Robustness and an application of a one-dimensional window-map based on rotation dynamics

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Received March 30, 2012; Revised June 4, 2012; Published October 1, 2012

Abstract: This paper studies robustness and an application of a one-dimensional window-map based on rotation dynamics. The map with a flat part that is called a window exhibits various superstable periodic orbits (SSPOs) and bifurcations. Using theoretical analysis, we clarify existence regions of the period of the SSPOs in parameter space. We also consider an application to an analog-to-digital converter and clarify output characteristics based on theoretical calculation. Next, we consider the case where the window has a small slope as parameter perturbation and show typical phenomena and robustness. Finally, we discuss bifurcation phenomena and robustness for the window-map based on chaos.

Key Words: one-dimensional map, bifurcation, superstable periodic orbit, rotation, Analog-to-digital converter

1. Introduction

A one-dimensional map with nonlinearity exhibits rich periodic, nonperiodic orbits and bifurcation phenomena. Such maps have been studied in a large number of papers. Replacing a constant term to a part of an equation of the map, feature of the map has a flat part. When an orbit of the map starts from the flat part and return to it, the orbit becomes a superstable periodic orbit (SSPO). The SSPOs and relating to bifurcation in such maps have been observed in a variety of mathematical model and a model based on real systems in various fields [1–10]. The SSPOs have interesting properties: superstable for the initial state but sensitive for parameter variation. In fact, as the parameter of the system varies, very rich SSPOs and complex bifurcations occur [8].

In this paper, we study the dynamics and robustness of the one-dimensional window-maps. We also consider an application to an analog-to-digital converter (ADC) and its characteristics [9]. In section 2, we consider the dynamics and an application of the window-map based on rotation. We refer to a flat (zero-slope) part as a window and the map with a flat part as a flat window-map (FWmap). In the FWmap, if the orbit starting from the window returns to the window, it becomes the SSPOs. As the parameter of the system varies, the FWmap exhibits various SSPOs and bifurcations. Using
theoretical analysis, we clarify existence regions of the period of the SSPOs in the parameter space. It should be noted these results are important for consideration of encoding characteristic of the ADC.

Next, we consider an application of the FWmap to the ADC. The ADC encodes an analog input to a digital output and have been studied in many researchers [11–13]. Based on the dynamics of the FWmap, we can realize the ADC that an analog input and digital output sequence can correspond to a parameter and SSPO. We show typical operation and encoding characteristics. This ADC realizes the rate-encoding and higher resolution than a Σ − Δ modulator [14]. An estimative analog value of the digital output is given by a fraction with the denominator that is equal to a period of the SSPO. Using algebraic calculations, we obtain the following results: the estimative analog value of the digital output can be calculated theoretically. It is especially clarified that the appearance of the estimative analog value corresponds to Farey series as the parameter of the window varies.

Next, we consider the case where the window has a small slope. In real system, we can think that the flat window fluctuates and is destroyed by parameter mismatch, noise influence and so on. In this paper, we assume that the small slope of the window is a kind of parameter perturbation and its map is called a sloping window-map (SWmap). We show typical phenomena and bifurcation of the SWmap. Since the SWmap does not have the flat part, the SSPOs do not occur. However, if the slope on the window is small, the SWmap based on rotation exhibits a SSPO-like stable periodic orbit. Note that superstability is destroyed but the map keeps stability. In fact, we have confirmed the dynamics of the FWmap based on rotation in experiments, using the spiking neuron model [9].

In section 3, we consider the dynamics of the FWmap and SWmap based on chaos. As compared with the FWmap based on rotation, the FWmap based on chaos exhibits more complicated SSPOs and bifurcation. However there is the case where stability of the SWmap based on chaos is destroyed because the orbit is expanding except the window. We show above description, using typical phenomena and Lyapunov exponent in numerical simulations.

We describe significance and novelty of this paper as the following. We have clarified theoretical classification of the SSPOs for the FWmap based on rotation. This result is important for consideration of bifurcation phenomena with superstability as well as consideration of an application to the ADC. In particular, we obtain that the output of the ADC can be completely encoded by the SSPOs. This result can be developed into an improvement in performance of the ADC. Note that [9] and [14] do not clarify classification of the period of the SSPOs and output characteristic of the ADC theoretically. Also, [9] does not discuss a comparison with the FWmap based on chaos in main subjects. Applications of the FWmap based on chaos have been studied in [15] and [16]. However, the system is not robustness for small parameter perturbation. It can be hard for us to observe the stable periodic orbit with long period in the FWmap based on chaos and it can be hard to apply to practical systems. In the FWmap based on rotation, stability including superstability is robustness for small parameter perturbation. Therefore, the FWmap based on rotation can be applied to practical systems such as the ADC.

Fig. 1. Feature of the one-dimensional window-map based on rotation.
In this section, we focus on a one-dimensional map that is described by Eq. (1).

$$x_{n+1} = F(x_n) = F_1(x_n) \mod 1, \quad F_1(x_n) = \begin{cases} x_n + u & \text{for } x_n \notin W, \\ h + s(x_n - w_c) & \text{for } x_n \in W, \end{cases}$$

(1)

where $x_n$ and $n$ are a state variable and discrete time $n = (1, 2, \cdots)$. $W = [w_c - \epsilon, w_c + \epsilon]$ is a control part of this system and we call $W$ “a window” in this paper where $W \subset I \equiv [0, 1]$. $\epsilon$, $w_c$, $h$, and $s$ denote a half width, center, height and slope of the window, respectively. $u$ is a control parameter. For simplicity, we assume

$$0 < \epsilon < 0.5, \quad 0 < u < 1, \quad 0 < s \ll 1, \quad h \equiv u + \epsilon, \quad w_c \equiv \epsilon, \quad (W = [0, 2\epsilon]).$$

(2)

This system has three parameters $\epsilon$, $u$, and $s$. Feature of the map is shown in Fig. 1. Since $F_1$ is modulo one, $x_n \in I$ is satisfied for an initial state $x_1 \in I$ and $n > 1$. We consider two cases where $s = 0$ and $s \neq 0$ hereafter.

### 2.1 The dynamics with a flat window

In this subsection, we consider the case of $s = 0$. Figure 2 shows typical feature of the maps for $s = 0$. In this paper, we refer to the map with $s = 0$ as the flat-window-map (FWmap) and Fig. 2 is the FWmap based on rotation. When $s = 0$, the FWmap has a flat segment on the window. In order to consider the dynamics of the map, we give some definitions here.

**Definition1:** A point $x_f$ is said to be a fixed point (or one periodic point) if $x_f \in I$ and $x_f = F(x_f)$ are satisfied. Points $x_p$ are said to be $k$ periodic points if $x_p \in I$, $x_p = F^k(x_p)$ and $x_p \neq F^j(x_p)$ are satisfied.
satisfied where $1 < j < k$ and $F^k$ is $k$-fold composition of $F$. Periodic points sequence \{x_1, x_2, \ldots, x_k\} is said to be $k$ periodic orbit.

**Definition 2:** If $|DF^k(x_p)| > 1$, $|DF^k(x_p)| < 1$ or $|DF^k(x_p)| = 0$ are satisfied, $k$ periodic orbit is said to be unstable, stable or superstable for initial states where $DF \equiv \frac{d}{dx}F$. In particular, stable and superstable periodic orbits are abbreviated to SPOs and SSPOs. The SPO and SSPO with period $k$ are abbreviated to $k$-SPO and $k$-SSPO.

**Definition 3:** When there are some positive integers $k$ such as $F(I) \subseteq I$ and $|DF^k(x_1)| \geq 1$ for almost $x_1$ are satisfied, the map $F$ exhibits chaos [17].

Figure 2 exhibits 8, 3, 5 and 7-SSPOs, respectively. In the FWmap, the orbit starting from $W$ returns to $W$ and it becomes the SSPO. Except for the window, the map of Eq. (1) is the rotation map (unit circle map) that exhibits periodic or quasi-periodic orbits for rational or irrational parameter $u$. However, since measure of rational $u$ is zero, the orbits become quasi-periodic orbits and covers an interval $I$ for almost $u$. So the orbit of the FWmap based on rotation can hit the window after transient time and return to the window. Figure 3(a) and (b) show a bifurcation diagram and period of the SSPOs for $u$. The orbit is almost SSPO and we can see that there are the SSPOs with various period and corresponding to bifurcations as $u$ varies.

In order to classify the period of the SSPO here, let $D_1$ and $D_N$ be defined as existence regions of 1-SSPO and $N$-SSPO, respectively. Assuming an initial state $x_1 \in W$, we obtain the following theorems using algebraic calculations.

**Theorem 1:** The 1-SSPO can exist in $D_1 = \{(u, \epsilon) \mid 0 < u < \epsilon, 1 - \epsilon < u < 1\}$.

Proof: $x_2 = x_3$ can be satisfied and the orbit becomes the 1-SSPO if
Fig. 4. Existence regions of the period \( N \) of the SSPOs for \( N \leq 9 \). (\( s = 0 \)). The number denotes \( N \).

\[ x_2 = (u + \epsilon \mod 1) \in W. \]  

(3)

Considering modulo 1, Eq. (3) is guaranteed if two inequalities \( 0 < x_2 < 2\epsilon \) and \( 1 < x_2 < 1 + 2\epsilon \) are satisfied. Considering the range \( 0 < u < 1 \) and substituting \( u + \epsilon \) for \( x_2 \) to two inequalities, Eq. (3) is satisfied when \( 0 < u < \epsilon \) and \( 1 - \epsilon < u < 1 \). Q.E.D.

**Theorem 2:** The \( N \)-SSPO can exist in \( D_N = \{(u, \epsilon) \mid 0 < u < \frac{\epsilon}{N}, \frac{m-\epsilon}{N} < u < \frac{m+\epsilon}{N}, \frac{N-\epsilon}{N} < u < 1, (u, \epsilon) \notin \bigcup_{l=1}^{N-1} D_l \} \) where \( l \) is a positive integer and \( m = (1, 2, \cdots N-1) \).

**Proof:**

\[ x_{N+1} = ((Nu + \epsilon \mod 1) \in W \] and \( x_m \notin W, \)  

(4)

Considering modulo 1, Eq. (4) is guaranteed if \( N + 1 \) inequalities \( 0 < x_{N+1} < 2\epsilon, \cdots m < x_{N+1} < m+2\epsilon, \cdots N < x_{N+1} < N+2\epsilon \) are satisfied. Considering the range \( 0 < u < 1 \) and substituting \( Nu + \epsilon \) for \( x_{N+1} \) to \( N + 1 \) inequalities, Eq. (4) is satisfied when \( 0 < u < \frac{\epsilon}{N}, \frac{m-\epsilon}{N} < u < \frac{m+\epsilon}{N}, \frac{N-\epsilon}{N} < u < 1, \) and \((u, \epsilon) \notin \bigcup_{l=1}^{N-1} D_l \). Q.E.D.

Figure 4 shows existence regions of the \( N \)-SSPO for \( N \leq 9 \). For example, \( D_N = \{(u, \epsilon) \mid 0.099 < u < \frac{0.099}{4}, \frac{0.909}{3} < u < \frac{0.909}{4} \} \) is given when \( \epsilon = 0.091 \). Using the above theorems, existence regions of the \( N \)-SSPO can be clarified theoretically. For smaller \( \epsilon \), the SSPOs with long period can exist since the window is narrow and the orbit tends to be hard to hit the window. It should be noted that these results are important for consideration of the analog-to-digital converter (ADC) characteristics that are shown in Subsection 2.2. It should be also noted that [9] does not clarify existence regions of the SSPOs theoretically.

### 2.2 An application to the analog-to-digital converter

The FWmap based on rotation dynamics can be an application to the ADC with the rate encoding [9]. The ADC encodes an analog input to a digital output sequence. In this ADC, an analog input and digital output sequence correspond to the parameter \( u \) and the SSPO, respectively. When the input \( u \) and the initial state \( x_1 \) are given, the digital output sequence \( \{y_n\} \) corresponding to the SSPO can be obtained where \( y_n \) is defined by

\[
y_n = \begin{cases} 
Q(x_n) & \text{for } x_n \notin W \\
Q(w_c) & \text{for } x_n \in W
\end{cases} \quad Q(x) = \begin{cases} 0 & \text{for } x < 1 - u \\
1 & \text{for } x \geq 1 - u.
\end{cases}
\]

(5)
Fig. 5. Encoding characteristics of the ADC ($\epsilon = 0.091, s = 0$). (a) Conversion characteristic, (b) Error characteristic. Black regions denote an error characteristic (value). In other words, we have painted the graph with a vertical black line as error value for an input $u$.

Fig. 6. Distribution of $\tilde{u} = \frac{n}{N}$ in the input and parameter space ($s = 0$).

On the window $W$, the output depends on the center of the window $w_c$ and is independent of $x_n$. In Fig. 2 (a), for example, the digital output sequence $\{y_n\} = \{0, 0, 0, 0, 0, 0, 1\}$ is obtained for $u = 0.12$ and $\epsilon = 0.091$. It should be noted that the digital output sequence $\{y_n\}$ has a code length $N$ that is equal to a period of the SSPO.

In order to evaluate characteristics of the ADC here, we define an estimative analog value $\tilde{u}$ of the digital output (In other words, $\tilde{u}$ is a decoded analog value using an appropriate decoding system):

$$\tilde{u} = \frac{1}{N} \sum_{n=1}^{N} y_n.$$  \hspace{1cm} (6)

That is, $\tilde{u}$ is given by a fraction with the period $N$ and ratio of the output 1. This ADC can realize the rate encoding. In Fig. 2 (a), $\tilde{u} = \frac{1}{8} = 0.125$ is calculated for $u = 0.12$. Other values in Fig. 2 can be calculated:

Fig. 2 (b) : $\{y_n\} = \{0, 0, 1\}, \tilde{u} = \frac{1}{3} \approx 0.333$ for $u = 0.31$.

Fig. 2 (c) : $\{y_n\} = \{0, 1, 1, 1, 1\}, \tilde{u} = \frac{4}{5} = 0.8$ for $u = 0.79$. 

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Fig. 2 (d): \( \{ y_n \} = \{ 0, 1, 1, 1, 1, 1 \} \), \( \ddot{u} = \frac{6}{7} \approx 0.857 \) for \( u = 0.86 \).

Each \( \ddot{u} \) is an approximation value for each \( u \). Figure 5 shows an encoding characteristic of the ADC. Vertical axes of Fig. 5(a) and (b) denote the estimative analog value \( \ddot{u} \) and the error \( |\ddot{u} - u| \), respectively. Figure 5(a) has discontinuous step characteristic though \( \ddot{u} \) increases as \( u \) increases. In Fig. 5(b), a black region denotes an error characteristic (value). In other words, we have painted the graph with a vertical black line as error value for an input \( u \). It should be noted that this ADC has higher resolution than a basic ADC: \( \Sigma - \Delta \) modulator [14].

In order to clarify encoding characteristics of the ADC, let \( D_{(0,1)}, D_{(1,1)} \) and \( D_{(m,N)} \) be defined as existence regions that can output \( \ddot{u} = \frac{0}{1}, \ddot{u} = \frac{1}{1} \) and \( \ddot{u} = \frac{m}{N} \) where \( N \) is a code length of the digital output and \( m \) is a positive integer. Referring Theorems 1 and 2, we can derive the following theorems.

**Theorem 3:** The ADC of Eq. (1) with \( s = 0 \) can output \( \ddot{u} = \frac{0}{1} \) for \( D_{(0,1)} = \{ (u, \epsilon) \mid 0 < u < \epsilon \} \) and \( \ddot{u} = \frac{1}{1} \) for \( D_{(1,1)} = \{ (u, \epsilon) \mid 1 - \epsilon < u < 1 \} \).

Proof: \( D_{(0,1)} \) exhibits the 1-SSPO and \( N = 1 \) by Theorem 1. So Eq. (5) guarantees that the digital output \( \{ y_n \} = \{ 0 \} \) if \( w_c < 1 - u \) is satisfied since \( x_n \in W \). Considering the range \( 0 < \epsilon < 0.5 \) and assuming \( u < \epsilon \), an inequality \( w_c = \epsilon < 1 - u \) is guaranteed. Also, \( D_{(1,1)} \) exhibits the 1-SSPO and \( N = 1 \). So Eq. (5) guarantees that the digital output \( \{ y_n \} = \{ 1 \} \) if \( w_c > 1 - u \) is satisfied since \( x_n \in W \). Assuming the range \( 1 - \epsilon < u \), an inequality \( w_c = \epsilon > 1 - u \) is guaranteed. Q.E.D

**Theorem 4:** The ADC of Eq. (1) with \( s = 0 \) can output an rational fraction \( \ddot{u} = \frac{m}{N} \) for \( D_{(m,N)} = \{ (u, \epsilon) \mid \frac{m-1}{N-1} < u < \frac{m+1}{N+1}, (u, \epsilon) \notin \bigcup_{l=1}^{N-1} D(k,l) \} \) where \( k = (0, 1, \cdots l) \) and \( l \) is a positive integer.

Proof: \( D_{(m,N)} \) exhibits the \( N \)-SSPOs by Theorem 2. We introduce dummy variables \( z_n \) and \( z_{n+1} = F_1(z_n) \). When an initial state \( z_1 \in W \) is given, \( z_{N+1} = Nu + \epsilon \). Assuming the range \( \frac{m-1}{N-1} < u < \frac{m+1}{N+1} \), an inequality \( m < z_{N+1} < m + 2\epsilon \) is satisfied. Since a number of the digital output “1” is equal to an integer part of \( z_{N+1} \), \( \{ y_n \} \) has the code length \( N \) and output “1” of \( m \) times. Therefore, \( \ddot{u} = \frac{m}{N} \) is calculated when Theorem 4 is satisfied. Q.E.D

Using Theorems 3 and 4, we can calculate the estimative analog value \( \ddot{u} \) for the input \( u \) and parameter \( \epsilon \) theoretically. Figure 6 shows distribution of \( \ddot{u} \) in the input and parameter space. We can see interesting properties: appearance of \( \ddot{u} \) corresponds to Farey series as \( \epsilon \) decreases. That is, all of the SSPO is encoded and classified, based on a rational fraction of Farey series. Note that these results are not shown in [9] and [14].

### 2.3 The dynamics with a sloping window

Equation (1) with \( s = 0 \) exhibits various SSPOs when the parameters vary. In real systems, however, there are parameter mismatch, an effect on nonideal characteristic of circuit elements, noisy influence and so on. By effects of these, we think that the flat part on the window fluctuates and the zero-slope is destroyed. In this paper, we consider basic case where the window has a small slope \( s \neq 0 \) as parameter perturbation.

Typical feature of the map and phenomena of Eq. (1) with \( s = 0.01 \) are shown in Fig. 7. We refer to this map as a sloping window-map (SWmap). Figure 7 is the SWmap based on rotation dynamics. Although the SWmap does not have a flat part and does not exhibit the SSPOs, the orbit is contracting when \( s < 1 \). Therefore the SWmap based on rotation dynamics exhibits various SPOs. Moreover, if the parameter \( s \) is nearly equal to 0, feature of the SPO is similar to that of the SSPO and mostly a period is equal to that of the SSPO as shown in Figs. 2 and 7. A bifurcation diagram for \( u \) is shown in Fig. 8. We can see that this diagram is similar to Fig. 3 and the period of the SPO is equal to that of the SWmap for most of \( u \). These results can describe as the following: superstability of the FWmap based on rotation dynamics is destroyed but stability is kept for small parameter perturbation. Roughly speaking, stability including superstability is robustness.

It should be noted that [9] has considered the bifurcation phenomena for wider parameter range of \( s \). Referring to [9], we can describe basic bifurcation phenomena as the following. If \( |s| > 1 \) or \( |s| < 1 \), the SWmap based on rotation exhibits chaos or the SPO. If \( s = 0 \) or \( s = 1 \), the SWmap based on
rotation exhibits the SSPO or quasi-periodic orbits. However, it is very hard to clarify theoretically whether the period of the SSPO cannot exist how much the parameter $s$ varies. This consideration is in future problems.

Figure 9 shows characteristics of the ADC as $s = 0.01$ in numerical simulations. When $s \neq 0$, the SWmap exhibits the SPO. Since it is hard to that we decide whether the SPO converges or not, in this case, we assume $\tilde{u}$ as Eq. (7).

$$\tilde{u} = \frac{1}{M} \sum_{n=1}^{M} y_n,$$

(7)
Fig. 9. Encoding characteristics of the ADC in the SWmap based on rotation dynamics ($\epsilon = 0.091, s = 0.01, M = 400$). (a) Conversion characteristic. (b) Error characteristic. Black regions denote an error characteristic (value).

where $M$ is a large integer and Fig. 9 is $M = 400$. We can see that the encoding characteristic of the ADC is similar to that of $s = 0$ even if the slope on the window fluctuates. In fact, using the spiking neuron model that realizes the dynamics of the FWmap, we have fabricated a test circuit and have observed typical phenomena of the FWmap based on rotation dynamics [9]. We think that the FWmap based on rotation dynamics is robustness for small parameter perturbation and can be an application to real systems.

3. A one-dimensional window-map based on chaos

In this section, we consider a one-dimensional window-map based on chaotic dynamics. The objective map is described by

$$x_{n+1} = F(x_n) = F_2(x_n) \mod 1, \quad F_2(x_n) = \begin{cases} 2x_n & \text{for } x_n \notin W, \\ h + s(x_n - w_c) & \text{for } x_n \in W, \end{cases}$$

where $x_n, n, w_c, \epsilon, h, s$ and $W = [w_c - \epsilon, w_c + \epsilon]$ are the same definitions in Section 2. We assume parameter ranges and conditions for simplicity:

$$0 < \epsilon < 0.5, \quad 2\epsilon < h < 1 - s\epsilon, \quad 0 < s < 1, \quad w_c \equiv \frac{h}{2}.$$  

This system has three parameters $h, \epsilon$ and $s$. If $2\epsilon > h$, $W \subset I$ does not satisfy since $w_c - \epsilon$ becomes a negative value. Feature of the map is shown in Fig. 10. The map is equivalent to the cut map except
the window $W$. It is well known that the cut map exhibits chaos for most of the initial state. We consider two cases where $s = 0$ and $s \neq 0$ hereafter.

3.1 The dynamics with a flat window

We consider the case of $s = 0$ and call this map the FWmap based on chaos. Typical phenomena are shown in Fig. 11(a) and (b) that exhibit the 3 and 9-SSPOs. This map has the slope 2 and the orbit is expanding, except for the window. However, as shown in Fig. 11(a) and (b), the orbit becomes the SSPO if it starting from $W$ return to $W$ regardless of hitting times on the slope 2. Figure 11(c) shows a bifurcation diagram for $h$. Although we can see that this figure shows the complicated bifurcation diagram as compared with Fig. 3(a), the orbit for almost $h$ becomes the SSPO. It should be noted that there are rich SSPOs in the FWmap based on chaos. The cut map with a flat segment is found in chaos-control dynamics of the electrical circuit [16]. [16] has proposed an algorithm that clarifies existence regions of the period of the SSPOs and theoretical results have been given.

3.2 The dynamics with a sloping window

We consider the case of $s \neq 0$ and we call this map the SWmap based on chaos. Figure 12 shows typical phenomena. Since the slope on the window is not zero, this SWmap does not exhibit the SSPOs. Figure 12(a) exhibits the 3-SPO, however, Fig. 12 (b) exhibits chaotic orbit. That is, we can describe that the 9-SSPO disappears and stability is destroyed when $s = 0.01$. In other words, the SSPO is changed into chaotic orbit as $s$ varies from 0. There is the case where the orbit is expanding even if $s < 1$, since the slope except the window is 2. Figure 13 shows a bifurcation diagram and corresponding to Lyapunov exponent for $h$. Lyapunov exponent $\lambda$ is defined as the following.
Fig. 12. Typical SW maps based on chaos ($\epsilon = 0.1, s = 0.01$). (a) 3-SPO ($h = 0.3$), (b) Chaotic orbit ($h = 0.736$).

Fig. 13. Bifurcation diagram and Lyapunov exponent of the SW map based on chaos ($\epsilon = 0.1, s = 0.01$). (a) Bifurcation diagram, (b) Lyapunov exponent corresponding to Fig. (a) ($N = 8000$).

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \ln |DF_2(x_n)|,$$

where $N$ is a very large positive integer in numerical simulations. Stable and chaotic orbits have negative and positive Lyapunov exponent $\lambda$, respectively. Although the map exhibits various SSPOs in Fig. 11(c), we can see that chaotic region occurs in Fig. 13(a). Roughly speaking, the SSPO with long period tends to be changed into chaotic orbit.

The cut map is deeply related to the ADC system with the binary encoding [15]. In this ADC, an analog input corresponds to an initial value of the state variable. Using the FW map of Eq (8) with
\( s = 0 \), the analog input may be encoded as the digital output sequence via the SSPO. However, in a real system, the zero-slope on \( W \) is destroyed. We think that the SPO with long period cannot observe and unstable orbit may observe in the experiments.

4. Conclusions
This paper has been studied robustness and an application of a one-dimensional window-maps based on rotation dynamics. The FWmap exhibits a variety of the SSPO and bifurcation. The orbits become the SSPOs for almost parameters. Using theoretical analysis, we have clarified the period of the SSPOs in the parameter space.

Next, we have considered an application of the FWmap to the ADC. The controlling parameter and SSPO correspond to the analog input and digital output, respectively. The FWmap can realize the ADC with the rate encoding. We have clarified distribution of estimative analog value corresponding to the output theoretically. Decreasing a width of the window, appearance of estimative analog value is the same as appearance of Farey series. Next, as parameter perturbation, we have considered the dynamics with small slope on the window. The SWmap based on rotation dynamics exhibits the SPOs. If the slope on the window is much smaller than 1, feature of the SPO is similar to that of the SSPO: the FWmap keeps stability and is robustness for small parameter perturbation. Characteristics of the ADC are also not influenced by small parameter perturbation. Therefore we think that the FWmap is able to be application to real systems.

On the other hands, the FWmap based on chaos exhibits rich SSPOs and complicated bifurcation. However, even if the slope on the window is smaller than 1, there is the case where the orbits become chaotic orbit: stability of the FWmap based on chaos is destroyed for parameter perturbation.

We describe future problems as the following: more detailed analysis for bifurcation and classification of the SPO in the SWmap, implementation of the dynamics of the FWmap and SWmap, consideration of other applications and so on.

Acknowledgments
We would like to thanks Professor Toshimichi Saito for helpful advice and comments greatly.

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