Paper

The Stabilizing mechanism for an interrupted dynamical system with periodic threshold

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Abstract: This study mathematically analyzes an interrupted dynamical system (IDS) with a periodic threshold. First, we describe a simple IDS, which is dependent on its own state and a periodic interval, and explain the behavior of the waveform. Then, we define the discrete map (return map) of the system and calculate the bifurcation diagrams. Finally, we focus on the dynamical structure of the return map in the system with a periodic threshold and discuss the stabilizing mechanism, especially its effect in a wide parameter space. The stabilizing effect is verified by the laboratory experiment.

Key Words: interrupted dynamical system, periodic threshold, fixed threshold, bifurcation, return map, stability

1. Introduction

A piecewise-smooth system (PWS) has a discrete model for N regions. Each of the discrete models has an interface at the border. Although the map of the PWS that includes the border is continuous, the border is a singular point. We know that the PWS often exhibits complex behavior owing to its composite dynamical structure. Analysis of the nonlinear phenomena in the PWS is one of the interesting research topics [1]. Previous studies mainly focused on the dynamical mechanism of the local bifurcation. However, a new type of the bifurcation phenomenon called the border-collision bifurcation was discovered in the 1990's [2, 3]. This contributed in the development of the nonlinear theory.

The discrete model of the interrupted dynamical system (IDS) can often be used to construct the

PWS. The IDS has two or more different dynamical systems that can be interfaced with a switch. There are various types of IDSs in the engineering field. For example, some IDSs are dependent on the state (state-dependent IDS), while others are dependent on the state and a periodic interval (state- and time-dependent IDS). In the electrical engineering field, we can easily realize these IDSs using switching devices such as switch, diode, transistor, and relay. In particular, the DC/DC or AC/DC switching regulators are the typical examples of the state- and time-dependent IDS [4–6]. We know that their detailed analysis is extremely important not only from the practical viewpoint but also from the physical viewpoint. There are many studies reporting the nonlinear phenomena in the IDS from the physical viewpoint [7–13]. We have also proposed a simple model, which simulates the dynamic behavior of the current mode controlled DC/DC converter, and have essentially investigated the bifurcation phenomena including the border-collision bifurcation [14, 15].

For the past few decades, techniques for chaos control have been developing in the engineering field. In particular, the OGY chaos control method [16] and delayed feedback control (DFC) scheme [17] are well-known chaos control techniques. Moreover, [18–20] imply that a periodic threshold has a stabilizing effect on the state-dependent IDS. References [21, 22] report on the initial research of the stabilizing effect of the periodic threshold in the state- and time-dependent IDS. They indicate that the periodic threshold is effective in stabilizing the chaotic attractor in the IDSs. However, previous studies have only discussed a local stabilizing effect around a fixed point.

This study provides a global analysis of the stabilizing mechanism in a state- and time-dependent IDS with a periodic threshold. First, we show simple state- and time-dependent IDS and explain its dynamics. Then, we define the discrete model (return map) for the following analysis. Using the return map, we derive the one- and two-parameter bifurcation diagrams and discuss the bifurcation phenomena. Finally, we focus on the dynamical structure of the return map and globally clarify the stabilizing mechanism. Moreover, based on the two-parameter bifurcation diagram, we show the stabilizing effect of the periodic threshold from a wide parameter space. The mathematical results are verified by laboratory experiment.

2. An interrupted dynamical system

2.1 System description and behavior of the waveform

We propose a simple IDS as shown in Fig. 1. In the following analysis, we use the dimensionless values:

$$t' = \frac{t}{RC}, \quad T' = \frac{T}{RC}, \quad x = \frac{v}{E}, \quad x_r(t) = \frac{v_r(t)}{E}. \quad (1)$$

Note that we rewrite t' and T' with t and T for the sake of simplicity. The circuit equation is described as follows:

$$\frac{dx}{dt} = \begin{cases} -x + 1, & \text{system A} \\ -x, & \text{system B} \end{cases}. \quad (2)$$

The solution of Eq. (2) is derived as follows with the initial value x_k at $t = kT$:

$$x(t) = \begin{cases} \varphi(t, x_k) = (x_k - 1)e^{-(t-kT)} + 1, & \text{system A} \\ \phi(t, x_k) = x_k e^{-(t-kT)}, & \text{system B} \end{cases}. \quad (3)$$

Fig. 1. A circuit model.

Figure 2 shows the examples of the waveforms in the system with periodic and fixed thresholds, respectively. In the figure, $x_r(t)$ denotes the threshold:

$$x_r(t) = a \sin \Omega t + b, \quad (4)$$

where a , b , and $\Omega = 2\pi/T$ denote the amplitude, baseline, and angular velocity, respectively. Note that $a = 0$ corresponds to the system with a fixed threshold. If the waveform reaches the threshold $x_r(t)$, the system changes from A to B. After that the system changes from B to A when the next clock pulse appears. The clock pulse is ignored if it appears when the system is in A. We assume that the switching action does not occur until $x(t) > x_r(t)$ is satisfied; this switching rule has been proved in a laboratory experiment. Moreover, we do not consider the situation where the clock pulse appears when the waveform reaches the periodic threshold for the sake of the simplicity because such an event occurs very rarely. In addition, we assume the following condition to prevent two or more switching actions in one clock interval:

$$0 < b - x(kT), \quad k = 0, 1, 2, \dots \quad (5)$$

2.2 Return map

There are two types of the waveforms during the clock interval. One belongs to system A until the next clock pulse appears. Thus, the return map G is defined as follows:

$$\begin{aligned} G: \mathbf{R} &\rightarrow \mathbf{R} \\ x_k &\mapsto x_{k+1} = \varphi((k+1)T, x_k) \end{aligned} \quad (6)$$

The other one, which has the initial value x_k at $t = kT$, reaches the threshold $x_r(t)$ at $t = kT + t_\alpha(x_k)$. Thus, the following map $\hat{G}_1$ is defined as

$$\begin{aligned} \hat{G}_1: \mathbf{R} &\rightarrow \mathbf{\Pi} \\ x_k &\mapsto x_r(kT + t_\alpha(x_k)) = \varphi(kT + t_\alpha(x_k), x_k), \end{aligned} \quad (7)$$

where $\mathbf{\Pi}$ denotes the threshold. Then, the waveform reaches x_{k+1} at $t = (k+1)T$. Thus, we defined the following map $\hat{G}_2$:

Fig. 2. Examples of the waveforms.

$$\begin{aligned} \hat{G}_2 : \quad \mathbf{\Pi} &\rightarrow \mathbf{R} \\ x_r(kT + t_\alpha(x_k)) &\mapsto x_{k+1} = \phi((k+1)T, x_r(kT + t_\alpha(x_k))) \end{aligned} \quad (8)$$

Consequently, the following return map is defined for the other case:

$$\begin{aligned} G : \quad \mathbf{R} &\rightarrow \mathbf{R} \\ x_k &\mapsto x_{k+1} = \hat{G}_2 \circ \hat{G}_1 \end{aligned} \quad (9)$$

At the next stage, we rigorously define the return map based on Eqs. (6), (7), (8), and (9). Here, we describe the border D_p to classify the waveform during the clock interval. Let the slopes of the waveform and the threshold be $dx(t)/dt$ and $dx_r(t)/dt$, respectively.

$$\frac{dx(t)}{dt} = -(x_k - 1)e^{-t} \quad (10)$$

$$\frac{dx_r(t)}{dt} = a\Omega \cos \Omega t \quad (11)$$

If the initial value x_k is on the border D_p , i.e., $x_k = D_p$, the slopes $dx(t)/dt$ and $dx_r(t)/dt$ satisfy $dx(t)/dt = dx_r(t)/dt$ at $t = \hat{t}_\alpha(x_k)$. Therefore, the border D_p is derived as follows:

$$D_p = 1 - a\Omega e^{\hat{t}_\alpha(x_k)} \cos \Omega \hat{t}_\alpha(x_k), \quad (12)$$

where we numerically calculate a time $\hat{t}_\alpha(x_k)$. In particular, the border D_p also satisfies the following condition (see Eqs. (3) and (4)):

$$(D_p - 1)e^{-\hat{t}_\alpha(x_k)} - a \sin \Omega \hat{t}_\alpha(x_k) + 1 - b = 0. \quad (13)$$

By substituting Eq. (12) into Eq. (13), we get

$$a \sin \Omega \hat{t}_\alpha(x_k) + a\Omega \cos \Omega \hat{t}_\alpha(x_k) + b - 1 = 0. \quad (14)$$

A time $\hat{t}_\alpha(x_k)$ is derived by solving Eq. (14). Note that the border D_f , for the circuit with fixed threshold, is defined as follows [14]:

$$D_f = (b - 1)e^T + 1. \quad (15)$$

The system belongs to A during the clock interval if $x_k \leq D_p$ is satisfied. Thus, the return map is given by

$$x_{k+1} = (x_k - 1)e^{-T} + 1. \quad (16)$$

On the other hand, the system changes from A to B during the clock interval if $x_k > D_p$ is satisfied. Here, the system belongs to A during a time $t_\alpha(x_k)$. A time $t_\alpha(x_k)$ is defined by solving the following equation:

$$(x_k - 1)e^{-t_\alpha(x_k)} + 1 - a \sin \Omega t_\alpha(x_k) - b = 0. \quad (17)$$

Note that we mathematically define a time $t_\alpha(x_k)$ in the system with fixed threshold as follows:

$$t_\alpha(x_k) = \log \frac{x_k - 1}{b - 1}. \quad (18)$$

Then, the system changes to B and it stays as B during a time $t_\beta(x_k)$.

$$t_\beta(x_k) = T - t_\alpha(x_k) \quad (19)$$

Thus, the return map is given by

$$x_{k+1} = (x_k - 1)e^{-T} + e^{-T+t_\alpha(x_k)}. \quad (20)$$

Consequently, we can completely define the return map as follows:

$$x_{k+1} = G(x_k) = \begin{cases} g_1(x_k) = (x_k - 1)e^{-T} + 1, & x_k \leq D_p \\ g_2(x_k) = (x_k - 1)e^{-T} + e^{-T+t_\alpha(x_k)}, & x_k > D_p \end{cases}. \quad (21)$$

Note that we assume that the border D_p is included in the return map $g_1(x_k)$ because this supposition makes it physically possible to reconstruct the return map of the laboratory experiment. The derivatives of the return maps $dg_1(x_k)/dx_k$ and $dg_2(x_k)/dx_k$ are derived as follows:

$$\frac{dg_1(x_k)}{dx_k} = e^{-T}, \quad (22)$$

$$\frac{dg_2(x_k)}{dx_k} = e^{-T} + \frac{\partial t_\alpha(x_k)}{\partial x_k} e^{-T+t_\alpha(x_k)} = e^{-T} \left(1 - \frac{1}{(1-x_k)e^{-t_\alpha(x_k)} - a\Omega \cos \Omega t_\alpha(x_k)} \right). \quad (23)$$

The stable period- m solution is defined as follows:

$$G^m(x_0) = x_0, \quad G^l(x_0) \neq x_0 \quad (l < m), \quad (24)$$

where x_0 denotes a fixed point. Figure 3 shows an example of the waveform and its return map in the system with fixed and periodic thresholds; when $a = 0.0$, $t = 0.0 - 10.0$ and when $a = 0.02$, $t = 10.0 - 20.0$ in the waveform. Figure 4 shows a conceptual diagram for understanding the behavior of the waveform around the border. The red points in the figure correspond to the borders D_p for the system with periodic threshold and D_f for the system with fixed threshold, respectively. This figure demonstrates the reason why the return map of the system with periodic threshold becomes discontinuous around the border. Here, we focus on the solutions x_{k+1} at $t = (k+1)T$ (see the orange dotted line areas in Fig. 4). In the system with periodic threshold, the solutions x_{k+1} never get close to each other even if we choose the initial values x_k around the border D_p (see Fig. 4(a)), and therefore, the return map becomes discontinuous at the border interface.

3. Theoretical and experimental results

3.1 Bifurcation analysis in the system with periodic threshold

There are two types of the bifurcation phenomena in the system with periodic threshold: period-doubling bifurcation and border-collision bifurcation. Figure 5 shows examples of the one-parameter

Fig. 3. An example of the waveform and its return map ($T = 0.5$, $b = 0.8$).

Fig. 4. A conceptual diagram for understanding behavior of the waveform around the border.

bifurcation diagram. These one-parameter bifurcation diagrams are derived by increasing or decreasing a parameter b . The blue areas in Fig. 5 denote the region in which the period-2 solution and the period-3 solution coexist. The symbols P and B denote a bifurcation point of the period-doubling bifurcation and the border-collision bifurcation, respectively. Figure 6 shows the appearance of the bifurcation phenomena. It is clear that the period-2 solution becomes unstable through the period-doubling bifurcation. The period-3 solution is generated when a part of the solution collides with the border D_p (see Fig. 6(a)). The period-doubling bifurcation in the system with periodic threshold is defined as follows:

$$\prod_{k=1}^m \frac{dG(x_k)}{dx_k} + 1 = 0. \quad (25)$$

On the other hand, the border-collision bifurcation occurs if a part of the solution collides with the border D_p . We observe that a part of the period-3 solution collides with the border D_p , and the period-2 solution is generated (see the blue dotted line area in Fig. 6(b)). The border-collision bifurcation is defined as follows:

$$G^m(D_p) - D_p = 0. \quad (26)$$

Using Eqs. (25) and (26), we calculate the bifurcation sets of the period-doubling bifurcation and the border-collision bifurcation. Figure 7 shows the two-parameter bifurcation diagrams in the system with fixed and periodic thresholds, respectively. In these figures, blue lines denote the bifurcation sets of the period-doubling bifurcation and pink lines denote the bifurcation sets of the border-collision bifurcation. The numbers denote the periodicity of the solution. Gray areas denote the existence region of the over period-5 solutions including the non-periodic solution. There are some interesting bifurcation structures in the circuit with fixed and periodic thresholds. For example, the period- m solution bifurcates to the period- $2m$ solution through the period-doubling bifurcation according to the bifurcation theory. However, we observe that the period- m solution bifurcates to the non-periodic solution through the period-doubling bifurcation (see Fig. 7). This is because the border-collision bifurcation appears just after the period-doubling bifurcation. Note that the bifurcation structure in the circuit with fixed threshold is investigated in [14].

Fig. 5. One-parameter bifurcation diagrams ($T = 0.7$, $a = 0.05$).

Fig. 6. Appearance of the bifurcation phenomena ($T = 0.7$, $a = 0.05$).

3.2 Stabilizing mechanism of the periodic threshold

We discuss the stabilizing mechanism of the periodic threshold based on the bifurcation diagrams and the return maps. It is clear that the periodic threshold affects the existence regions of the solution. For example, existence region of the period-1 solution is clearly enlarged in the system with periodic threshold. Here, note that the coexistence region of the period-1 and period-2 solutions is not included in the existence region of the period-1 solution because the period-1 solution may suddenly bifurcate to the period-2 solution if a perturbation occurs in the coexistence region of the period-1 and period-2 solutions. Moreover, we find that the gray area is reduced in the system with periodic threshold (see Fig. 7). These results indicate that the periodic threshold has a stabilizing activity in a wide parameter space for this class of IDS.

At the next stage, we investigate the stabilizing mechanism of the periodic threshold more specifically. Figure 8 shows an example of the one-parameter bifurcation diagram. We change the parameter a from 0.0 to 0.1. Here, a denotes the amplitude of the periodic threshold. The red and blue points correspond to the unstable period-1 and period-2 solutions. Figure 9 shows the examples of the waveform and its return map; these figures correspond to Fig. 8 with $a = 0.0$, $a = 0.05$, and $a = 0.1$, respectively. Note that $a = 0.0$ corresponds to the system with fixed threshold. We observe that the non-periodic solution bifurcates to the period-2 solution through the border-collision bifurcation. Then, the period-2 solution is stabilized to the period-1 solution as the parameter a increases. It is clear that the periodic threshold affects the dynamical structure of the return map (see Fig. 9). We find that the system tends to be stable if the periodic threshold is added in the system dynamics, because slopes of the return maps are globally stabilized by the periodic threshold (see Eqs. (22), (23), and Fig. 9). In addition, the periodic threshold affects not only the global structure of the return map but also the local stability of the fixed point. Tables I and II show the stability of the

Fig. 7. Examples of the two-parameter bifurcation diagram.

Fig. 8. An example of the one-parameter bifurcation diagram for understanding the stability effect of the periodic threshold ($T = 1.0$, $b = 0.9$).

period-1 and period-2 solutions, respectively. We derive the characteristic multiplier λ by calculating m -times composite of the derivatives for each fixed point (see Eqs. (22) and (23)). In particular, the characteristic multiplier λ is derived as follows:

$$\lambda = e^{-mT} \left\{ 1 - \frac{1}{(1 - x_k)e^{-t_\alpha(x_k)} - a\Omega \cos \Omega t_\alpha(x_k)} \right\}. \quad (27)$$

The tables indicate that the absolute value of the characteristic multiplier reduces as the parameter a increases. Consequently, we conclude that the periodic threshold locally and globally stabilize the system dynamics in the state- and time-dependent IDS.

3.3 Laboratory experiment

We realize the IDS shown in Fig. 1 in the laboratory. In the circuit model, the capacitance voltage behaves intermittently with the switching action. The switch is controlled by a flip-flop whose inputs are connected to the output of the comparator and the clock pulse from the generator.

The circuit equation of Fig. 1 is given by

$$RC \frac{dv}{dt} = \begin{cases} -v + E, & \text{switch-A} \\ -v, & \text{switch-B} \end{cases}. \quad (28)$$

Note that Eq. (28) can be rewritten as Eq. (2) using the dimensionless values shown in Eq. (1). The circuit parameters are as follows:

$$R = 10[\text{k}\Omega], \quad C = 0.1[\mu\text{F}], \quad E = 3.0[\text{V}]. \quad (29)$$

Figures 10 shows an example of the waveform and its return map in the laboratory experiment. These figures correspond to Fig. 9. Thus, it can be said that the theoretical results are verified by the laboratory experiment.

4. Conclusions

In this study, we have investigated the effect of the periodic threshold in an IDS whose switching action is dependent on the state and a periodic interval. First, we introduced a simple IDS model, and then explained its dynamics. Next, we mathematically defined the return map of the system. Using the return map, we derived one- and two-parameter bifurcation diagrams in systems with fixed and periodic thresholds, respectively. Finally, we studied the stabilizing mechanism of the periodic threshold based on the return map. Moreover, we discussed the effect of the periodic threshold from a wide parameter space. We found that the periodic threshold has a stabilizing effect for the IDS in a wide parameter space. This stabilizing method is simple and effective method for the IDS. For use this stabilizing method, we only add a small periodic amplitude into the threshold. In addition, the IDS, studied in this paper, simulates the switching dynamics of the frequency-mode-controlled DC/DC converter. Thus, the analytical results may contribute to the practical converters. Future work should be focused onto investigating the stabilizing effect of the periodic threshold in the high-dimensional switching circuits.

Fig. 9. Stabilization effect of the periodic threshold in the return map ($T = 1.0$, $b = 0.9$).

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Table I. Stability analysis of period-1 waveform ($T = 1.0$, $b = 0.9$).

a	Characteristic multiplier λ	Remark
0.0000	-3.3109	Unstable period-1 solution
0.0750	-1.3673	Unstable period-1 solution
0.0900	-0.8761	Stable period-1 solution
0.1000	-0.6618	Stable period-1 solution

Table II. Stability analysis of period-2 waveform ($T = 1.0$, $b = 0.9$).

a	Characteristic multiplier λ	Remark
0.0000	-1.2180	Unstable period-2 solution
0.0020	-1.0663	Unstable period-2 solution
0.0035	-0.9731	Stable period-2 solution
0.0060	-0.8462	Stable period-2 solution

(a) Non-periodic waveform ($a = 0.0[\text{V}]$)

(b) Period-2 waveform ($a = 0.15[\text{V}]$)

(c) Period-1 waveform ($a = 0.3[\text{V}]$)

Fig. 10. Stabilizing effect of the periodic threshold in the laboratory experiment ($T = 1.0[\text{ms}]$, $b = 2.7[\text{V}]$).

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