to have the same charge and flux at the instants $\alpha_0^{(k)}$ and $\beta_0^{(k)}$ ($\alpha_1^{(k)}$ and $\beta_1^{(k)}$) when $x(t)$ reaches and leaves the boundary 0 (1), respectively. In other words, $x(t) = 0$ ($x(t) = 1$) corresponds to charge $q(t)$ and flux $\varphi(t)$ lower (greater) than q_{th_0} and φ_{th_0} (q_{th_1} and φ_{th_1}), respectively.

The description of an example where conditions (27) and (28) are applicable is presented in Section 6.

Thereby, the following boundary condition-based window function can be defined under the assumption that Eqs. (27) and (28) are fulfilled:

$$\tilde{F}(x(t), \eta i(t)) = \begin{cases} 1 & \forall t : C_1 \text{ holds} \\ 0 & \forall t : C_2 \text{ or } C_3 \text{ holds} \end{cases} \quad (31)$$

where conditions C_k ($k = 1, 2, 3$) are expressed in terms of the current $i(t)$ as follows

$$C_1 = \left\{ t : x(t) \in (0, 1) \text{ or } \left(x(t) = 0 \text{ and } \int_{-\infty}^t i(t') dt' \geq \eta q_{th_0} \right) \text{ or } \left(x(t) = 1 \text{ and } \int_{-\infty}^t i(t') dt' \leq \eta q_{th_1} \right) \right\}, \quad (32)$$

$$C_2 = \left\{ t : x(t) = 0 \text{ and } \int_{-\infty}^t i(t') dt' < \eta q_{th_0} \right\}, \quad (33)$$

$$C_3 = \left\{ t : x(t) = 1 \text{ and } \int_{-\infty}^t i(t') dt' > \eta q_{th_1} \right\}. \quad (34)$$

Throughout this work the model adopted for the memristor is the set of Eqs. (11)–(12) with window (31). In comparison with other windows such as (13) or (14)–(15), the advantage of Eq. (31) is that it allows a straightforward analytical integration to Eqs. (11)–(12), still retaining an adequate accuracy level in qualitatively reproducing memristor dynamics. Note that the proposed model is equivalent to William's linear drift model augmented with Biolek's realistic boundary conditions.

4.2 Investigation of the BCM

Use of the proposed window function expressed by Eq. (31) permits the analytical integration of Eqs. (11)–(12), whose solutions may be given in closed form under any condition in Eqs. (32)–(34).

⁵The case when conditions (27) and (28) are not satisfied is more complex and it will be investigated in a further manuscript.

4.2.1 Dynamics under condition C_1

Let us assume that condition (32) is satisfied. This implies that $\tilde{F}(x, \eta i) = 1$ (see (31)).

Using Eq. (11) to express i in terms of $\frac{dx}{dt}$, inserting such expression into Eq. (12), using Eq. (10) and time integrating both sides of the resulting equation, the time-dependence of flux φ through state variable x is found to be:

$$(\varphi \circ x)(t) = \varphi(0) + \frac{Q_0 R_{off}}{\eta} [x(t) - x(0)] - \frac{Q_0 \Delta R}{2\eta} [x^2(t) - x^2(0)], \quad (35)$$

where $x(0) = \int_{-\infty}^0 dx(t)$ and $\varphi(0) = \int_{-\infty}^0 v(t)dt$ denote the initial values for normalized length of conductive layer and flux respectively.

This expression holds whenever $x(t) \in (0, 1)$, i.e. $q(t) \in \eta(q_{th_0}, q_{th_1})$ and $\varphi(t) \in \eta(\varphi_{th_0}, \varphi_{th_1})$. By abuse of notation, with $q(t) \in \eta(q_{th_0}, q_{th_1})$ we mean:

- if $\eta = +1$ then $(q_{th_0} < q_{th_1})$ (see Eq. (20)), i.e. $q(t) \in (q_{th_0}, q_{th_1})$
- if $\eta = -1$ then $(q_{th_1} < q_{th_0})$ (see Eq. (20)), i.e. $q(t) \in (-q_{th_1}, -q_{th_0})$.

Similar considerations hold for $\varphi(t) \in \eta(\varphi_{th_0}, \varphi_{th_1})$.

Note that q_{th_0} and q_{th_1} are expressed in terms of initial conditions $x(0)$ and $q(0)$ by means of Eqs. (19) and (20).

From Eq. (35) it follows that limit values flux respectively assumes at $x = 0$ and at $x = 1$, thus φ_{th_0} and φ_{th_1} , are given by the following expressions in terms of initial conditions $x(0)$ and $\varphi(0)$:

$$\varphi_{th_0} = \varphi(0) - \frac{Q_0 R_{off}}{\eta} x(0) + \frac{Q_0 \Delta R}{2\eta} x^2(0), \quad (36)$$

$$\varphi_{th_1} = \varphi(0) + \frac{Q_0 R_{off}}{\eta} [1 - x(0)] - \frac{Q_0 \Delta R}{2\eta} [1 - x^2(0)] = \varphi_{th_0} + \frac{Q_0 (R_{off} + R_{on})}{2\eta} \quad (37)$$

It should be realized that Eqs. (36) and (37) respectively indicate minimum and maximum (maximum and minimum) values flux may take for $\eta = 1$ ($\eta = -1$).

Remark 2 It is worth observing that, under the assumptions expressed by Eqs. (27) and (28), the existence of limit values defined by Eqs. (19)–(20) and (36)–(37), within which charge and flux are constrained to lie for condition (32) to be maintained, is due to the finite physical dimension of the nano-device. Thereby, C_1 results in $q(t) \in \eta(q_{th_0}, q_{th_1})$ and $\varphi(t) \in \eta(\varphi_{th_0}, \varphi_{th_1})$. In addition, C_2 (C_3) defines the complementary conditions, i.e. $q(t) \in \eta(-\infty, q_{th_0})$ and $\varphi(t) \in \eta(-\infty, \varphi_{th_0})$ ($q(t) \in \eta(q_{th_1}, +\infty)$ and $\varphi(t) \in \eta(\varphi_{th_1}, +\infty)$).

Inserting Eq. (17) into Eq. (35), the memristor flux-charge mathematical relationship is thus obtained:

$$(\varphi \circ q)(t) = \varphi(0) + R(0)[q(t) - q(0)] - \frac{\eta \Delta R}{2Q_0} [q(t) - q(0)]^2, \quad (38)$$

where, using Eqs. (10) and (19), initial memristance $(R \circ x)(0)$ is expressed by

$$R(0) = (R \circ x)(0) = R_{on}x(0) + R_{off}[1 - x(0)] = R_{off} - \Delta R x(0) = R_{off} - \eta \Delta R \frac{q(0) - q_{th_0}}{Q_0}. \quad (39)$$

Inverting Eq. (38) and discarding the solution lying outside closed interval $\eta[q_{th_0}, q_{th_1}]$, the expression for charge in terms of flux is found to be:

$$(q \circ \varphi)(t) = q(0) + \frac{Q_0 R(0)}{\eta \Delta R} - \frac{Q_0 R(0)}{\eta \Delta R} \sqrt{1 - \frac{2\eta \Delta R}{Q_0 [R(0)]^2} [\varphi(t) - \varphi(0)]}. \quad (40)$$

Note that the existence condition for the radical does not add any further constraint on the allowable interval of flux values $\eta[\varphi_{th_0}, \varphi_{th_1}]$ guaranteeing the fulfillment of condition (32). Differentiating Eq. (40) with respect to φ , the expression for the memristor conductance is thus derived:

$$(W \circ \varphi)(t) = \frac{d(q \circ \varphi)(t)}{d\varphi(t)} = \frac{1}{R(0) \sqrt{1 - \frac{2\eta \Delta R}{Q_0 R^2(0)} [\varphi(t) - \varphi(0)]}}. \quad (41)$$

The expression for the memristor resistance, obtained through charge differentiation of Eq. (38), is given by

$$(M \circ q)(t) = \frac{d(\varphi \circ q)(t)}{dq(t)} = R(0) - \frac{\eta \Delta R}{Q_0} [q(t) - q(0)]. \quad (42)$$

Manipulating Eq. (40) yields

$$R(0) \sqrt{1 - \frac{2\eta \Delta R}{Q_0 R^2(0)} [\varphi(t) - \varphi(0)]} = R(0) - \frac{\eta \Delta R}{Q_0} [q(t) - q(0)]$$

from which it follows that $(W \circ \varphi)(t) = [(M \circ q)(t)]^{-1}$.

Finally the memristor current-voltage relationship is obtained through application of the chain rule, i.e.

$$i(t) = \frac{dq(t)}{dt} = \frac{dq(\varphi)}{d\varphi} \frac{d\varphi(t)}{dt} = W(\varphi)v(t).$$

4.2.2 Dynamics under condition C_2

Let us here assume that condition (33) is satisfied. In this case $\tilde{F}(x, \eta i) = 0$ (see Eq. (31)) and thus $x = 0$ is an equilibrium point for Eq. (11). Under such condition the thin oxide film is fully-undoped and works as a linear resistor with resistance R_{off} . Integrating Eq. (12) for all $t_0 \in (\alpha_0^{(k)}, \beta_0^{(k)})$ with $R(t_0) = R_{off}$ (recall Eq. (21) and use Eq. (10)) and initial conditions $\varphi(\alpha_0^{(k)}) = \varphi_{th_0}$ and $q(\alpha_0^{(k)}) = q_{th_0}$, the flux-charge relationship and its inverse function may be derived:

$$(\varphi \circ q)(t) = R_{off} (q(t) - q_{th_0}) + \varphi_{th_0}, \quad (43)$$

$$(q \circ \varphi)(t) = \frac{\varphi(t) - \varphi_{th_0}}{R_{off}} + q_{th_0}, \quad (44)$$

where q_{th_0} and φ_{th_0} are expressed by Eqs. (19) and (36) respectively. The allowable flux and charge intervals for condition (33) to be maintained respectively are $\varphi(t) < \varphi_{th_0}$ and $q(t) < q_{th_0}$, i.e. this holds for all $t \in (\alpha_0^{(k)}, \beta_0^{(k)})$ and at $t = \beta_0^{(k)}$ condition (27) occurs.

Furthermore, the memristor current-voltage relationship is expressed by $v(t) = R_{off} i(t)$.

4.2.3 Dynamics under condition C_3

This analysis can be carried out, *mutatis mutandis*, as in the previous section. Assuming the fulfillment of condition (34), Eq. (31) yields $\tilde{F}(x, \eta i) = 0$. As a result, $x = 1$ is an equilibrium point for Eq. (11). The film is fully-doped and acts as a linear resistor with resistance R_{on} . Integration of Eq. (11) for all $t_1 \in (\alpha_1^{(k)}, \beta_1^{(k)})$ with $R(t_1) = R_{on}$ (recall Eq. (22) and use Eq. (10)) and initial conditions $\varphi(\alpha_1^{(k)}) = \varphi_{th_1}$ and $q(\alpha_1^{(k)}) = q_{th_1}$ yields the expression for the flux-charge relationship and its inverse function:

$$(\varphi \circ q)(t) = R_{on} (q(t) - q_{th_1}) + \varphi_{th_1}, \quad (45)$$

$$(q \circ \varphi)(t) = \frac{\varphi(t) - \varphi_{th_1}}{R_{on}} + q_{th_1}, \quad (46)$$

where q_{th_1} and φ_{th_1} are given by Eqs. (20) and (37) respectively. In order for condition (34) to hold, flux and charge are constrained to lie within intervals $\varphi(t) > \varphi_{th_1}$ and $q(t) > q_{th_1}$, i.e. this holds for all $t \in (\alpha_1^{(k)}, \beta_1^{(k)})$ and at $t = \beta_1^{(k)}$ condition (28) occurs.

Also, the memristor current-voltage relationship is given by $v(t) = R_{on} i(t)$.

The combination of the charge-flux relations found under conditions C_1 , C_2 and C_3 allows one to show the polarity-dependent asymmetric q - φ nonlinearity of the HP memristor.

4.3 Polarity-dependent asymmetry

Combining Eqs. (40), (44) and (46), the expression for charge in terms of flux is finally expressed by

$$(q \circ \varphi)(t) = q(\varphi(t)) = \begin{cases} q(0) + \frac{Q_0 R(0)}{\eta \Delta R} - \frac{Q_0 R(0)}{\eta \Delta R} \sqrt{1 - \frac{2\eta \Delta R}{Q_0 [R(0)]^2} [\varphi(t) - \varphi(0)]} & \text{if } C_1 \text{ holds,} \\ \frac{\varphi(t) - \varphi_{th_0}}{R_{off}} + q_{th_0} & \text{if } C_2 \text{ holds,} \\ \frac{\varphi(t) - \varphi_{th_1}}{R_{on}} + q_{th_1} & \text{if } C_3 \text{ holds.} \end{cases} \quad (47)$$

The flux-charge mathematical relationship may be obtained by using Eqs. (38), (43) and (45):

$$(\varphi \circ q)(t) = \varphi(q(t)) = \begin{cases} \varphi(0) + R(0)[q(t) - q(0)] - \frac{\eta \Delta R}{2Q_0} [q(t) - q(0)]^2 & \text{if } C_1 \text{ holds,} \\ R_{off}(q(t) - q_{th_0}) + \varphi_{th_0} & \text{if } C_2 \text{ holds,} \\ R_{on}(q(t) - q_{th_1}) + \varphi_{th_1} & \text{if } C_3 \text{ holds.} \end{cases} \quad (48)$$

From Eqs. (47) and (48), it is readily derived that $q(\varphi(t))$ and $\varphi(q(t))$ are parabolic if C_1 holds, that is within the region $\eta [q_{th_0}, q_{th_1}] \times \eta [\varphi_{th_0}, \varphi_{th_1}]$ of the plane φ - q and q - φ , respectively. Outside this region $q(\varphi(t))$ and $\varphi(q(t))$ are linear. In particular, if $\eta = +1$:

- the slope of $q(\varphi(t))$ is $(R_{off})^{-1}$ and $(R_{on})^{-1}$ for $(-\infty, \varphi_{th_0})$ and $(\varphi_{th_1}, +\infty)$, respectively;
- the slope of $\varphi(q(t))$ is R_{off} and R_{on} for $(-\infty, q_{th_0})$ and $(q_{th_1}, +\infty)$, respectively.

By contrast, if $\eta = -1$ (compare Figs. 3(b) and 3(c) to see the exchange of R_{off} and R_{on} in this case):

- the slope of $q(\varphi(t))$ is $(R_{on})^{-1}$ and $(R_{off})^{-1}$ for $(-\infty, -\varphi_{th_1})$ and $(-\varphi_{th_0}, +\infty)$, respectively;
- the slope of $\varphi(q(t))$ is R_{on} and R_{off} for $(-\infty, -q_{th_1})$ and $(-q_{th_0}, +\infty)$, respectively.

Let us consider an exemplifying numerical example. For simplicity we assume $x(0) = 0$, $q(0) = 0$ and $\varphi(0) = 0$. It follows that $q_{th_0} = q(t_0) = 0$, $q_{th_1} = q(t_1) = \eta Q_0$ and $(R \circ x)(0) = R(0) = R_{off}$. Using Eqs. (36)–(37) this also implies that expressions for flux at the physical boundaries reduce to the following simpler forms:

$$\varphi_{th_0} = \varphi(t_0) = 0 \quad (49)$$

$$\varphi_{th_1} = \varphi(t_1) = \frac{Q_0}{2\eta} (R_{off} + R_{on}). \quad (50)$$

As for the parameter setting, we select typical values reported in [21]: $D = 10 \text{ nm}$, $\mu = 10^{-10} \frac{\text{cm}^2}{\text{Vs}}$ and $R_{off} = 100 R_{on}$. Also, choosing $R_{on} = 1 \text{ k}\Omega$ implies $R_{off} = 100 \text{ k}\Omega$, $\Delta R = 99 \text{ k}\Omega$ and $Q_0 = 10^{-5} \text{ C}$. From Eqs. (49)–(50) it follows that $\varphi(t_0) = 0 \text{ Vs}$, and $\varphi(t_1) = \pm 0.505 \text{ Vs}$ (the positive sign is associated to $\eta = +1$).

Figures 5(a)–5(b) respectively depict q versus φ , as it is mathematically given in Eq. (47), for $\eta = -1$ and for $\eta = +1$ (solid curves). The asymmetry of the two devices is evident from the characteristics.

In each of these plots, the linear parts of the solid curve and their dashed-line extensions determine the two-half-line-based PWL approximation to Eq. (47).

The simple PWL approximation to Eq. (47), often adopted in literature [22], consists of the tangent lines to the function defined by Eq. (47) at flux limit values under condition (32), i.e. at points $O = (\varphi(t_0), q(t_0))$ and $Q = (\varphi(t_1), q(t_1))$. These lines, expressed by Eqs. (46) and (44), have angular coefficients respectively equal to $\frac{1}{R_{on}}$ and $\frac{1}{R_{off}}$ and meet at point $P = (\varphi_c, q_c)$, whose coordinates may be expressed in the following form:

$$\varphi_c = \frac{Q_0}{2\eta} R_{off}, \quad (51)$$

$$q_c = \frac{Q_0}{2\eta}. \quad (52)$$

Fig. 5. Memristor charge-flux nonlinearity (solid curve) and its PWL approximation (linear parts of the solid curve and their dashed line extensions) with η set to -1 (plot(a)) and $+1$ (plot(b)).

The angular coefficients of the PWL approximation to Eq. (47) respectively equal $\frac{1}{R_{on}} = 10^{-3} S$ and $\frac{1}{R_{off}} = 10^{-5} S$. Using Eqs. (51) and (52), the coordinates of the non-differentiability point of the PWL approximation to Eq. (47) are $\varphi_c = \pm 0.5 V s$ and $q_c = \pm 0.5 \cdot 10^{-5} C$ (the positive sign for both φ_c and q_c is associated to $\eta = +1$).

Derivation of the plot of the inverse function $\varphi = \varphi(q)$, whose analytical description is reported in Eq. (48), and of its PWL approximation follows similar guidelines.

Remark 3 It is worth noting that if $x(0) \neq 0$, $q(0) \neq 0$ and $\varphi(0) \neq 0$ then $q_{th_0} = q(t_0) \neq 0$ and $\varphi_{th_0} = \varphi(t_0) \neq 0$, that is axis in Fig. 5 are just translated.

Although the memristor is inherently-asymmetric, complementary behaviors arise in devices with opposite polarity (see Fig. 5). Monotone-increasing⁶ odd-symmetric charge–flux nonlinearity, suitable for chaos-based applications, can be obtained through suitable interconnections of asymmetric memristors (as those fabricated at Hewlett-Packard Labs).

Interconnections of different memristors having flux–charge characteristics defined by Eqs. (47) and (48) permit to obtain complex memristor elements⁷. We focus on series and parallel interconnection of two identical memristor with opposite polarities (the case with same polarities is trivial). In particular, the following two cases are possible:

- (a) two identical memristors with opposite polarities connected in series, i.e. anti-series combination;
- (b) two identical memristors with opposite polarities connected in parallel, i.e. anti-parallel combination.

The flux–charge characteristic for the case (a) is derived by means of Eq. (48), where the memristor with $\eta = +1$ is parabolic on $[q_{th_0}, q_{th_1}]$ whereas the other with $\eta = -1$ is parabolic on $[-q_{th_1}, -q_{th_0}]$.

⁶As shown in Theorem 1 of [1], monotonicity implies passivity.

⁷As shown in Theorem 2 of [1]: “A one-port containing memristors is equivalent to a memristor”.

On the other hand, for the case (b) the charge-flux is obtained from Eq. (47), where the memristor with $\eta = +1$ is parabolic on $[\varphi_{th_0}, \varphi_{th_1}]$ whereas the other with $\eta = -1$ is parabolic on $[-\varphi_{th_1}, -\varphi_{th_0}]$. Over the others intervals the flux-charge results to be linear. As shown in the next section, such properties are crucial to obtain odd-symmetric charge-flux nonlinearity of anti-series and anti-parallel combinations.

5. Memristive elements with symmetry property

In this section we shall exploit the complementary dynamics of memristor with opposite polarity to devise a couple of memristor combinations characterized by a monotone-increasing odd-symmetric charge-flux relationship.

5.1 Anti-parallel combination

Let us consider two memristors, m_j , $j = 1, 2$. Combining them in anti-parallel, as it is shown in Fig. 6(a), the following constraints, due to Kirchhoff's Voltage and Current Laws, link charge and flux variables of the memristors at any time:

$$q_{ap} = q_1 + q_2, \quad (53)$$

$$\varphi_{ap} = \varphi_1 = \varphi_2, \quad (54)$$

where q_1 and q_2 (φ_1 and φ_2) are the charges through (the fluxes across) m_1 and m_2 respectively, while q_{ap} and φ_{ap} respectively represent the charge and flux of the passive bipole resulting from their combination (we use subscript ap to denote anti-parallel connection).

Using Eqs. (53)–(54), the charge-flux constitutive relation for the anti-parallel memristor combination may feature different expressions. However, we are just interested in a simple case. We assume identical memristors. i.e. for $j = 1, 2$ we set $D_j = D$, $\mu_j = \mu$, $R_{j,on} = R_{on}$ and $R_{j,off} = R_{off}$, this implies $\Delta R_j = R_{off} - R_{on} = \Delta R$ and $Q_{j,0} = \frac{D^2}{\mu R_{on}} = Q_0$. With reference to Fig. 6(a) we have $\eta_2 = -\eta_1 = \eta$, with η set here to $+1$.

We also assume $x_j(0) = 0$, which yields $(R_j \circ x_j)(0) = R_{off}$ and $q_j(0) = 0$. Finally, we take $\varphi_j(0) = 0$. Taking Eqs. (19)–(20) and (49)–(50) into account, it is derived

$$\begin{aligned} q_1(t_0) &= q_{th_0,1} = q_{th_0,2} = q_2(t_0) = 0 \\ -q_1(t_1) &= -q_{th_1,1} = q_{th_1,2} = q_2(t_1) = Q_0 \\ \varphi_{as}(t_0) &= \varphi_1(t_0) = \varphi_{1,th_0} = \varphi_{2,th_0} = \varphi_2(t_0) = 0 \\ -\varphi_{as}(t_1) &= -\varphi_1(t_1) = -\varphi_{1,th_1} = \varphi_{2,th_1} = \varphi_2(t_1) = \frac{Q_0(R_{off} + R_{on})}{2\eta} = \varphi_{th_1} \end{aligned}$$

After a number of algebraic calculations the combined memristive element is found to be characterized by the following charge q_{ap} -flux φ_{ap} relationship:

$$q_{ap}(\varphi_{ap}) = \begin{cases} \varphi_{ap} \left(\frac{1}{R_{off}} + \frac{1}{R_{on}} \right) + \frac{\Delta R}{R_{on}} \frac{Q_0}{2\eta} & \forall \varphi_{ap} \in (-\infty, -\varphi_{th_1}), \\ -\frac{Q_0}{\eta \Delta R} \frac{R_{off}}{R_{off}} + \frac{\varphi_{ap}}{R_{off}} + \frac{Q_0}{\eta \Delta R} \sqrt{1 + \frac{2\eta \Delta R}{Q_0 R_{off}^2} \varphi_{ap}} & \forall \varphi_{ap} \in (-\varphi_{th_1}, 0], \\ +\frac{Q_0}{\eta \Delta R} \frac{R_{off}}{R_{off}} + \frac{\varphi_{ap}}{R_{off}} - \frac{Q_0}{\eta \Delta R} \sqrt{1 - \frac{2\eta \Delta R}{Q_0 R_{off}^2} \varphi_{ap}} & \forall \varphi_{ap} \in [0, \varphi_{th_1}], \\ \varphi_{ap} \left(\frac{1}{R_{off}} + \frac{1}{R_{on}} \right) - \frac{\Delta R}{R_{on}} \frac{Q_0}{2\eta} & \forall \varphi_{ap} \in (\varphi_{th_1}, +\infty). \end{cases} \quad (55)$$

The charge-flux characteristic described by Eq. (55) is depicted in Fig. 7(a) (solid curve), where after some algebraic calculations the coordinates of points O , Q and Q' are found to be:

$$O = (\varphi_{as}(t_0), q_1(t_0) + q_2(t_0)) = (0, 0)$$

$$Q = -Q' = (\varphi_2(t_1), q_2(t_1) + (q_1 \circ (-\varphi_1))(t_1)) = \left(\frac{Q_0}{2\eta} (R_{on} + R_{off}), \frac{Q_0}{2\eta} \frac{R_{on} + 3R_{off}}{R_{off}} \right)$$

where using Eq. (47) it is derived

Fig. 6. Anti-parallel (plot(a)) and anti-series (plot(b)) combination of HP memristors.

$$(q_1 \circ (-\varphi_1))(t_1) = q_1(-\varphi(t_1)) = -\frac{Q_0}{2\eta} \frac{R_{off} + R_{on}}{R_{off}}.$$

It may be proved that the PWL approximation to Eq. (55) has the same expression as in Eq. (1), i.e.:

$$q_{ap}(\varphi_{ap}) = b_{ap}\varphi_{ap} + \frac{a_{ap} - b_{ap}}{2}(|\varphi_{ap} + \varphi_{ap,c}| - |\varphi_{ap} - \varphi_{ap,c}|), \quad (56)$$

where

$$a_{ap} = \frac{2}{R_{off}}, \quad (57)$$

$$b_{ap} = \frac{1}{R_{off}} + \frac{1}{R_{on}}, \quad (58)$$

$$\varphi_{ap,c} = \varphi_{2,c} = \frac{R_{off}Q_0}{2\eta}. \quad (59)$$

The PWL charge-flux characteristic for the anti-parallel combination is also depicted in Fig. 7(a) (linear parts of the continuous characteristic and their dashed-line extensions), where points P and P' have the following coordinates: $P = -P' = (\varphi_{ap,c}, q_{ap,c})$ with $\varphi_{ap,c}$ expressed by Eq. (59) and $q_{ap,c} = \frac{Q_0}{\eta}$, as it may be derived from Eq. (56).

We have thus demonstrated that under particular conditions the anti-parallel combination of two HP memristors permits the electronic implementation of a monotone-increasing odd-symmetric charge-flux nonlinearity suitable for chaos-based applications.

5.2 Anti-series combination

Let us study the connection of memristors in anti-series (Fig. 6(b)). Here application of Kirchoff's Voltage and Current Laws to the memristor combination at all times yields:

$$\varphi_{as} = \varphi_1 + \varphi_2, \quad (60)$$

$$q_{as} = q_1 = q_2, \quad (61)$$

where subscript *as* stands for anti-series connection.

Making use of Eqs. (60)–(61) it may be demonstrated that the most general relation linking flux and charge for the anti-series memristor combination may feature a number of different expressions. However, our interest lies in a simple case. In fact, as in the anti-parallel case, we focus on two identical and let $x_j(0) = 0$, $q(0) = 0$ and $\varphi_j(0) = 0$, i.e. charge and flux have the same bounds as in the previous section.

Fig. 7. Charge-flux characteristics (solid curves) and their PWL approximations (linear parts of the solid curves and their dashed-line extensions) for anti-parallel (plot (a)) and anti-series (plot (b)) combination of two memristors of Fig. 6.

With reference to Fig. 6(b), we do have $\eta_2 = -\eta_1 = \eta$, with η set here to +1. Some algebraic calculations yield the flux φ_{as} -charge q_{as} mathematical relationship (subscript *as* indicates anti-series connection):

$$\varphi_{as}(q_{as}) = \begin{cases} (R_{off} + R_{on}) q_{as} + \frac{Q_0}{2\eta} \Delta R & \forall q_{as} \in (-\infty, -q_{th_1}), \\ 2R_{off} q_{as} - \frac{\Delta R \eta}{2Q_0} q_{as}^2 & \forall q_{as} \in [-q_{th_1}, 0], \\ 2R_{off} q_{as} + \frac{\Delta R \eta}{2Q_0} q_{as}^2 & \forall q_{as} \in [0, q_{th_1}], \\ (R_{off} + R_{on}) q_{as} - \frac{Q_0}{2\eta} \Delta R & \forall q_{as} \in (q_{th_1}, +\infty), \end{cases} \quad (62)$$

The flux-charge characteristic expressed by Eq. (62) is shown in Fig. 7(b) (solid curve). Coordinates of points O, Q and Q' are given by: $0 = (q_2(t_0), \varphi_2(t_0) + \varphi_1(t_0)) = (0, 0)$ and

$$Q = -Q' = (q_2(t_1), \varphi_2(t_1) + (\varphi_1 \circ (-q_1))(t_1)) = \left(\frac{Q_0}{\eta}, \frac{Q_0}{2\eta} (R_{on} + 3R_{off}) \right).$$

The PWL approximation to Eq. (62) is given by:

$$\varphi_{as}(q_{as}) = \frac{1}{b_{as}} q_{as} + \frac{1}{2} \left(\frac{1}{a_{as}} - \frac{1}{b_{as}} \right) (|q_{as} + q_{as,c}| - |q_{as} - q_{as,c}|), \quad (63)$$

where

$$a_{as} = \frac{1}{2R_{off}}, \quad (64)$$

$$b_{as} = \frac{1}{R_{off} + R_{on}}, \quad (65)$$

$$q_{as,c} = q_{2,c} = \frac{Q_0}{2\eta} \quad (66)$$

The PWL approximation to Eq. (62) is also shown in Fig. 7(b) (linear parts of the solid curve and their dashed-line extensions). The coordinates of *P* are given by: $P = (q_{as,c}, \varphi_{as,c})$, where $q_{as,c}$ is expressed by Eq. (66) and, using Eq. (63), $\varphi_{as,c} = \eta Q_0 R_{off}$.

Therefore yet another monotone-increasing odd-symmetric charge-flux characteristic suitable for memristor-based chaotic circuits may be practically realized by connecting two memristors in anti-series.

Remark 4 Due to the passivity property of memristive elements [1, 2], it is impossible to realize an even-symmetric charge-flux characteristic with some combination of such two-terminal nano-scale devices.

6. Chaotic circuit with anti-parallel memristor combination

This section is devoted to the brief validation of the theoretical results presented in this manuscript. The circuit equations for the oscillator of Fig. 8, obtained from the oscillator of Fig. 1 by replacing the artificially-conceived symmetric memristor characterized by the memductance function (3) with the anti-parallel combination of HP nano-scale devices shown in Fig. 6(a), are:

$$\begin{aligned}
C_1 \frac{dv}{dt} &= \frac{v_2 - v}{R_1} - \frac{v}{R_{m1}} - \frac{v}{R_{m2}} + \frac{v}{R_2}, \\
C_2 \frac{dv_2}{dt} &= -\tilde{i} - \frac{v_2 - v}{R_1}, \\
L \frac{d\tilde{i}}{dt} &= v_2 - r\tilde{i}, \\
\frac{1}{D} \frac{dl_1(t)}{dt} &= \frac{\eta_1}{Q_0} i_1 \tilde{F} \left(\frac{l_1}{D}, \eta_1 i_1 \right), \\
\frac{1}{D} \frac{dl_2(t)}{dt} &= \frac{\eta_2}{Q_0} i_2 \tilde{F} \left(\frac{l_2}{D}, \eta_2 i_2 \right),
\end{aligned} \tag{67}$$

where $\tilde{F}$ is described by Eq. (31), l_j ($j = 1, 2$) designates the length of conductive layer for memristor m_j , $i_j = v(R_{mj})^{-1}$ and $R_{mj} = R_{off} - \Delta R(l_j/D)$. Defining state variables as $x_1 = v$, $x_2 = v_2$, $x_3 = -R_1\tilde{i}$, $x_4 = l_1(D)^{-1}$ and $x_5 = l_2(D)^{-1}$, expressing system parameters α , β , γ and ξ as in Section II, using normalized time variable $\tau = t(\bar{t})^{-1}$, with system time scale given by $\bar{t} = R_1 C_2$, state equations associated to Eq. (67) are found to be:

$$\begin{aligned}
\frac{dx_1}{d\tau} &= \alpha \left[x_2 - \left(1 + \frac{R_1}{R_{m1}} + \frac{R_1}{R_{m2}} - \xi \right) x_1 \right], \\
\frac{dx_2}{d\tau} &= x_3 - x_2 + x_1, \\
\frac{dx_3}{d\tau} &= -\beta x_2 - \gamma x_3, \\
\frac{dx_4}{d\tau} &= \frac{1}{\eta_1 I_0} \left(\frac{x_1}{R_{m1}} \right) \tilde{F} \left(x_4, \eta_1 \frac{x_1}{R_{m1}} \right), \\
\frac{dx_5}{d\tau} &= \frac{1}{\eta_2 I_0} \left(\frac{x_1}{R_{m2}} \right) \tilde{F} \left(x_5, \eta_2 \frac{x_1}{R_{m2}} \right),
\end{aligned} \tag{68}$$

where $I_0 = Q_0(\bar{t})^{-1}$, $R_{m1} = R_{off} - \Delta R x_4$ and $R_{m2} = R_{off} - \Delta R x_5$.

Setting values for circuit devices R_1 , R_2 , r , C_1 , C_2 and L as in Section II, system parameters α , β , γ and ξ are numerically expressed as in numerical simulation of Fig. 2. Equating a and b from Eq. (3) to Eqs. (57)–(58) and recalling Eqs. (6)–(7), the expressions for R_{on} and R_{off} are found to be:

$$R_{off} = \frac{2R_1}{\tilde{a}}, \tag{69}$$

$$R_{on} = \frac{R_1 R_{off}}{\tilde{b} R_{off} - R_1}. \tag{70}$$

Setting $\tilde{a}$ and $\tilde{b}$ as in Section II, the values for the off- and on-resistances respectively are $R_{off} = 10.198 \text{ k}\Omega$ and $R_{on} = 2.3534 \text{ k}\Omega$. Choosing for μ and D the same values specified in Section 4.3, taking $\eta_1 = 1$ and $\eta_2 = -1$, chaos arises in dynamical system (68) (see Fig. 9, where initial conditions for state vector $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5] \in \mathbb{R}^5$ are set to $[0.1, -0.1, 0.1, 0, 0]$).

Fig. 8. Oscillator of Fig. 1 with the artificially-conceived memristor with memductance defined by Eq. (3) replaced with the anti-parallel combination of HP nano-scale elements of Fig. 6(a).

Fig. 9. Chaotic behavior observed in the oscillator of Fig. 8. Compare it to the dynamics shown in Fig. 2. Note that the use of window (31) in the memristor model for both m_1 and m_2 ensures that conditions (27)–(28) are fulfilled.

7. Conclusions and future research developments

One of the necessary aspects for the occurrence of chaos in memristor-based chaotic circuits is the presence of a monotone-increasing odd-symmetric charge-flux nonlinearity. At present there is no evidence that a practical realization for such symmetric nonlinearity may actually exist in nature. In fact, the physical memristor fabricated at Hewlett-Packard Labs in 2008 is inherently asymmetric. However, the asymmetrical behaviors of real memristors with opposite polarity are complementary. As a result, at least in principle, it is possible to devise appropriate memristor combinations exhibiting the symmetry feature suitable for chaos-based applications. This was achieved in this work through analytical integration of a simple yet accurate memristor model.

Agreement between chaotic behaviors of a couple of modified Chua's oscillators, in which the nonlinear diode is substituted in the first case with an artificially-conceived symmetric memristor and in the second case with one of the proposed memristor combinations validate our claims.

Finally, note that a circuit model for the symmetric charge-flux relationship expressed by Eq. (1), was recently derived in Fig. 7(b) of [24] by connecting in anti-parallel two basic sub-circuits. According to our findings, those sub-circuits are electronic implementations of the charge-flux characteristics of memristors with opposite polarity.

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